PHYSICAL REVIEW D VOLUME 24, NUMBER 8 15 OCTOBER 1981

Slowly rotating fluid spheres in general relativity with and without radiation

Selçuk Ş. Bayin*
Department of Physics, The University of Michigan, Ann Arbor, Michigan 48109
and Department of Physics, The University of Victoria, Victoria, British Columbia V8W 2Y2, Canada
(Received 12 January 1981)

We introduce slow rotation to some of the solutions given by Vaidya, which correspond to radiating fluid spheres in general relativity. We give several new analytic solutions, some of which correspond to uniform rotation and others to differentially rotating fluid spheres. We also study the stationary field equations for slowly rotating and nonradiating fluid spheres, and present a new analytic solution and also generalize the solution corresponding to the $P = \alpha\rho$ equation of state. Some of the solutions mentioned above could be physically reasonable throughout the star while the rest could be used to represent portions of it. We discuss the physical properties of these solutions and the boundary conditions in detail.

I. INTRODUCTION

In some of my recent papers[1-3] I have discussed static and radiating fluid spheres in general relativity and constructed several analytic models that could be used to represent some of the compact objects observed in nature. It is interesting that almost every object in the sky exhibits some form of rotation, and today there is even the possibility of the Universe itself being endowed with a slight rotation. In this regard we have recently studied slowly rotating Friedmann universes in detail.[4] Now we will extend our previous work by studying slowly rotating and nonradiating fluid spheres.

During the last 15 years rotating objects have been studied quite extensively. Among these Butterworth and Ipser[5] have numerically studied the structure and stability of rapidly rotating fluid spheres with various amounts of uniform and differential rotation. Hartle and Thorne[6] have studied uniformly rotating white dwarfs and neutron stars up to second order in angular velocity by using Harrison-Wheeler and the Tsuruta-Cameron $V\gamma$ equation of state. Other papers related to the numerical approach on this subject can be found in Abramovicz and Wagoner,[7] where they have presented the analytic theory of slowly and uniformly rotating general-relativistic bodies and discussed conditions of stability.

It is well known that all known pulsars satisfy conditions of slow rotation, i.e., tangential velocity of all fluid elements are much less than the speed of light and the centrifugal force is much less than the gravitational force. In this approximation when we write the field equations to first order in angular momentum, they reduce to the original unperturbed equations with spherical symmetry, plus two additional equations to be solved for $\Omega(r, t)$, which represents the dragging of inertial frames.

In Sec. III we present several new analytic solutions corresponding to slowly rotating and radiating fluid spheres. These solutions are found for the radiating models given by Vaidya[8] (solutions III, IV, and IX), and some of them correspond to uniform rotation while the others to differential rotation. Previously time-dependent and rotating systems have been studied by Silk and Wright.[9] Also Chandrasekhar and Friedman[10] have studied the stability of axisymmetric systems to axisymmetric perturbations in general relativity. However, to the best of my knowledge there are no published exact analytic solutions corresponding to slowly rotating and radiating fluid spheres.

Next we consider nonradiating and slowly rotating fluid spheres. So far the only analytic solutions are given by Adams *et al.*,[11] where they have considered the $P = \alpha\rho$ equation of state, and the two recent solutions found by Whitman,[12] which correspond to the generalization of Tolman's sixth solution and to incompressible matter with the equation of state $\rho = \text{const}$. All these solutions were found for uniform rotation. In Sec. IV we present a new solution for uniform rotation, corresponding to the polytropelike equation of state given by us.[1] This solution is given in terms of Bessel functions and could be used to represent portions of stars. We also reconsider for $P = \alpha\rho$ and give the general solution for Ω with the proper number of integration constants. We discuss the physical properties of these solutions and the boundary conditions in detail. For the exterior of the radiating and slowly rotating fluid spheres we take the Kerr-Vaidya[13,14] metric to first order in angular velocity.

Physical effects of the dragging of inertial frames could in principle be observed. These include the modification of planetary orbits,[15] effects on the spin axis of gyroscopes, rotation of the plane of polarization of polarized light, and modification of the phase[16] and the bending of light by the rotating object. Also an observer at infinity will observe a rotating and radiating star slightly displaced from its true position.[13] Even though this is inconsequential for single stars, for binary systems it has a net effect of giving the wrong semimajor axis, and hence will affect the calculation of mass from Kepler's third law. Even though these effects are expected to be small for normal stars they could be important in binary systems, where one or both of the stars are pulsars.[17]

II. THE FIELD EQUATIONS

To establish the metric consider a general perturbation h_{ik} of a given solution for radiating fluid spheres as

$$g_{ik}=g_{ik}^{(0)}+h_{ik}\,,$$

where $(x^0,x^1,x^2,x^3)=(t,r,\theta,\phi)$.[18] With the assumption of axial symmetry we can choose h_{00}, h_{11}, h_{22}, h_{33}, and h_{03} to be the only nonvanishing components of h_{ik}. The perturbations h_{11}, h_{22}, and h_{33} of the diagonal components come into play in the case of distortion, which is not the subject of the present study, and hence they are set to zero. Also in the case of rotational perturbations h_{00} will be of second order relative to h_{03}. This is because the effect of rotation is to take $d\phi\to d\phi-\Omega dt$. Moreover to first order in Ω deviations from spherical symmetry can be neglected. Hence the perturbed metric can be written as

$$\begin{aligned}ds^2=&f(r)^2g(t)^2dt^2-h(r)^2m(t)^2dr^2\\&-l(r)^2m(t)^2(r^2d\theta^2+r^2\sin^2\theta\,d\phi^2)\\&+2l(r)^2m(t)^2\Omega(r,t)r^2\sin^2\theta\,d\phi\,dt\,.\end{aligned}\tag{2.1}$$

Since we are going to take Vaidya's solutions as our base metric we have taken $g_{ik}^{(0)}$ as separable.

