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Charged spheres in general relativity

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The coupled Einstein-Maxwell field equations are solved by quadratures for spherically symmetric static systems containing charge. In particular, we show how interior metrics can be derived which reduce to classical solutions for neutral distributions of matter when the charge becomes vanishingly small. A number of simple analytic solutions are expressed in order to indicate how charge can change the overall character of these objects. The stability of charged systems is considered. We find the stability of the Schwarzschild interior solution is enhanced by the inclusion of charge, and that an increase in the charge further reduces the critical radius for which instability sets in. The application of this analysis to the solution of Pant and Sah indicates their model is unstable.

I. INTRODUCTION

A number of papers have recently appeared dealing with both equilibrium solutions for charged fluid spheres and the stability criteria for such models. Bonner[1-3] showed it was possible to have equilibrium solutions with a vanishing pressure. He maintained that if a cloud of hydrogen gas were to have a charge equivalent to the loss of one electron in 10^{18}, the Coulombic repulsion by itself could hold the cloud in equilibrium without the aid of an isotropic particle pressure. Glazer,[4] by an analysis of the pulsation equation for small oscillations of a charged fluid sphere, showed that Bonner's solution is unstable to such radial pulsations.

A model such as this is of course also unstable to a change in the net charge, as was pointed out by Bonner himself.[3] If the charge is increased, the cloud must expand due to an increase in the Coulombic repulsion, and a reduction in the net charge reduces the interparticle repulsion so that it can no longer prevent collapse of the cloud. The collapse of such spheres of charged dust was investigated by Novikov[5] and Bardeen.[6]

Glazer[7] presented an approximate solution representing a charged interior for which he investigated the properties with respect to small radial pulsations. His results indicate that charge contained in the interior may increase the stability of the solution.

The problem of the stability of a homogeneous distribution of matter containing a net surface charge was considered by Stettner.[8] He showed that a fluid sphere of uniform density with a modest surface charge is more stable than the same system without charge. His solution is also stable towards an increase in the net surface charge.

In this paper we present a solution by quadratures for arbitrary charge and mass distributions in the form of static spheres, and show how charged analogs to neutral solutions can be derived. We express a number of simple solutions to indicate how the incorporation of charge can alter the overall character of the solution. Two of these are considered in detail.

The stability of charged metrics is considered. We express Glazer's pulsation equation in a more convenient form and obtain the variational base compatible with Chandrasekhar's analysis for uncharged spheres[9,10] and apply it to one of the solutions presented here. We show the inclusion of charge increases the stability of this model. The pulsation equation in differential form is applied to the charged solution of Pant and Sah.[11] We find satisfaction of the boundary condition on the Lagrangian displacement at the origin is incompatible with densities and pressures that are everywhere positive definite.

II. FIELD EQUATIONS

A useful coordinate system for metrics with spherical symmetry is that of Schwarzschild. In these coordinates the line element can be written as

$$ds^2=\gamma^2(r)(cdt)^2-\tau^{-1}(r)dr^2-r^2d\Omega^2\,, \tag{2.1}$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\varphi^2\,. \tag{2.1$'$}$$

In terms of the contracted Riemann tensor R^μ_ν and Ricci scalar R, the combined Einstein-Maxwell equations can be expressed as

$$R^\mu_\nu-\tfrac{1}{2}\delta^\mu_\nu R=-8\pi\frac{G}{c^4}T^\mu_\nu\,. \tag{2.2}$$

The energy-momentum tensor T^μ_ν can be decomposed into a part pertaining to the matter contribution M^μ_ν and a part expressing the contribution from the electromagnetic field:

$$T^\mu_\nu=M^\mu_\nu+E^\mu_\nu\,. \tag{2.3}$$

If the interior can be described by an isotropic pressure and mass-energy density ϵ, then M^μ_ν takes the form

$$M^\mu_\nu = (P+\epsilon)u_\nu u^\mu - P\delta^\mu_\nu \,, \tag{2.4}$$

where

$$u^\mu = (\gamma^{-1}, 0, 0, 0) \tag{2.4$'$}$$

is the four-velocity of the fluid element at the point r within the interior.

