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Estimate of eikonal scattering amplitudes via multiplicity distributions

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Methods for the use of measured high-energy multiplicities as input to elastic-scattering analyses are suggested, in order to test the validity of a general, multiperipheral-type formula which correlates high-energy multiplicity and scattering information. The discussion is based on the assumption of simple, theoretically based ansatzes for the absorptive eikonal function, used as a vehicle for the input of multiplicity information, and also on the use of multiplicity data to determine, in part, the appropriate form of the ansatz adopted.

Six years ago, a method was described for the construction of particle multiplicities resulting from high-energy hadronic collisions, when the corresponding elastic amplitude is specified in eikonal approximation.[1] The main prop of that analysis, Eq. (1) below, was derived[2] for a pair of Abelian field theories, and rested on two essential assumptions: (A) The phase space of each emitted particle increases with rapidity $y=\ln s$, where s is the total c.m. (energy)^2. (B) The most important s dependence of each inelastic $|$amplitude$|^2$ is due to all possible rescattering graphs (evaluated in eikonal approximation) of each amplitude, together with the phase-space factors of (A). These assumptions are general enough to encompass a wide class of models, including all "tower graphs" of arbitrary complexity; and they lead to the formula

$$\sigma_{\text{in}}(\xi)=\int d^2b\, e^{-2\rho(s,b)}\left[e^{2\rho(s^{\xi},b)}-1\right], \tag{1}$$

where

$$\sigma_{\text{in}}(\xi)=\sum_n \xi^n\sigma_n,\quad \sigma_{\text{in}}(1)=\sigma_{\text{tot}}-\sigma_{\text{el}}$$

and $\rho(s,b)$ is the (assumed) purely absorptive eikonal function. The σ_n represent topological cross sections for the production of n identical bosons in the reaction $p_1+p_2\to p_2'+p_2'+\sum_{i=1}^n k_i$.

Following recent work of several groups,[3] one expects the eikonal representation to remain a valid and useful formalism when the underlying theory is non-Abelian quantum chromodynamics (QCD). Binding effects, turning the initial and final quark-antiquark lines into hadrons of zero color, are expected to involve relatively low-energy phenomena, which will not generate significant energy dependence to high-energy reactions beyond that given by the eikonal estimates for the scattering and production of pairs and/or triplets of color-singlet quark-antiquark lines. Hence one expects a more complete theory to reproduce the the essential impact-parameter forms of the Abelian theories, modulo some isotopic complications;[4] and in particular, one may imagine that Eq. (1) remains approximately valid even when the underlying field theory is QCD. Some four years after the last compilation of multiplicity data,[5] and hopefully not too soon before multiplicity experiments are contemplated at CERN ISR energies, it may be worthwhile to comment on the relation between particle multiplicities and the elastic eikonal, as expressed by Eq. (1). The purpose of this paper is to point to the inverse process of Ref. 1, as a way of empirically calculating the eikonal function in terms of the energy dependence of three multiplicity moments and the differential cross section, thereby providing experimental tests of the general multiperipheral analysis (GMA) which underlies the construction of Eq. (1).

Of course, one may simply Fourier transform the square root of the measured differential cross section $d\sigma_{\text{el}}/dt\,(s,t)$, $-t=\vec{q}^{\,2}$, to obtain estimates of $\rho(s,b)$,

$$\frac{d\sigma_{\text{el}}}{dt}=\frac{1}{4\pi}\,|T|^2,$$

$$T=i\int d^2b\, e^{i\vec{q}_\perp\cdot\vec{b}}\left[1-e^{-\rho(s,b)}\right],$$

a procedure that has been described by many authors, for example Miettinen,[6] who included a real as well as an imaginary part of T. One is left with a set of curves for ρ, typically run at a few values of energy over a range of impact parameters. The method described here contains the additional input information of energy-dependent multiplicities, which with certain reservations permits the prediction of ρ at one energy if it is known at another. By comparison with the corresponding ρ values measured from the differential cross section, one can then test the validity of the GMA assumptions (A) and (B) above.