To first order in $\Omega(r,t)$ only the following components of the Ricci tensor are nonzero:

$$R_1^1,\ R_2^2\equiv R_3^3\,,\ R_0^0\,,\ R_1^0,\ R_0^3\,,\ \text{and } R_1^3\,.\tag{2.2}$$

The first four components do not involve $\Omega(r,t)$ and their solution gives $g_{ik}^{(0)}$. This determines the pressure and density distributions, which are not perturbed to first order in Ω.[2-4,9] The remaining field equations related to R_0^3 and R_1^3 can be written as

$$R_{03}=-8\pi(T_{03}-\tfrac{1}{2}g_{03}T)\,,\tag{2.3}$$

$$R_{13}=-8\pi T_{13}\,.\tag{2.4}$$

We take the energy-momentum tensor to be the perfect-fluid energy-momentum tensor plus the energy-momentum tensor for expanding radiation. The second part of this tensor can be used to represent either photons or neutrinos.[3] Hence,

$$T^{ik}=(P+\rho)U^iU^k-Pg^{ik}+\sigma V^iV^k\,,$$

where

$$\begin{aligned}&U^iU_i=1\,,\quad U^0\neq 0,\quad U^1=U^2=0\,,\quad U^3=\frac{\omega}{fg}\,,\\&V^iV_i=0\,,\quad V^0\neq 0\,,\quad V^1\neq 0,\quad V^3\neq 0,\quad \text{and } V^2=0\,.\end{aligned}\tag{2.5}$$

We define the null vector V^i as $dx^i/d\tau$, where $d\tau=f(r)g(t)\,dt$.[19] This gives

$$V^0=\frac{1}{fg},\quad V_0=fg\,,\tag{2.6}$$

and

$$V^3=\frac{\omega}{fg}\,,\quad V_3=\frac{l^2m^2r^2\sin^2\theta}{fg}(\Omega-\omega)\,,\tag{2.7}$$

where $\omega=d\phi/dt$. With these equations (2.3) and (2.4) become

$$\begin{aligned}R_{03}=&-8\pi[(P+\rho)l^2m^2r^2\sin^2\theta(\Omega-\omega)\\&+\tfrac{1}{2}(P-\rho)l^2m^2r^2\sin^2\theta\Omega+\sigma V_0V_3]\,,\end{aligned}\tag{2.8}$$

and

$$R_{13}=0\,,\tag{2.9}$$

where R_{03} and R_{13} are

$$\begin{aligned}R_{03}=-\sin^2\theta\bigg[\bigg(&\frac{f'l'lr^2}{fh^2}+\frac{4ll'r}{h^2}+\frac{l'^2r^2}{h^2}+\frac{f'l^2r}{fh^2}+\frac{l^2}{h^2}-\frac{h'l'lr^2}{h^3}-\frac{h'l^2r}{h^3}+\frac{ll''r^2}{h^2}+\frac{l^2m\dot{m}\dot{g}r^2}{f^2g^3}\\&-\frac{2l^2\dot{m}^2r^2}{f^2g^2}-\frac{l^2\ddot{m}mr^2}{f^2g^2}-1\bigg)\Omega+\bigg(-\frac{h'l^2r^2}{2h^3}-\frac{1}{2}\frac{f'l^2r^2}{fh^2}+\frac{2ll'r^2}{h^2}+\frac{2l^2r}{h^2}\bigg)\Omega'+\frac{1}{2}\frac{l^2r^2}{h^2}\Omega''\bigg]\,,\end{aligned}\tag{2.10}$$

$$R_{13}=-\tfrac{1}{2}\sin^2\theta\left(\frac{m^2l^2r^2}{f^2g^2}\right)\left[\Omega'\left(\frac{3\dot{m}}{m}-\frac{\dot{g}}{g}\right)+\dot{\Omega}'\right].\tag{2.11}$$

Finally the equations to be solved for $\Omega(r,t)$ can be given as

$$\begin{aligned}\Omega''+&\left(-\frac{h'}{h}-\frac{f'}{f}+\frac{4l'}{l}+\frac{4}{r}\right)\Omega'\\&=16\pi h^2m^2(P+\rho+\sigma)(\Omega-\omega)\end{aligned}\tag{2.12}$$

and

$$\Omega'\left(\frac{3\dot{m}}{m}-\frac{\dot{g}}{g}\right)+\dot{\Omega}'=0\,.\tag{2.13}$$

As we can easily see (2.13) will give the time dependence of $\Omega(r,t)$, while (2.12) will determine the

r dependence. Equation (2.13) can be integrated immediately to give

$$\Omega(r,t)=C(r)\frac{g(t)}{m^3(t)}, \tag{2.14}$$

where $C(r)$ is an arbitrary function of r which will be determined from Eq. (2.12). Before we proceed we will summarize the general properties of the solutions given by Vaidya as[8,20]

$$P(r,t)=P(r)\frac{1}{m^2(t)}, \quad \rho(r,t)=\rho(r)\frac{1}{m^2(t)}, \tag{2.15}$$

and

$$\sigma=-\frac{1}{4\pi}\frac{f'}{f^2h}\left(\frac{\dot m}{m}\right)\frac{1}{m^2}, \quad \text{where } \dot m=-s_0 m, \tag{2.16}$$

and s_0 is a separation constant. Note that since we are using comoving coordinates along the radial direction the boundary of the star has to be independent of time. This requires $m(t)=g(t)$.[2,3] Finally using Eqs. (2.15) and (2.16), and substituting (2.14) into (2.12) we obtain

$$\frac{C''/C+(4l'/l+4/r-h'/h-f'/f)C'/C}{16\pi h^2[P(r)+\rho(r)+f's_0/4\pi f^2h]}-1=-\frac{\omega(r,t)m^2(t)}{C(r)}. \tag{2.17}$$

$\omega(r,t)$ is the rotation function of the star, which has to be supplied to the field equations, like an equation of state. However, from (2.17) it is seen that its functional form is restricted by the field equations.[4] Since the left-hand side of (2.17) is only a function of r the right-hand side should also be either a function of r or a function of t alone (and hence should be equal to a constant). From this it follows that $\omega(r,t)$ has to be a separable function. Let $\omega(r,t)=a(r)b(t)$ and consider the following possibilities:

(a) The right-hand side is only a function of r. This includes uniform rotation and (2.17) is given as

$$C''+\left(\frac{4l'}{l}+\frac{4}{r}-\frac{h'}{h}-\frac{f'}{f}\right)C'=16\pi h^2\left[P+\rho+\frac{f's_0}{4\pi f^2h}\right][C-b_0a(r)], \tag{2.18}$$

where $b(t)=b_0/m^2(t)$.

(b) The right-hand side is only a function of t and hence equal to a constant. For this case (2.17) becomes

$$C''+\left(\frac{4l'}{l}+\frac{4}{r}-\frac{h'}{h}-\frac{f'}{f}\right)C'=16\pi h^2\left(P+\rho+\frac{f's_0}{4\pi f^2h}\right)(1-a_0b_0)C, \tag{2.19}$$

where $a(r)=a_0C(r)$ and $m^2b(t)=b_0$. When the constants a_0 and b_0 are unity, we have perfect dragging.