The electromagnetic contribution to the stress-energy tensor can be written as

$$E^\mu_\nu = -\frac{1}{4\pi}(F_{\nu\alpha}F^{\mu\alpha} - \tfrac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta}) \,. \tag{2.5}$$

The electromagnetic field tensor $F_{\mu\nu}$ satisfies Maxwell's equations

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \,, \qquad F^{\mu\nu}{}_{;\nu} = \frac{4\pi}{c}J^\mu \,, \tag{2.6}$$

where J^μ is the current density. If the object is both static and spherically symmetric, Eq. (2.6) reduces to

$$\frac{\partial}{\partial r}(r^2\gamma\tau^{-1/2}F^{01}) = 4\pi\rho r^2 \,. \tag{2.7}$$

The charge density ρ defined in Eq. (2.7) is related to the proper charge density ρ^* by

$$\rho^* = \rho\tau^{-1/2} \,. \tag{2.8}$$

Equation (2.7) can be immediately integrated. If we define

$$Q(r_0) = 4\pi\int_0^{r_0} \rho r^2 dr \,, \tag{2.9a}$$

then Eq. (2.7) implies that

$$F_{01}F^{01} = -Q^2(r)r^{-4} \equiv -8\pi k(r) \,. \tag{2.9b}$$

We see that Eq. (2.9a) expresses the total charge (in esu) contained in the interior below the coordinate point $r = r_0$.

In terms of the quantities defined above, the field equations can be expressed as a function of the coordinate $x = (r/R)^2$ as

$$8\pi\frac{G}{c^4}(P-k)R^2x = \tau\left(1 + 4x\gamma^{-1}\frac{d\gamma}{dx}\right) - 1 \,, \tag{2.10a}$$

$$8\pi\frac{G}{c^4}(P+k)R^2x = 4\tau x\left(\gamma^{-1}\frac{d\gamma}{dx} + x\gamma^{-1}\frac{d^2\gamma}{dx^2}\right) + x\frac{d\tau}{dx}\left(1 + 2x\gamma^{-1}\frac{d\gamma}{dx}\right), \tag{2.10b}$$

$$8\pi\frac{G}{c^4}(\epsilon+k)R^2x = 1 - \tau - 2x\frac{d\tau}{dx} \,, \tag{2.10c}$$

$$x\frac{d}{dx}(P-k) = -x(P+\epsilon)\gamma^{-1}\frac{d\gamma}{dx} + 2k \,. \tag{2.10d}$$

Equation (2.10d) is the only nontrivial Bianchi identity for this system of equations.

III. SOLUTION BY QUADRATURES

Equation (2.10c) can be immediately integrated. Letting x_0 refer to any point within the sphere up to and including the boundary defined by $x_0 = 1$, we find

$$\tau(x_0) = 1 - 4\pi R^3\frac{G}{c^4}x_0^{-1/2}\int_0^{x_0}(\epsilon+k)x^{1/2}dx \,. \tag{3.1}$$

As in the uncharged case, the function τ depends only on the fluid content lying below the coordinate value $x = x_0$.

Forming the difference between Eqs. (2.10a) and (2.10b), we obtain an expression which does not depend on P or ϵ. After simplification the resulting differential equation becomes

$$\frac{d^2\gamma}{dx^2} + \tfrac{1}{2}\tau^{-1}\frac{d\tau}{dx}\frac{d\gamma}{dx} + (4x^2\tau)^{-1}\left(1 + x\frac{d\tau}{dx} - 16\pi\frac{G}{c^4}R^2kx\right)\gamma = 0 \,. \tag{3.2}$$

When $k=0$, Eq. (3.2) reduces to the usual equation expressing pressure isotropy discussed by Matese and Whitman.[12] In like fashion we consider two transformations on Eq. (3.2) which will allow a solution by quadratures.