The information obtained from multiplicities can be used in essentially two different ways: (I) to suggest the form of impact parameter b and rapidity $y=\ln(s/s_0)$ dependence of $\rho(s,b)\equiv L(y,b)$, expecting on the basis of many previous theoretical analyses that the eikonal can be written in the form $L(y,b)=U(y)\,W(b^2f(y))$, where U, f, and W are functions to be determined; and (II) having selected the form of $f(y)$, to make an independent determination of $U(y)$. Here s_0 denotes a constant of dimension GeV^2, typically chosen near the lower end of the measured CM (energy)^2 values; for simplicity it is suppressed below, where every function of s is understood to be given in terms of the dimensionless ratio s/s_0. In both cases, an initial specification of ρ for all b at fixed y_0 is needed, and the multiplicity data then generates the variable y dependence. The aim of the analysis is to provide functions of U and/or f which can be compared with those given by scattering data at different energies.

It is simplest to begin by first treating case (II), assuming prior knowledge of—or theoretical prejudices for—$f(y)$, using the measured L at some y_0 to determine W, and then applying multiplicity data to fix the form of $U(y)$. Perhaps the simplest possible situation occurs when the ansatz $L(y,b)=U(y)\,W(b^2)$, $f\equiv 1$, is chosen, as suggested by the old Cheng-Wu analysis.[7] This choice of ansatz can be experimentally tested by determining $L(y,b)$ at two different values of y, and of b, so that the quantity $L(y,b_1)/L(y,b_2)$ is reasonably independent of y, while $L(y_1,b)/L(y_2,b)$ is independent of b. If this test is not successful, then a better ansatz is needed, as described below. Since only the imaginary part of the scattering amplitude is considered in this discussion, "successful" would presumably mean accuracy to within 10%.

From Eq. (1) and the definitions

$$\sigma_{\rm in}\langle n\rangle=\left.\frac{\partial\sigma_{\rm in}(\xi)}{\partial\xi}\right|_{\xi=1}\equiv R(y)\,,$$

$$\sigma_{\rm in}\langle n(n-1)\rangle=\left.\frac{\partial^2\sigma_{\rm in}}{\partial\xi^2}\right|_{\xi=1}\equiv R(y)\,Q(y)\,,$$

$$Q(y)\equiv\langle n(n-1)\rangle/\langle n\rangle\,,$$

there follows the differential equation

$$U''+2r\,U'^2=\frac{1}{y}\,Q(y)\,U'\,,\tag{2}$$

where

$$r\equiv\int d^2b\,W^2(b^2)\Big/\int d^2b'\,W(b'^2)\,.$$

In the breakup of L into the product $U(y)\,W(b^2)$ there is clearly the freedom to choose the overall normalization of either U or W, and the normlization of the latter is determined, in part, by r.

For any r, (2) can be solved by quadratures,

$$U(y)=U(y_0)+\frac{1}{2r}\ln\left[1+2r\,U'(y_0)\times\int_{y_0}^{y}dy'\exp\left(\int_{y_0}^{y'}\frac{dx}{x}\,Q(x)\right)\right]\tag{3}$$

in terms of two independent integration constants $U(y_0)$ and $U'(y_0)$, defined at some specified y_0. We expect that $U(y)>U(y_0)$, for $y>y_0$; but in general, both y_0 and r are arbitrary. In fact, both r and $U'(y_0)$ will be fixed in terms of initial scattering data, and multiplicity moments, at y_0.