III. SOLUTIONS FOR RADIATING FLUID SPHERES

First we consider case (a) with uniform rotation. We take

$$\omega(r,t)=C_0m(t)^{-2}, \quad \text{where } C_0=\text{const.}$$

With this substitution Eq. (2.18) becomes

$$\overline{C}''+\left(\frac{4l'}{l}+\frac{4}{r}-\frac{h'}{h}-\frac{f'}{f}\right)\overline{C}'=\overline{C}16\pi h^2\left[P(r)+\rho(r)+\frac{f's_0}{4\pi f^2h}\right], \tag{3.1}$$

where $\overline{C}=C-C_0$. We now consider Vaidya's third solution which is given as

$$f(r)=\frac{r^{s+1}}{AC_1(B+r^{2q})^{n-1}},$$
$$h(r)=\frac{r^s}{A(B+r^{2q})^n}, \tag{3.2}$$
$$l(r)=\frac{r^s}{AD(B+r^{2q})^n},$$

and $m(t)=e^{-s_0t}$. Among s, A, C_1, B, q, n, D, s_0, which are constants, the following relations exist:

$$s+1=q(2n^2-1)/n,$$
$$C_1s_0=2nqB, \tag{3.3}$$

and

$$n^2D^2=2q^2(1-2n^2).$$

We then have

$$8\pi P(r,t)=\frac{1}{h^2m^2}\frac{q^2(1-2n)}{r^2n^2}\frac{(1-2n)r^{2q}+(1+2n)B}{r^2q+B}, \tag{3.4}$$

$$8\pi\rho(r,t)=\frac{1}{h^2m^2}\frac{q^2}{r^2n^2}\frac{(1-4n^2)r^{2q}+(1+4n^2)B}{r^{2q}+B}, \tag{3.5}$$

$$8\pi T_1^0=-\frac{1}{h^2m^2}\frac{4C_1Bq^2}{r^3}\frac{(2n-1)r^{2q}+(2n^2-1)B}{(r^{2q}+B)^3}. \tag{3.6}$$

For the above solution Eq. (3.1) becomes

$$\overline{C}''+\left(\frac{2s+3}{r}+\frac{r^{2q-1}}{B+r^{2q}}2q(-2n-1)\right)\overline{C}'=\left\{\left[\frac{2q^2}{n^2}(2-4n)r^{4q}+\left(\frac{4q^2}{n^2}B+\frac{2q^2}{n^2}(2-4n)B+4s_0C_1(s+1)-8qs_0C_1(n-1)\right)r^{2q}+\left(\frac{4q^2}{n^2}B^2+4s_0C_1(s+1)B\right)\right]\Big/r^2(B+r^{2q})^2\right\}\overline{C}. \tag{3.7}$$

From Eq. (3.7), which in general could be given as

$$\bar{C}'' + X_1(r)\bar{C}' + X_2(r)\bar{C} = 0 , \tag{3.8}$$

we can eliminate the first-order derivative by substituting

$$\bar{C} = \nu(r) \exp\left(-\frac{1}{2}\int X_1\, dr\right), \tag{3.9}$$

where

$$\nu'' = -(X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^{\,2})\nu . \tag{3.10}$$

For Eq. (3.7) this gives

$$\bar{C}(r) = \nu(r) r^{-(2s+3)/2}(B + r^{2q})^{(2n+1)/2} . \tag{3.11}$$

The function $\nu(r)$ comes from the solution of (3.10), where

$$\begin{aligned} X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^{\,2} \equiv \Bigg\{ & r^{4q}\Big[-\frac{2q^2}{n^2}(2-4n) + \tfrac{1}{2}(2s+3) + q(2n+1)(2q-1) - 2q^2(2n+1) - \tfrac{1}{4}(2s+3)^2 \\ & \quad + (2s+3)(2n+1)q - q^2(2n+1)^2\Big] \\ & + r^{2q}\Big[-\frac{4q^2}{n^2}B - \frac{2q^2}{n^2}B(2-4n) - 4s_0C_1(s+1) + 8qs_0C_1(n-1) + B(2s+3) \\ & \quad + q(2n+1)(2q-1)B - \frac{B}{2}(2s+3)^2 + (2s+3)(2n+1)qB\Big] \\ & - \frac{4q^2}{n^2}B^2 - 4s_0C_1(s+1)B + \tfrac{1}{2}B^2(2s+3) - \frac{B^2}{4}(2s+3)^2\Bigg\} \Big/ r^2(B + r^{2q})^2 . \end{aligned} \tag{3.12}$$

This in general cannot be solved analytically. However, if we assume

$$X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^{\,2} = \frac{D_0}{r^2} \quad (D_0 = \text{constant}), \tag{3.13}$$

we can obtain the following solution[21]:

$$\nu(r) = \sqrt{r}(A_1y_1 + A_2y_2) , \tag{3.14}$$

(i) $D_0 - \frac{1}{4} = r_0^{\,2} > 0$; $y_1 = \cos r_0 \ln r$, $y_2 = \sin r_0 \ln r$,

(ii) $r_0^{\,2} < 0$; $y_1 = r^{r_0}$, $y_2 = r^{-r_0}$,

(iii) $r_0^{\,2} = 0$; $D_0 = \frac{1}{4}$; $y_1 = 1$, $y_2 = \ln r$.

For the assumption (3.13) we have the following three relations in addition to (3.3) to be satisfied by the nine arbitrary parameters in the solution[22]:

$$-\frac{2q^2}{n^2}(2-4n) + \tfrac{1}{2}(2s+3) + q(2n+1)(2q-1) - 2q^2(2n+1) - \tfrac{1}{4}(2s+3)^2 + (2s+3)(2n+1)q - q^2(2n+1)^2 = D_0 , \tag{3.15}$$

$$\begin{aligned} -\frac{4q^2B}{n^2} - \frac{2q^2(2-4n)B}{n^2} - 4s_0C_1(s+1) + 8qs_0C_1(n-1) + B(2s+3) + q(2n+1)(2q-1)B \\ - \frac{B}{2}(2s+3)^2 + (2s+3)(2n+1)qB = 2D_0B , \end{aligned} \tag{3.16}$$

$$-\frac{4q^2}{n^2}B^2 - 4s_0C_1(s+1)B + \tfrac{1}{2}B^2(2s+3) - \tfrac{1}{4}B^2(2s+3)^2 = D_0B^2 . \tag{3.17}$$