We define a pair of functions g and ϕ such that

$$\gamma(x) = g^{1/2}\phi \,, \tag{3.3}$$

where g is considered arbitrary, and introduce this definition into Eq. (3.2), simplifying results in

$$\frac{d^2\phi}{dx^2} + p(x)\frac{d\phi}{dx} + q(x)\phi = 0 \,, \tag{3.4}$$

where

$$p(x) = \frac{1}{2}\frac{d}{dx}\ln|g^2\tau| \tag{3.4a}$$

and the function $q(x)$ is defined through the relation

$$4x^2\tau g^2 q = (g^2 + xgg')\tau' - [g^2 - x(g')^2 - 2xgg'']\tau + g^2\left(1 - 16\pi R^2\frac{G}{c^4}kx\right). \tag{3.4b}$$

The prime refers to differentiation with respect to the coordinate x. If we now make the change of independent variable

$$z(x) = \int |q(x)|^{1/2}dx \,, \tag{3.5}$$

Equation (3.4) becomes

$$\frac{d^2\phi}{dz^2} + \mathrm{sgn}(q)\phi + \frac{1}{2}\frac{d}{dz}\ln|g^2\tau q|\frac{d\phi}{dz} = 0 \,, \tag{3.6}$$

where

$$\mathrm{sgn}(q) = q/|q| . \tag{3.6'}$$

Since g is an arbitrary function, there is no loss of generality if we further require g to satisfy the integrability condition on Eq. (3.6):

$$g^2\tau q = -\tfrac{1}{4}\beta \equiv \text{constant}. \tag{3.7}$$

Introducing this into Eq. (3.6) we find

$$\phi = \begin{cases} A\ \sin z + B\cos z\,, & \beta < 0 \\ A\ \sinh z + B\cosh z\,, & \beta > 0\,, \\ A\int g^{-1}\tau^{-1/2}\,dx + B\,, & \beta = 0 \end{cases}$$

where

$$z = \tfrac{1}{2}|\beta|^{1/2}\int g^{-1}\tau^{-1/2}dx . \tag{3.8}$$

Equation (3.7) is viewed as a differential equation for the function $\tau(x)$ in terms of the as yet unspecified generator $g(x)$. Since this is a first-order linear equation, a quadratures solution is easily obtained. We find

$$\tau(x) = -e^{-F}\left\{D + \int_0^x\left[\beta x + g^2\left(x^{-1} - 16\pi R^2\frac{G}{c^4}k\right)\right] \times (g^2 + xgg')^{-1}e^F dx\right\}, \tag{3.9}$$

where

$$F = -\int(g^2 + x^2g'^2 - 2x^2gg'')x^{-1}(g^2+xgg')^{-1}dx$$

$$= \ln|(g+xg')g^{-1}x^{-1}| + \int xg''(g+xg')^{-1}dx . \tag{3.9'}$$

We observe that for every integrable choice of functions $g(x)$ and $k(x)$ Eq. (3.9) yields $\tau(x)$. Equation (3.8) gives $z(x)$ and $\phi(z)$ which in turn gives us $\gamma(x)$. The pressure and density distributions throughout the interior can then be obtained by virtue of Eq. (2.10).

IV. CHARGED ANALOGS

An examination of Eq. (3.9) indicates that if $k=0$, τ must satisfy

$$\tau \equiv \bar{\tau} = -x\bar{g}(\bar{g} + x\bar{g}')^{-1}e^{-\bar{f}}\left(\bar{D} + \bar{\beta}\int \bar{g}^{-2}e^{\bar{f}}dx + \int x^{-2}e^{\bar{f}}dx\right), \tag{4.1}$$

where

$$\bar{f} = \int x\bar{g}''(\bar{g}+x\bar{g}')^{-1}dx . \tag{4.1a}$$

We have used the integration by parts expressed by Eq. (3.9′). The bar above the functions in Eq. (4.1) indicates uncharged quantities. Obviously $\bar{\tau}$ and the associated $\bar{\gamma}$ from Eqs. (3.8) and (3.3) represent static neutral solutions. Hence if we take

$$g \equiv \bar{g} = \bar{\gamma}^2(x) \tag{4.2}$$

as the solution generator, Eq. (3.9) can be decomposed in terms of Eq. (4.1). We find

$$\tau - \bar{\tau} = -x\bar{g}(\bar{g} + x\bar{g}')^{-1}e^{-\bar{f}}\left[(D - \bar{D}) + (\beta - \bar{\beta})\int \bar{g}^{-2}e^{\bar{f}}dx - 16\pi R^2\frac{G}{c^4}\int x^{-1}e^{\bar{f}}k\,dx\right]. \tag{4.3}$$