To do this, one observes from the definitions of $\langle n\rangle$ and r that

$$2rU'(y_0)=\frac{1}{y_0}\,\langle n(y_0)\rangle\,\sigma_{\rm in}(y_0)\,\frac{\int d^2b\,W^2}{(\int d^2b'\,W)^2}\,,\tag{4}$$

an expression independent of the normalization of $W(b^2)$. Reference to the scattering data, which in principle generates $L(y,b)$ from the differential cross section, then permits the replacement of (4) by

$$2rU'(y_0)=\frac{1}{y_0}\,\langle n(y_0)\rangle\,\sigma_{\rm in}(y_0)\,\frac{\int d^2b\,L^2(y_0,b)}{[\int d^2b'\,L(y_0,b')]^2}\,.\tag{5}$$

Similarly, the constant r may be evaluated directly from the scattering data, at any value of y,

$$\frac{1}{r}=U(y)\int d^2b\,L(y,b)\left[\int d^2b'\,L^2(y,b')\right]^{-1}.\tag{6}$$

The most useful form results when (6) is evaluated at y_0, for the substitution of (5) and (6) into (3) yields

$$U(y)=U(y_0)\times\left\{1+u_0\ln\left[1+v_0\int_{y_0}^{y}dy'\exp\left(\int_{y_0}^{y'}\frac{dx}{x}\,Q(x)\right)\right]\right\},\tag{7}$$

where u_0, v_0, are constants determined from data at y_0,

$$u_0=\frac{1}{2}\int d^2b\,L(y_0,b)\left[\int d^2b'\,L^2(y_0,b')\right]^{-1},$$

and $v_0=2rU'(y_0)$ of (5). Hence, $U(y)$ is determined, to within the constant $U(y_0)$, in terms of scattering and multiplicity data. The function $W(b^2)$ is then given, to within the same normalization constant $U(y_0)$, by the experimentally measured $L(y_0,b)$: $W(b^2)=L(y_0,b)/U(y_0)$.

This analysis requires the measurement of $L(y,b)$ at just one value of rapidity y_0, and uses the multiplicity data to interpolate to large values of y. Other forms are possible, but they depend on knowing $L(y,b)$, or its impact-parameter integrals, as a function of y. For example, evaluating (6) at variable y generates

$$U(y)=U(y_0)\times\left\{1-u(y)\ln\left[1+v_0\int_{y_0}^{y}dy'\exp\left(\int_{y_0}^{y'}\frac{dx}{x}Q(x)\right)\right]\right\}^{-1}, \tag{8}$$

with

$$u(y)=\frac{1}{2}\int d^2b\,L(y,b)\bigg/\int d^2b'\,L^2(y,b')\,.$$

A somewhat simpler, first-order formalism is available if one is willing to specify the rapidity dependence of $\sigma_{\rm in}$ as well as that of $\langle n\rangle$, rather than the y dependence of $\langle n^2\rangle$ and $\langle n\rangle$, as above. From the relation

$$2y\int d^2b\frac{\partial}{\partial y}L(y,b)=\langle n\rangle\sigma_{\rm in}\,, \tag{9}$$

it easily follows that

$$U(y)=U(y_0)\left[1+\frac{1}{2}\,\frac{\int_{y_0}^{y}\frac{dy'}{y'}R(y')}{\int d^2b\,L(y_0,b)}\right]. \tag{10}$$

Within the context of this simplest choice for L, Eqs. (10) and (7) are equivalent, and this equivalence provides a consistency check for the validity of the ansatz and/or the validity of Eq. (1). It should be noted that, independently of any choice of ansatz, Eq. (9) provides a direct test of the GMA multiplicities, within the eikonal context.