Next we consider solution IV in Vaidya's paper, which is given as

$$f(r) = \frac{(B + r^{2q})^{1-n}}{AC_1}, \quad h(r) = \frac{1}{Ar(B + r^{2q})^n}, \quad l(r) = \frac{(B + r^{2q})^{1-n}}{ADr^{2q+1}} , \tag{3.18}$$

and $m(t) = e^{-s_0 t}$, where

$$C_1 s_0 = B(3-4n)q/(n-1)\,, \quad D^2 = 4q^2(2n^2-1)\,. \tag{3.19}$$

We then have

$$8\pi P(r,t) = \frac{1}{h^2 m^2}\frac{4q^2}{r^2(B+r^{2q})^2}\left[(n-1)^2 r^{4q} + B r^{2q} - B^2(2n-1)(6n-5)/4(n-1)^2\right], \tag{3.20}$$

$$8\pi\rho(r,t) = \frac{1}{h^2 m^2}\frac{4q^2}{r^2(B+r^{2q})^2}\left[(n^2-1)r^{4q} + B r^{2q}(1-6n) + 3B^2(2n-1)(6n-5)/4(n-1)^2\right], \tag{3.21}$$

$$8\pi T_1^4 = \frac{4C_1 q^2 B}{r^3(B+r^{2q})^3}\frac{1}{h^2m^2}(3-4n)r^{2q}\,. \tag{3.22}$$

For this solution Eq. (3.1) can be given as

$$\bar{C}'' + \left[\frac{(1-8q)}{r} + \frac{r^{2q-1}(6q-4nq)}{(B+r^{2q})}\right]\bar{C}' = \left[\left(8q^2(2n^2-2n)r^{4q} + \left[8q^2(2B-6Bn) + 8s_0 C_1 q(1-n)\right]r^{2q} + \frac{4B^2(2n-1)(6n-5)q^2}{(n-1)^2}\right)\Big/ r^2(B+r^{2q})^2\right]\bar{C}\,. \tag{3.23}$$

We could again eliminate the first-order derivative by the substitution (3.9), which gives

$$\bar{C}(r) = \nu(r) r^{4q-1/2}(r^{2q}+B)^{n-3/2}\,, \tag{3.24}$$

where $\nu(r)$ satisfies (3.10) and $X_2 - \frac{1}{2}X_1' - \frac{1}{4}X_1^2$ is given as

$$X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^2 = \left[r^{4q}(-20q^2n^2 + 12q^2 n - q^2 + \tfrac{1}{4}) + r^{2q}\left(-30q^2 + 36nq^2 - \frac{8s_0C_1}{B}q(1-n) + \tfrac{1}{2}\right)B + \left(-\frac{4B^2q^2(2n-1)(6n-5)}{(n-1)^2} - \frac{B^2(8q-1)}{2} - \frac{B^2(1-8q)^2}{4}\right)\right]\Big/ r^2(B+r^{2q})^2\,. \tag{3.25}$$

Again if we assume

$$X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^2 = \frac{D_0}{r^2} \quad (D_0 = \text{constant})\,,$$

we can obtain the solution given in (3.14). Besides (3.19) we also have the following relations to be satisfied by the arbitrary parameters in the solution:

$$-20q^2n^2 + 12q^2n - q^2 + \tfrac{1}{4} = D_0\,, \tag{3.26}$$

$$-30q^2 + 36nq^2 - 8s_0C_1 q\frac{(1-n)}{B} + \tfrac{1}{2} = 2D_0\,, \tag{3.27}$$

$$-4q^2\frac{(2n-1)}{(n-1)^2}(6n-5) - \frac{(8q-1)}{2} - \frac{(1-8q)^2}{4} = D_0\,. \tag{3.28}$$

These two solutions corresponding to the solutions III and IV given by Vaidya could be physically reasonable throughout the star.[22,8]

Finally in this section we are going to consider solution IX given by Vaidya as

$$h(r) = \frac{1}{Ar(B+r^{2q})^n}\,, \quad l(r) = \frac{1}{ADr(B+r^{2q})^{n-1/2}}\,, \quad f(r) = \frac{r^{-2q}}{AC_1(B+r^{2q})^{n-1}}\,, \tag{3.29}$$

and $m(t) = e^{-s_0 t}$, where

$$C_1 s_0 = (1-2n)q\,, \quad D^2 = -4q^2 B\,. \tag{3.30}$$

We then have

$$8\pi P(r,t) = \frac{1}{h^2m^2}\frac{1}{r^2}\frac{4q^2B}{r^{2q}+B}\,, \tag{3.31}$$

$$8\pi\rho(r,t) = -\frac{1}{h^2m^2}\frac{4q^2B}{r^2(r^{2q}+B)}\,, \tag{3.32}$$

$$8\pi T_1^0 = \frac{1}{h^2m^2}\frac{4C_1q^2(1-2n)}{r^{3-4q}}\frac{nr^{2q}+B}{(r^{2q}+B)^3}\,. \tag{3.33}$$

For this solution Eq. (3.1) becomes

$$\bar{C}'' + \left[\frac{1+2q}{r} + \frac{r^{2q-1}}{B+r^{2q}}(2q-4qn)\right]\bar{C}' + \frac{4s_0C_1 r^{2q}}{r^2(B+r^{2q})^2}(2qnr^{2q} + 2qB)\bar{C} = 0\,. \tag{3.34}$$

The first-order derivative can again be eliminated by the substitution (3.9), which gives

$$\bar{C}(r) = \nu(r) r^{q-1/2}(B+r^{2q})^{n-1/2}\,, \tag{3.35}$$

where $\nu(r)$ satisfies (3.10) and $X_2 - \frac{1}{2}X_1' - \frac{1}{4}X_1^2$ is given as

$$X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^2 = \left[r^{4q}(8s_0C_1qn + \tfrac{1}{4} - 4q^2 + 8q^2n - 4q^2n^2) + r^{2q}(8s_0C_2Bq + \frac{B}{2} - 6Bq^2 + 8Bq^2n) + \frac{B^2}{2}(1+2q) - \frac{B^2}{4}(1+2q)^2\right] \Big/ r^2(B + r^{2q})^2 . \quad (3.36)$$

Again if we assume

$$X_2 - \tfrac{1}{2}X_1' - \tfrac{1}{4}X_1^2 = \frac{D_0}{r^2}, \quad (3.37)$$

we can obtain the solution given in (3.14). This gives the following relations besides (3.30) to be satisfied among seven parameters[22]:

$$8s_0C_1qn + \tfrac{1}{4} - 4q^2 + 8q^2n - 4q^2n^2 = D_0, \quad (3.38)$$

$$8s_0C_1qB + \frac{B}{2} - 6Bq^2 + 8Bq^2n = 2BD_0, \quad (3.39)$$

$$\frac{B^2}{2}(1+2q) - \frac{B^2}{4}(1+2q)^2 = B^2D_0. \quad (3.40)$$

The equation of state for this solution is $P + \rho = 0$. In this respect it is not suitable to represent interiors of radiating and slowly rotating fluid spheres. However, with the inclusion of the cosmological constant it could be useful in cosmology. We would like to point out that inclusion of the cosmological constant does not alter the equations to be solved for $\Omega(r, t)$.