We can use Eq. (4.3) to obtain charged interiors which reduce to specific uncharged solutions. We introduce into Eq. (4.3) the functions $\bar{\gamma}$ and $\bar{\tau}$ from a classical neutral model and obtain the resultant $\tau(x,k)$. Further specifications as to how the charge is distributed within the sphere lead to a specific $\tau(x)$. Introducing this along with the generator $\bar{g}$ into Eq. (3.8) will give us $\gamma(x)$ by virtue of Eq. (3.3).

The structure of Eq. (4.1), however, indicates that even in the limit of vanishing charge, the resulting solution $\tau(x,0)$ can be considerably more general than the original.[12] The input classical solution will, however, be contained in the results.

V. EXAMPLE SOLUTIONS

We note that the incorporation of charge has introduced a degree of arbitrariness, and in order to obtain specific solutions, we must specify in some manner how the charge is to be distributed throughout the interior. The simplest choice is to assume it to be uniform. If we restrict ourselves in this fashion, Eq. (2.9) yields

$$k(x) = \tfrac{2}{9}\pi\rho^2R^2x \equiv \frac{c^4k_0^{\,2}}{16\pi GR^2}x . \tag{5.1}$$

With this assumption we consider a number of solutions using the analysis of the previous section.

A. Flat space

An examination of Eq. (4.3) will show that

$$\bar{\gamma} = \bar{\tau} = 1 \tag{5.2}$$

results in simple integrals. Performing the indicated operations leads to

$$\tau(x) = 1 - Dx + (k_0^{\,2} - \beta)x^2 . \tag{5.3}$$

A glance at Eq. (3.8) indicates that six different solutions can be derived from this expression. We now consider these.

Case (*i*): $D=0$. If we take the matter content of the space to be zero and consider a fluid of zero mass and nonvanishing charge, Eq. (2.10c) indicates that $D=0$. From Eqs. (5.1), (5.3), and

(2.10c), we find

$$\beta = \tfrac{11}{10}k_0^2 \tag{5.4}$$

yields the charged analog to flat space. Upon introducing this information into Eqs. (3.8) and (3.3) we find

$$y(z) = A\,\sinh(z) + B\cosh(z)\,, \tag{5.5}$$

where

$$z = \beta^{1/2}\sin^{-1}(\beta^{1/2}x)\,. \tag{5.5a}$$

This solution corresponds to the classical Poincaré-Lorentz fluid particle.[13]

Case (ii): $\beta = k_0^2$. If this condition holds we find

$$z(x) = -D^{-1}k_0\tau^{1/2}, \tag{5.6}$$

and $\gamma(z)$ is a hyperbolic function with this argument.

Case (iii): $\beta < 0$. When this inequality holds, the function $z(x)$ takes the form

$$z(x) = (k_0^2 - \beta)^{-1/2}\ln\left|2(k_0^2-\beta)^{1/2}\tau^{1/2} + 2(k_0^2-\beta)x - D\right|\,. \tag{5.7}$$

Equation (3.8) then requires $\gamma(x)$ to involve trigonometric functions with a logarithmic argument.

Case (iv): $0 < \beta < k_0^2$. For the parameter β in this range, the solution is nontranscendental. We find

$$\gamma(x) = Au^{-\omega} + Bu^{\omega}\,,$$

where

$$\omega^2 = \tfrac{1}{4}\beta(k_0^2 - \beta)\,,$$

and

$$u = 2(k_0^2-\beta)^{1/2}\tau^{1/2} + 2(k_0^2-\beta)x - D\,. \tag{5.8}$$

Case (v): $\beta > k_0^2$. We now find $\gamma(x)$ is in the form of a hyperbolic function of an inverse sine. The Poincaré-Lorentz fluid is contained in this solution as a limiting case.

Case (vi): $\beta = 0$. The sixth solution occurs when the parameter β vanishes. In this case $\gamma(x)$ takes the form

$$\gamma(x) = k_0^{-1}A\,\ln\left|2k_0\tau^{1/2} + 2k_0^2x - D\right| + B\,. \tag{5.9}$$

This is a particularly interesting solution. We will show that in the limit of vanishing charge, it reduces to the Schwarzschild interior solution.

B. Tolman type-IV solution

If we let

$$\bar{y} = (1+ax)^{1/2}\,,\qquad \bar{\tau} = (1+ax)(1-cx)(1+2ax)^{-1}\,, \tag{5.10}$$

we again find Eq. (4.3) can be integrated. The result is

$$\tau(x) = (1+ax)(1+2ax)^{-1} \times\left[1 - Dx + a^{-1}\beta x(1+ax)^{-1} + k_0^2x^2\right]. \tag{5.11}$$

This particular solution is a charged analog for the Tolman type-IV solution.[14] Reasonable forms for $\gamma(x)$ can be obtained under the conditions

$$D = k_0^2\,, \tag{5.12}$$

$$\beta = a^2\,, \tag{5.13}$$

$$\beta = 0\,,\quad D = k_0^2\,. \tag{5.14}$$

These three charged solutions have the same mathematical structure as the generalized Tolman type-IV solutions of Matese and Whitman.[12] They differ only in the interpretation of the solution parameters and equations of state obtainable from Eq. (2.10).

C. Charged Adler solution

A solution which is a charged analog to Adler's[15] solution can also be obtained. If we let

$$\bar{\gamma} = 1 + ax\,,\quad \bar{\tau} = 1 - Dx(1+3ax)^{-2/3} \tag{5.15}$$

a solution can be derived provided we choose $\beta = 0$ in Eq. (4.3). We find

$$\tau(x) = 1 - Dx(1+3ax)^{-2/3} + \tfrac{1}{5}k_0^2x(2+ax)\,. \tag{5.16a}$$

A simple form for $\gamma(x)$ can be obtained if we require $A = 0$ in Eq. (3.8). For this case

$$\gamma(x) = B(1+ax)\,. \tag{5.16b}$$

All of the above solutions for charged systems are nonsingular and well behaved within some range of various parameters. As a final example of charged analog solutions we consider a specific singular solution.

D. A singular solution

If we let $\bar{\gamma}$ and $\bar{\tau}$ be defined by

$$\bar{\gamma} = x^{n/2}(1+ax^{1-n})\,,\qquad \bar{\tau} = m^{-1} - Dx^{m/(n+1)}\left(1 + \frac{3-n}{1+n}ax^{1-n}\right)^{-2m/(m+2)}, \tag{5.17}$$

where

$$m = 1 + 2n - n^2\,,$$

we find that Eq. (4.3) can be integrated with the restriction $\beta = 0$. To be compatible with the singular nature of this solution, we will consider a charge density which is inverse square in the coordinate r. Inspection of Eq. (2.9) shows the resulting k is inversely proportional to x. Hence for $\beta = 0$, $k \propto x^{-1}$ we find

$$\tau = (1 - k_0^2)m^{-1} - Dx^{m/(n+1)}\left(1 + \frac{3-n}{1+n}ax^{1-n}\right)^{-2m/(m+2)} . \qquad (5.18)$$

From Eq. (3.8), we see $\gamma(x)$ is again complicated unless we take $A = 0$. For this choice

$$\gamma = Bx^{n/2}(1 + ax^{1-n}) . \qquad (5.18')$$

This solution reduces to one by Whitman[16] in the limit of vanishing charge, and to the charged solution of Pant and Sah[11] for $D = a = 0$.

E. Surface charge

In conclusion we consider conducting spheres for which the charge resides entirely on the surface. An examination of Eq. (3.1) shows that the function $\tau(x)$ involves only the content of the sphere below the coordinate point $x = x_0 \leq 1$. This being the case, $\tau(x)$ is completely unaltered at every interior point if the charge density is of the form

$$\rho(x) = \rho_0 \delta(1 - x) . \qquad (5.19)$$

It then follows that Eq. (3.8) also remains unchanged at every interior point of the fluid. The only difference between systems with surface charge and their neutral counterparts then resides in the evaluation of the constants of integration at the boundary where the charge discontinuity resides. This however is an effect of some consequence. Stettner[8] showed that the homogeneous interior solution of Schwarzschild was more adiabatically stable with the inclusion of charge confined to its surface.