In all these solutions the differential equation to be solved for $\nu(r)$ in (3.9) was of the form

$$\nu''(r) + \frac{D_0}{r^2}\nu = 0. \quad (3.41)$$

For differentially rotating fluid spheres, where $\omega(r,t) = a(r)m(t)^{-2}$ the corresponding equation becomes

$$\nu''(r) + \frac{D_0}{r^2}\nu = A(r)\exp\left(\frac{1}{2}\int X_1\,dr\right). \quad (3.42)$$

For $D_0 = \frac{1}{4}$, and $A(r)\exp(\frac{1}{2}\int X_1\,dr) = \frac{1}{4}r^{-3/2}$ we can give the following solution for $\nu(r)$:

$$\nu(r) = \sqrt{r}(A_1 + A_2\ln r) + \frac{\sqrt{r}}{8}(\ln r)^2, \quad (3.43)$$

where

$$A(r) = -16\pi h^2\left(P + \rho + \frac{f's_0}{4\pi f^2 h}\right)a(r). \quad (3.44)$$

Note that in general a differential equation of the form

$$\nu'' + \frac{D_0}{r^2}\nu = f(r)r^{b/2}, \quad (3.45)$$

where we define $D_0 = c + b/2 - b^2/4$, can be obtained from

$$r^2y'' + bry' + cy = f(r), \quad (3.46)$$

with the substitution

$$y = \nu(r)r^{-b/2}. \quad (3.47)$$

In this respect we are going to present various solutions to (3.46) which correspond to different forms of differential rotation and could be used in some of the cases given above:

$$\begin{aligned}
&\text{(i) } r^2y'' + ry' - y = ar^2;\quad y = \frac{1}{r}(c_1 + c_2r^2) + \frac{ar^2}{3},\\
&\text{(ii) } r^2y'' - ry' + y = r^2(3+r);\quad y = c_1r + c_2r\ln r + 3r^2 + r^3/4,\\
&\text{(iii) } r^2y'' - ry' + y = 3r^3;\quad y = c_1r + c_2r\ln r + 3r^3/4,\\
&\text{(iv) } r^2y'' + ry' + y = \ln r;\quad y = c_1\cos\ln r + c_2\sin\ln r + \ln r,\\
&\text{(v) } r^2y'' - 2ry' + 2y = r^3\sin r;\quad y = r(c_1 + c_2r) - r\sin r,\\
&\text{(vi) } r^2y'' - 2ry' + 2y = 2r\ln r;\quad y = r(c_1 + c_2r) - 2r\ln r - r(\ln r)^2,\\
&\text{(vii) } r^2y'' - 2ry' + 2y = r^5\ln r;\quad y = r(c_1 + c_2r) + \frac{r^5}{12}(\ln r - \tfrac{7}{12}),\\
&\text{(viii) } r^2y'' + 2ry' - 6y = 2 - r;\quad y = \frac{1}{r^3}(c_1 + c_2r^5) + \frac{r}{4} - \tfrac{1}{3},\\
&\text{(ix) } r^2y'' + 4ry' + 2y = e^r;\quad y = \frac{1}{r^2}(c_1 + c_2r) + \frac{e^r}{r^2},\\
&\text{(x) } r^2y'' - 2ary' + a(a+1)y = e^r r^{a+2};\quad y = c_1r^a + c_2r^{a+1} + e^r r^a.
\end{aligned} \quad (3.48)$$

Also note that when $f(r)=0$ (3.46) has the following general solution:

$$y(r)=\frac{1}{r^k}\left(\frac{C_0}{(1-A_1)}r^{A_1-1}+C_1\right), \quad (3.49)$$

where $k^2-(b-1)k+c=0$, $A_1=2k+2-b$, and $A_1\neq 1$. For $A_1=1$ we have

$$y(r)=\frac{1}{r^k}\left(C_0\ln\frac{1}{r}+C_1\right). \quad (3.50)$$

These could be used for homogeneous rotation where we redefine D_0 as

$$D_0=c+\frac{b}{2}-\frac{b^2}{4}, \quad \text{and } \nu=yr^{b/2}. \quad (3.51)$$

Finally we would like to point out that for perfect dragging, Eq. (2.19) could be easily integrated to give

$$C(r)=C_0\int\frac{hf}{l^4r^4}dr. \quad (3.52)$$

Note that in all these solutions the range of r is from 0 to R, where R is the radius of the star defined by $P(R)=0$, and the range of t is from 0 to ∞.[8] Since the solutions we give for $\Omega(s,t)$ correspond to slow rotation, the parameters have to be chosen such that the slow-rotation condition is satisfied; otherwise one has to consider higher-order terms and deviations from spherical symmetry.[22]

Finally the conditions for physical reasonableness are that L (luminosity), ρ, and P should be positive definite throughout the star, and P and ρ should be monotonic decreasing functions of r.[8]

IV. SOLUTIONS FOR NONRADIATING FLUID SPHERES

Nonradiating and slowly rotating fluid spheres have been studied quite extensively.[5-7] However, the only analytic solutions we have found in the literature were given by Adams *et al.*,[11] where they have considered the isothermal equation of state which corresponds to $P=\alpha\rho$, and the two recent solutions found by Whitman,[12] which correspond to the generalization of Tolman's sixth solution and to incompressible matter with the equation of state $\rho=$ const. All these solutions were found for $\omega=$ const. The problem with the solution corresponding to $P=\alpha\rho$ is that the pressure distribution does not vanish at finite radius. Hence it has to be joined to a shell over which pressure drops to zero. In this section we introduce rotation to the special case of the solution VI given by us,[1] which has the following equation of state,

$$P=K_0\rho^{1+1/n}+\sigma_1\rho, \quad (4.1)$$

where $\sigma_1=P_c/\rho_c$, K_0, and n are constants. We again consider uniform rotation where $\omega(r)=$ const. For this case equation (2.13) vanishes identically and (2.12) reduces to

$$\bar{\Omega}''+\left(-\frac{f'}{f}+\frac{3h'}{h}+\frac{4}{r}\right)\bar{\Omega}'=16\pi h^2(P+\rho)\bar{\Omega}, \quad (4.2)$$

where $\bar{\Omega}=\Omega-\omega_0$, and the line element is

$$ds^2=f(r)^2dt^2-h(r)^2[(dr^2+r^2d\theta^2+r^2\sin^2\theta(d\phi-\Omega\,dt)^2]. \quad (4.3)$$

The static solution that we will use is given as

$$f(r)^2=A(c_0r^2)^{-a/(1-c)}, \quad h(r)^2=B(c_0r^2)^{b/(1-c)}, \quad (4.4)$$

where $1-c\neq 0$.

The pressure and density distributions are given as

$$8\pi\rho(r)=\frac{1}{B}c_0^{\,n+1}(1-n^2)r^{2n}, \quad (4.5)$$

$$8\pi P(r)=\frac{1}{B}c_0^{\,n+1}\left[\frac{[b^2(n-1)+2ab](n+1)}{b^2}r^{2n}-\frac{2ac_0}{b}(1+n)^2r^{2n+2}\right], \quad (4.6)$$

where

$$\sigma_1=\frac{P_c}{\rho_c}=-1+\frac{2a}{b(1-n)},$$

and

$$c=\frac{\frac{1}{2}b^2-\frac{1}{2}a^2-ab+b-a}{b-a}. \quad (4.7)$$

The equation of state obtained by eliminating r among P and ρ is given as

$$P=\frac{2abc_0}{(1-c)2b+b^2}\left(\frac{c_0^{-b/(1-c)}}{(1-c)^2}\right)^{-(1-c)/(c-1-b)}\times\rho^{1+(1-c)/(c-1-b)}+\sigma_1\rho. \quad (4.8)$$

In order to have positive ρ we need $-1<n<1$, also for ρ to be a monotonic decreasing function of r we need $n<0$, otherwise this solution can only be used to represent portions of stars where density inversion takes place. The radius of the star is defined by $P(r)=0$. This gives

$$R=\frac{b^2(n-1)+2ab}{2ac_0(1+n)}. \quad (4.9)$$

Note that for $n>0$ pressure also vanishes at the center. Hence this solution is physically reasonable over an envelope over which pressure drops to zero monotonically. This solution could be used as the envelope for the solution with $P=\alpha\rho$.[11] Finally the rotation equation to be solved for $\bar{\Omega}$ becomes

$$2\left[\frac{2a}{1-c}\frac{1}{r^2}+\frac{2abc_0}{(1-c)^2}\right]\bar{\Omega}+\frac{1}{r}\left[\frac{3b}{1-c}+4+\frac{a}{1-c}\right]\bar{\Omega}_r+\bar{\Omega}_{rr}=0\,. \tag{4.10}$$

We define constants a_1, a_2, b_2 by

$$r^2\bar{\Omega}_{rr}+a_1 r\bar{\Omega}_r+(a_2+b_2r^2)\bar{\Omega}=0\,, \tag{4.11}$$

and substitute

$$\bar{\Omega}=r^kU(z)\,,\quad z=r\sqrt{b_2}\,,\quad 2k=1-a_1\,. \tag{4.12}$$

Then Eq. (4.11) becomes

$$z^2U''+zU'-(p^2-z^2)U=0\,,\qquad 4p^2=\left[3+\frac{(a+3b)(n+1)}{(-b)}\right]^2+\frac{16a(n+1)}{b}. \tag{4.13}$$

This is the Bessel equation of order p and can be expressed in terms of elementary functions for various values of p. The properties of Bessel functions are well known so we will just summarize the possible solutions for (4.13), which could be written as

$$U(z)=C_1Z_1(z)+C_2Z_2(z)\,. \tag{4.14}$$

(i) $p\neq$ integer. Then Z_1 and Z_2 are one of the following pairs:

$$J_p,\ J_{-p};\quad J_p,\ Y_p;\quad H_p^{(1)},\ H_p^{(2)}\,,$$

where

$$J_p(z)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\,\Gamma(1+k+p)}\left(\frac{z}{2}\right)^{p+2k},$$

$$Y_p(z)=\cos p\pi(J_p\cos p\pi-J_{-p})\,,$$

$$H_p^{(1)}(z)=J_p+iY_p\,,$$

$$H_p^{(2)}(z)=J_p-iY_p\,.$$

(ii) $p=n$ (integer). Then $J_{-n}=(-1)^nJ_n$. Hence the general solution could be taken as

$$U(z)=C_1J_n(z)+C_2Y_n(z)\,.$$

(iii) $p=n+\frac{1}{2}$; $n=$ integer (positve, negative, or zero). Then

$$U(z)=C_1J_p(z)+C_2J_{-p}(z)\,,$$

where

$$J_p(z)=\left(\frac{2}{\pi}\right)^{1/2}z^p\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sin z}{z}.$$

The result is

$$J_p(z)=P_n(z)\cos z+Q_n(z)\sin z\,,$$

where $P_n(z)$ and $Q_n(z)$ are polynomials in $1/z$.

Next we are going to reconsider the solution for $P=\alpha\rho$ equation of state. Even though a stellar model with this equation of state will not have finite radius, this solution is important in putting bounds on neutron star masses, since it represents the asymptotic form of the equation of state at ultrahigh densities.[23] Physically reasonable models can be obtained by matching this solution to an envelope over which pressure drops to zero. For this solution pressure and density distributions are given as

$$P(r)=\alpha^2(2\pi Dr^2)^{-1}\,, \tag{4.15}$$

$$\rho(r)=\alpha(2\pi Dr^2)^{-1}\,, \tag{4.16}$$

where $D=(1+\alpha)^2+4\alpha$. The differential equation in canonical coordinates for $\Omega(r)$ becomes

$$\bar{\Omega}''+\bar{\Omega}'\frac{(4-2\alpha_N)}{r}-\frac{8\alpha_N\bar{\Omega}}{r^2}=0\,, \tag{4.17}$$

where

$$\alpha_N=\alpha(1+\alpha)^{-1}\ \text{ and }\ \bar{\Omega}=\Omega-\text{const}\,. \tag{4.18}$$