VI. STABILITY

Following a line of reasoning similar to that of Chandrasekhar[9,10] for a neutral system undergoing small radial pulsations, we can obtain a variational base to test the adiabatic stability for charged interiors. The pulsation equation for the characteristic eigenfrequencies σ associated with the Lagrangian displacement $\xi(r)$ is

$$\sigma^2\tau^{-1}\gamma^{-2}(P+\epsilon)\xi = 4r^{-1}\frac{dP}{dr}\xi + 8\pi\frac{G}{c^4}(P+\epsilon)(P+k)\tau^{-1}\xi - (P+\epsilon)\left(\gamma^{-1}\frac{d\gamma}{dr}\right)^2\xi - \tau^{1/2}\gamma^{-2}\frac{d}{dr}\left[\gamma^3\tau^{-1/2}\hat{\gamma}Pr^{-2}\frac{d}{dr}(r^2\gamma^{-1}\xi)\right] . \qquad (6.1)$$

The function $\hat{\gamma}$ in Eq. (6.1) is the "ratio of specific heats." Equation (6.1) is subject to the boundary conditions

$$\xi = 0 \text{ at } r = 0 , \qquad (6.1a)$$

$$\frac{d}{dr}(r^2\gamma^{-1}\xi)\Big|_R = 0 . \qquad (6.1b)$$

Multiplying through by $r^2\gamma\tau^{-1/2}\xi$ and integrating over all r leads to the desired result. We find

$$\sigma^2\int_0^R \tau^{-3/2}\gamma^{-1}(P+\epsilon)r^2\xi^2dr = 4\int_0^R \gamma\tau^{-1/2}r\frac{dP}{dr}\xi^2dr + \frac{8\pi G}{c^4}\int_0^R \tau^{3/2}\gamma(P+\epsilon)(P+k)r^2\xi^2dr - \int_0^R \tau^{-1/2}\gamma(P+\epsilon)\left(\gamma^{-1}\frac{d\gamma}{dr}\right)^2 r^2\xi^2dr + \int_0^R \gamma^3\tau^{-1/2}\hat{\gamma}Pr^{-2}\left[\frac{d}{dr}(r^2\gamma^{-1}\xi)\right]^2 dr . \qquad (6.2)$$

What is usually done is to determine the conditions on $\hat{\gamma}$ for which σ^2 is positive for a given displacement $\xi(r)$ satisfying the above boundary conditions. It is assumed the fluid moves along an adiabat and $\hat{\gamma}$ is taken to be constant throughout the sphere.

As a particular application of Eq. (6.2) we consider the stability of the solution expressed by Eqs. (5.9) and (5.3) with $\beta = 0$. The solution can be written as

$$\tau = 1 - Dx + k_0^2x^2 ,$$

$$\gamma(x) = \gamma_s + k_0^{-1}A \ln\left|(D - 2k_0^2x - 2k_0\tau^{1/2}) \times (D - 2k_0^2 - 2k_0\gamma_s)^{-1}\right| , \qquad (6.3a)$$

where

$$\gamma_s = \gamma(1) . \qquad (6.3b)$$

The boundary conditions require

$$D = \frac{8\pi}{3}\frac{G}{c^4}R^2\epsilon_0 , \quad A = \tfrac{1}{8}(2D - 3k_0^2) , \quad \gamma_s = (1 - D + k_0^2)^{1/2} . \qquad (6.3c)$$

The density and pressure within the interior are given by

$$\epsilon = \epsilon_0 - 11k(x) , \qquad (6.3d)$$

$$P = 3k(x) - \tfrac{1}{3}\epsilon_0 + \frac{Ac^4}{2\pi R^2 G}\tau^{1/2}\gamma^{-1} , \qquad (6.3e)$$

where, as before, $x = (r/R)^2$. The constants in Eq. (6.3) were determined so that γ, τ, and P are continuous across the boundary.

In order to simplify the calculations, we consider these solutions in the limit $k_0 \ll 1$. Introducing this condition into Eq. (6.3) leads to the approximate solutions valid for small charge:

$$\tau = 1 - Dx \quad \gamma = \tfrac{1}{2}[(3+\alpha)\gamma_s - (1+\alpha)\tau^{1/2}], \qquad (6.4)$$

where

$$1+\alpha \equiv (1 - 2k_0\gamma_s D^{-1})^{-1} \simeq 1 + 2k_0\gamma_s D^{-1}. \qquad (6.4a)$$

We note in passing that Eq. (6.4) is precisely the uniform density solution in the limit $k_0 \to 0$.

Introducing these expressions into Eq. (6.2) results in

$$R^2\sigma^2 D^{-1}\int_0^{Z_s} y^{-3}z^4[(3+\alpha)y_s - \alpha y]dz = \tfrac{1}{4}\int_0^{Z_s} y^{-3}z^4[-2\alpha y^3 + 2(3+\alpha)y_s y^2 + 27\alpha y_s^2 y + \alpha y - (3+\alpha)y_s - 27(1+\alpha)y_s^3]dz + \tfrac{9}{8}\hat\gamma\int_0^{Z_s} y^{-1}(y-y_s)z^2[27(1+\alpha)y_s^2 - 6(3+5\alpha)y_s y + (3+7\alpha)y^2]dz. \qquad (6.5)$$

In the above equation, $y^2 \equiv \tau = 1 - Dx$ and $z = D^{1/2}rR^{-1}$. The subscript s again refers to surface values.

We observe that Eq. (6.5) contains only elementary functions and can be integrated exactly. The result however, is transcendental in D. We choose instead to consider Eq. (6.5) in the small mass limit. In this limit we find

$$R^2\sigma^2 D^{-1} = \tfrac{1}{2}[3\hat\gamma - 4 + \alpha\hat\gamma - \tfrac{1}{14}Z_s^2(19 + \tfrac{38}{3}\alpha)]. \qquad (6.6)$$

Instability will arise when $\sigma^2 < 0$. Requiring σ^2 to be non-negative or zero we find for $\hat\gamma$ close to $\frac{4}{3}$

$$\hat\gamma - \tfrac{4}{3} + \tfrac{4}{9}\alpha > \tfrac{19}{42}Z_s^2(1 + \tfrac{2}{3}\alpha), \qquad (6.7)$$

or, to this level of approximation

$$R > \frac{\frac{19}{21}(GM/c^2)(1+\frac{2}{3}\alpha)}{(\hat\gamma - \frac{4}{3}) + \frac{4}{9}\alpha}. \qquad (6.8)$$

If we define R_c to be the critical radius for instability in the charged case and R_0 the corresponding quantity for a neutral sphere, then

$$R_c/R_0 = \frac{(\hat\gamma - \frac{4}{3})}{(\hat\gamma - \frac{4}{3}) + \frac{4}{9}\alpha}(1 + \tfrac{2}{3}\alpha) < 1. \qquad (6.9)$$

We observe the critical radius before adiabatic instability sets in is not only smaller than for a neutral sphere, but an increase in the charge results in its further reduction. Written in terms of M and Q, the stability criteria can be expressed as

$$R \geq \frac{\frac{19}{21}(GM/c^2)[1 + (8/9G)^{1/2}Q/M]}{(\hat\gamma - \frac{4}{3}) + (32/81G)^{1/2}Q/M}. \qquad (6.10)$$

If, instead of being uniformly distributed throughout the interior, the same charge were to reside on the surface of the sphere of mass M, the stability criteria would be[8]

$$R \geq \frac{19}{21}\frac{GM}{c^2}\left[1 - \frac{283}{114G}\left(\frac{Q}{M}\right)^2\right](\hat\gamma - \tfrac{4}{3})^{-1} \qquad (6.11)$$

expressed in our notation. We note a surface charge of the same magnitude does not affect stability until the next higher order of approximation.