The general solution of this equation is given as

$$\bar{\Omega}(r)=\frac{1}{r^k}\left(C_0\frac{r^{A}l^{-1}}{(1-A_1)}+C_1\right), \tag{4.19}$$

where

$$k^2+k(2\alpha_N-3)-8\alpha_N=0\ \text{ and }A_1=2k-2+2\alpha_N\neq 1\,. \tag{4.20}$$

When $C_0=0$ this reduces to the solution given by Adams *et al.*[11]

V. BOUNDARY CONDITIONS

In 1963 Kerr[24] gave the following solution which is commonly accepted to represent the gravitational field exterior to rotating fluid spheres:

$$ds^2=\Sigma(d\theta^2+\sin^2\theta\,d\bar{\phi}^2)+2(du+a\sin^2\theta\,d\bar{\phi})(dr+a\sin^2\theta\,d\bar{\phi})-(1-2mr\Sigma^{-1})(du+a\sin^2\theta\,d\bar{\phi})^2\,, \tag{5.1}$$

where $\Sigma=R^2+a^2\cos^2\theta$, $u=\bar{t}+R$, and m and a are constants. Here m represents mass and a is a measure of the angular velocity.[25] One of the unsolved problems of theoretical physics is to find a source that generates the Kerr fluid.[26,27] However, it is well known that a slowly rotating perfect fluid sphere can be a source for the Kerr exterior solution with small a.[25] In this respect we are going to use metric (5.1) to first order in a as the exterior field to the nonradiating solution given in Sec. IV.

After a coordinate transformation and to first order in a (5.1) can be written as[25] [in Eqs. (5.1)–(5.3) the signature of the metric is $(-;+++)$]

$$ds^2=\left(1+\frac{m}{2r}\right)^4(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2) -\frac{(r-m/2)^2}{(r+m/2)^2}dt^2+\frac{4ma\sin^2\theta}{r(1+m/2r)^2}\,d\phi\,dt. \quad (5.2)$$

This will be matched to the line element for the interior which is given as

$$ds^2=B(c_0r^2)^{b/1-c}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2) -A(c_0r^2)^{-a/1-c}dt^2-B(c_0r^2)^{b/1-c}2r^2\sin^2\theta\,\Omega\,d\phi\,dt. \quad (5.3)$$

Three of the constants in $h(r)$, and $f(r)$ will be determined by matching them to the Schwarzschild exterior solution at the surface [this is evident from Eq. (5.2)].[1] The remaining two integration constants in $\Omega(r)$ will be chosen such that Ω and its first derivatives are continuous throughout the star.[12]

For the exterior of the solutions given in Sec. III we use the approximate solution given by Murenbeeld and Trollope, which represents radiating and slowly rotating fluid spheres.[13,14] This metric is given as

$$ds^2=\left(1-\frac{2m}{\bar r}\right)du^2+2\,d\bar r\,du+2\left(\frac{2ma}{\bar r}\right)\sin^2\theta\,d\phi\,du -2a\sin^2\theta\,d\bar r\,d\phi-\bar r^2d\theta^2-\bar r^2\sin^2\theta\,d\phi^2. \quad (5.4)$$

Where we have considered only the first-order terms in a, and a is a constant while m is a function of the retarded time $u=\bar t-\bar r$. The only nonvanishing components of the Ricci tensor are

$$R_{00}=\frac{2m'}{\bar r^2}\text{ and }R_{03}=-3m'a\sin^3\theta/\bar r^2. \quad (5.5)$$

If we require this metric to be a solution to the field equations

$$R_{ij}-\tfrac{1}{2}g_{ij}R=-8\pi T_{ij}, \quad (5.6)$$

we see that T_{ij} corresponds to an energy-momentum tensor for a flowing null fluid with

$$T_{ij}=qk_ik_j,\quad\text{where } k_ik^i=0. \quad (5.7)$$

We normalize k_i such that q is the energy density of the radiation measured by an observer with four-velocity v^i, so that[3]

$$q=v^iv^jT_{ij}. \quad (5.8)$$

In this observer's local Lorentz frame we take k^i as

$$k^i=(1;1,0,0). \quad (5.9)$$

From $v^iv_i=1$, one can obtain v^0 in terms of v^1, and v^3 where we take $v^2=0$.

With (5.8) and (5.6) we can write

$$q=-\frac{1}{8\pi}v^iv^jR_{ij}, \quad (5.10)$$

and

$$-8\pi q=(v^0)^2\frac{2m'}{\bar r^2}+v^0v^3(-)\frac{3m'a\sin^2\theta}{\bar r^2}. \quad (5.11)$$

From here we see that luminosity measured by an observer at rest at infinity is

$$L\underset{r\to\infty}{=}4\pi\bar r^2q=-\frac{dm}{du}. \quad (5.12)$$

Similarly one can show that at infinity radiated angular momentum per unit time is[13,25]

$$\frac{dJ}{du}=-\frac{dm}{du}a. \quad (5.13)$$

Finally we will close this section by discussing the coordinate transformations that put the interior and the exterior metrics into the same form. First we consider metric (5.4), and replace u by $\bar t-\bar r$, and then define

$$\gamma(\bar r,\bar t)dt=\frac{2am}{\bar r}d\bar t-\left(\frac{2ma}{\bar r}+a\right)d\bar r, \quad (5.14)$$

where $\gamma(\bar r,\bar t)$ is an integrating factor. This puts the exterior metric into the form

$$ds^2=\frac{\bar r^2\gamma^2}{4m^2a^2}\left(1-\frac{2m}{\bar r}\right)dt^2+2d\bar r\,dt\left[\frac{\gamma}{a}+\left(1-\frac{4m^2}{\bar r^2}\right)\frac{\bar r^2\gamma}{4m^2a}\right] -d\bar r^2\left[-1+\frac{2m}{\bar r}-\frac{\bar r^2}{4m^2}\left(1-\frac{2m}{\bar r}\right)\left(\frac{2m}{\bar r}+1\right)^2\right] -\bar r^2\,d\theta^2-\bar r^2\sin^2\theta\,d\phi^2+2\sin^2\theta\gamma\,d\phi\,dt. \quad (5.15)$$

The interior metric (2.1) can be cast into this form most easily by defining a new radial marker as[3]

$$\bar r=l(r)m(t)r. \quad (5.16)$$

This gives the metric

$$ds^2=-\frac{h^2}{(l'r+l)^2}(d\bar r)^2-\bar r^2(d\theta^2+\sin^2\theta\,d\phi^2)+\left[f^2g^2-\frac{h^2l^2r^2\dot m^2}{(l'r+l)^2}\right]dt^2+\frac{2h^2lr\dot m}{(l'+r+l)^2}dr'dt+2\bar r^2\Omega\sin^2\theta\,d\phi\,dt. \quad (5.17)$$