We now show the singular solution given by Eq. (5.18) is unstable near the origin. We do this by expressing the pulsation equation for this model and show that the associated boundary conditions are incompatible with non-negative pressures and densities. If we define

$$u(r) = r^2\gamma^{-1}\xi \qquad (6.12)$$

and introduce this into Eq. (6.1), we find after some simplification

$$r^2u'' + r[3r\gamma^{-1}\gamma' - \tfrac{1}{2}r\tau^{-1}\tau' - 2 + r(\ln|\hat\gamma P|)']u' + \frac{c^2}{v_c^2}\left[-4rP'(P+\epsilon)^{-1} + (r\gamma^{-1}\gamma')^2 - \frac{8\pi G}{c^4}(P+k)\tau^{-1}\right]u = 0. \qquad (6.13)$$

The quantity v_c^2 in (6.13) is the square of the sound speed. The ratio of specific heats $\hat\gamma$ is expressible as

$$\hat\gamma P = (P+\epsilon)\frac{dP}{d\epsilon} \equiv (P+\epsilon)\left(\frac{v_c}{c}\right)^2. \qquad (6.14)$$

Near the origin, the solution in question takes the limiting form[11]

$$\tau = D, \quad \gamma = ar^n, \quad \epsilon = \frac{c^4}{16\pi G r^2}[1 - D(1-n)^2],$$

$$P = \frac{c^4}{16\pi G r^2}[D(1+n)^2 - 1], \qquad (6.15)$$

$$k = \frac{c^4}{16\pi G r^2}[1 + D(n^2 - 2n - 1)].$$

Introducing these expressions into Eq. (6.13) and simplifying we find

$$r^2u'' - (4-3n)u' + \frac{2}{Dn}[1 - D(1-n)^2]u = 0, \qquad (6.16)$$

which is a Cauchy equation for the function $u(r)$. The indicial equation requires

$$2m = 5 - 3n \pm \left[(5-3n)^2 - \frac{8}{Dn}(n-1)^2\right]^{1/2}. \qquad (6.17)$$

By virtue of the relation between ξ and u, the boundary condition at $r=0$ for Eq. (6.16) is

$$m \geqslant 3 - n. \tag{6.18}$$

Introducing this condition into (6.17) leads to

$$D^{-1} \geqslant n^3 - 3n^2 + n + 1. \tag{6.19}$$

Positivity of the pressure requires that

$$D^{-1} \leqslant (n+1)^2. \tag{6.20}$$

A comparison of the two inequalities indicates they cannot be simultaneously satisfied unless $n < 0$. This is incompatible with both the pressure and density being positive definite, as can be seen from Eq. (6.15). The approximate solution discussed here is the charged solution of Pant and Sah.[11]

VII. CONCLUSION

We presented a solution by quadratures for charged systems and gave a number of new solutions to the coupled Einstein-Maxwell equations. Most of these interiors reduce to known solutions when appropriate limits are taken.

Six solutions were derived based on the $\tau(x)$ defined by Eq. (5.3). Of these *case* (*i*) is a generalization of the Poincaré-Lorentz fluid,[13,17] *case* (*v*) reduces to a neutral solution presented by Baylin,[18] *case* (*iii*) becomes a Tolman type-VII solution in the limit of vanishing charge,[14] and *case* (*vi*) tends to the uniform density solution of Schwarzschild in this limit. The solutions defined by *case* (*ii*) and *case* (*iv*) do not reduce to known solutions. We also presented three charged metrics based on the Tolman type-IV solution which reduce to interiors given by Matese and Whitman[12] for vanishing charge—a solution which is an analog to the Adler solution[15]; and we presented a metric which contains the charged solution of Pant and Sah[11] as a special case and reduces to a solution by Whitman[16] in the neutral limit. Our analysis indicates that there is no change in the mathematical structure of solutions if the charge is maintained at the boundary.

We considered the stability of systems containing an interior charge, and applied our analysis to two of the solutions presented here. We found the charged analog to the uniform-density solution was more stable then the neutral interior and further that stability was enhanced by an increase in the charge. The singular solution which is a generalization of that given by Pant and Sah was found to be unstable. Though an interior charge seems to have a stabilizing effect, even a singular charge distribution appears to be insufficient to stabilize a singular neutral solution.

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