VI. SUMMARY AND CONCLUSIONS

We have considered the field equations for radiating and slowly rotating fluid spheres. We saw that the field equations reduce to the regular unperturbed equations for P, ρ, and σ, plus two additional equations to be solved for $\Omega(r,t)$, which represents the dragging of inertial frames. Equation (2.13) determines the time dependence of Ω as $\Omega = C(r)e^{2s_0 t}$, while $C(r)$ comes from Eq. (2.12). We have considered Vaidya's solutions III, IV, and IX and presented three analytic solutions for $\Omega(r,t)$ corresponding to uniform rotation. The first two of these solutions could be physically reasonable throughout the star, while the third could either be used to represent portions of stars or could be used in cosmology with the inclusion of the cosmological constant. We have also considered Eq. (2.12) for differential rotation and presented several solutions. These could be used for all the three solutions given by Vaidya. Next we considered nonradiating and slowly rotating fluid spheres in general relativity and presented a solution for the polytropelike equation of state. This solution is given in terms of Bessel functions and could be used to represent portions of stars. We have also reconsidered the solution for $P=\alpha\rho$ equation of state and gave the general solution for $\Omega(r,t)$ so that it has the proper number of integration constants.

Once $\Omega(r,t)$ is found we can evaluate the rotational energy and the angular momentum J by using

$$J = \int h m^4 l^3 r^3 \sin^2\theta T^{03} dr\, d\phi\, d\theta \tag{6.1}$$

with $m(t) = g(t)$ and Eq. (2.5), we obtain

$$J = \int \frac{m^2 h l^3 r^3 \sin^2\theta}{f^2}[(P+\rho+\sigma)\omega - 2P\Omega]\, dr\, d\theta\, d\phi\ , \tag{6.2}$$

$$E_{\rm rot} = \frac{1}{2}\int \frac{m^2 h l^3 r^3 \sin^2\theta\omega}{f^2}[(P+\rho+\sigma)\omega - 2P\Omega]\, dr\, d\theta\, d\phi\ . \tag{6.3}$$

Equation (6.3) allows us to roughly estimate the value of the parameter s_0 that appears in $m(t) = e^{-s_0 t}$. Notice that the ratio of the luminosity due to the loss of rotational energy to the total rotational energy is

$$\frac{\dot{E}_{\rm rot}}{E_{\rm rot}} = -2s_0\ . \tag{6.4}$$

For a typical pulsar $\dot{E}_{\rm rot} \sim 10^{38}$ erg/sec, also taking 10^7 yr as a typical age for pulsars[28,29] we find $E_{\rm rot} \sim 10^{52}$ erg. This gives $s_0 \sim -10^{-15}$ sec^{-1}.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor F. I. Cooperstock for valuable comments. This work was partially supported by the Turkish Scientific and Research Council.

*Present address: Department of Physics, The Middle East Technical University, Ankara, Turkey.

[1] S. Ş. Bayin, Phys. Rev. D 18, 2745 (1978).

[2] S. Ş. Bayin, Phys. Rev. D 19, 2838 (1979).

[3] S. Ş. Bayin, Phys. Rev. D 21, 2433 (1980).

[4] S. Ş. Bayin and F. I. Cooperstock, Phys. Rev. D 22, 2317 (1980).

[5] E. M. Butterworth and J. R. Ipser, Astrophys. J. 204, 200 (1976).

[6] J. B. Hartle and K. S. Thorne, Astrophys. J. 153, 807 (1968).

[7] M. A. Abramowicz and R. V. Wagoner, Astrophys. J. 226, 1063 (1978).

[8] P. C. Vaidya, Phys. Rev. 83, 10 (1951).

[9] J. Silk and J. P. Wright, Mon. Not. R. Astron. Soc. 143, 55 (1969).

[10] S. Chandrasekhar and J. L. Friedman, Astrophys. J. 175, 379 (1972).

[11] R. C. Adams, J. M. Cohen, R. J. Adler, and C. Sheffield, Phys. Rev. D 8, 1651 (1973).

[12] P. G. Whitman and J. F. Pizzo, Astrophys. J. 230, 893 (1979).

[13] M. Murenbeeld and J. R. Trollope, Phys. Rev. D 1, 3220 (1970).

[14] P. C. Vaidya and L. K. Patel, Phys. Rev. D 7, 3590 (1972).

[15] R. H. Boyer and T. G. Price, Proc. Cambridge Philos. Soc. 61, 531 (1955).

[16] J. M. Cohen and D. R. Brill, Nuovo Cimento 56B, 209 (1968).

[17] J. H. Taylor, L. A. Fowler, and P. M. McCulloch, Nature 277, 437 (1979).

[18] The signature of our metric is (+; − − −), and we set $c = G = 1$. Also a prime denotes $\partial/\partial r$ and an overdot denotes $\partial/\partial t$.

[19] P. C. Vaidya, Astrophys. J. 144, 943 (1966).

[20] Using P for $P(r,t)$ and $P(r)$, also ρ for $\rho(r,t)$ and $\rho(r)$, will not cause any confusion.

[21] G. M. Murphy, *Ordinary Differential Equations and Their Solutions* (Van Nostrand, Princeton, N. J., 1960), p. 336.

[22] In all these solutions we have a system of simultaneous nonlinear equations, where the number of parameters is two more than the number of equations. Hence in principle one can chose two of the parameters to express the others. However at this point it is not clear which two should be chosen. For a review of the properties of implicit functions see W. Kaplan, *Advanced Calculus*, 2nd ed. (Addison-Wesley, Reading, Massachusetts, 1973), p. 144. Also note that these restrictions are purely a mathematical simplification.

[23] S. Ş. Bayin, Phys. Rev. D 21, 1503 (1980).

[24] R. Kerr, Phys. Rev. Lett. 11, 237 (1963).

[25] J. M. Cohen, J. Math. Phys. 8, 1477 (1967).

[26] S. W. Hawking and G. F. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1974), p. 161.

[27] A. Krasinski, Ann. Phys. (N.Y.) 112, 122 (1979).

[28] F. G. Smith, *Pulsars* (Cambridge University Press, Cambridge, England, 1977), p. 219.

[29] C. W. Allen, *Astrophysical Quantities* (The Athalone Press, University of London, London, England, 1976), p. 232.