A first-order formalism based on Eq. (9) provides perhaps the simplest way of incorporating multiplicity information if a more complicated ansatz for ρ is necessary. For example, if the choice $L(y,b)=U(y)\,W(b^2/c_1+c_2y)$ is made, with specified constants $c_{1,2}$, the same form of analysis that leads to Eq. (10) will generate

$$U(y)=U(y_0)\left(\frac{c_1+c_2y_0}{c_1+c_2y}\right)\left[1+\frac{1}{2}\,\frac{\int_{y_0}^{y}\frac{dy'}{y'}R(y')}{\int d^2b\,L(y_0,b)}\right], \tag{11}$$

and $W(b/c_1+c_2y_0)=L(y_0,b)/W(y_0)$. This choice $f(y)=(c_1+c_2y)^{-1}$, based on familiar theoretical forms, may or may not agree with the scattering data to within 10% as the rapidity values grow; but if constants $c_{1,2}$ can be found which do provide such agreement, Eq. (11) will then generate $U(y)/f(y)$, in terms of the constant $U(y_0)/f(y_0)$ and the multiplicity data over the range from y_0 to y.

Turning to the more complicated aspect of determining f as well as U from the multiplicity data, the rather general ansatz $L(y,b)=U(y)\times W(b^2f(y))$ will be adopted, leaving until the very end the question of testing the validity of that ansatz. It will be convenient to set $U(y)=f(y)Z(y)$, and to rescale the impact-parameter dependence $b^2f\to c^2$, so that

$$\alpha Z(y)=\int d^2b\,L(y,b)\,, \tag{12}$$

where α denotes the constant $\alpha\equiv\int d^2c\,W(c)$. Differentiating with respect to y, and substituting into Eq. (9), Eq. (12) can be rewritten as

$$\alpha Z(y)=\alpha Z_0+\frac{1}{2}\int_{y_0}^{y}\frac{dy'}{y'}R(y')\,, \tag{13}$$

where $Z_0=Z(y_0)$. [It is the function $\alpha Z(y)$ which corresponds, in this more general situation, to Eqs. (10) and (11).] If y_0 is taken to be one of the rapidities at which $L(y,b)$ has been constructed from the scattering data, then $\int d^2b\,L(y_0,b)$ specifies αZ_0; as in the case above, the combination $\alpha Z(y)=\alpha U(y)/f(y)$ is completely specified by the differential cross section at y_0 and the multiplicity data between y_0 and y.

In order to find an independent equation which will determine f (rather than U/f, as above), it is necessary to consider higher-multiplicity moments, obtained by calculating higher derivatives of $\sigma_{\rm in}(\xi)$ with respect to ξ. The complexity rapidly becomes formidable, and only the least complicated equation, derived from $d^2\sigma_{\rm in}(\xi)/d\xi^2|_{\xi=1}$, is illustrated here by the relation

$$\left(\frac{f'}{f}\right)^2+A\left(\frac{f'}{f}\right)+\frac{\gamma}{\beta}A^2=\frac{C}{f}\,, \tag{14}$$

where β and γ are the independent constants

$$\beta=\int d^2c\,W^2(c^2),\quad \gamma=\int d^2c\left[\frac{c}{2}\frac{\partial}{\partial c}W(c^2)\right]^2,$$

and

$$A(y)=\frac{\beta}{\gamma}\frac{Z'}{Z}\,,\quad C(y)=\frac{\alpha^2}{\gamma}(2y\,\alpha^2Z^2)^{-1}(\alpha Z'Q-\alpha yZ'').$$

In this relatively complicated equation, the functions $A(y)$ and $C(y)$ are known in terms of αZ and Q, while the constants appearing in them are also measurable quantities at some y_0,

$$\frac{\gamma}{\beta}=\frac{\int d^2b\left[\frac{b}{2}\frac{\partial}{\partial b}L(y_0,b)\right]^2}{\int d^2b'\,L^2(y_0,b')}\,,\qquad \frac{\alpha^2}{\gamma}=f_0\frac{\left[\int d^2b\,L(y_0,b)\right]^2}{\int d^2b'\left[\frac{b'}{2}\frac{\partial}{\partial b'}L(y_0,b')\right]^2}\,, \tag{15}$$

with the exception of the parameter $f_0=f(y_0)$ entering into α^2/γ. Since $C(y)$ is then proportional to f_0, it follows that Eq. (14) can only determine the ratio f/f_0; there will be (at least) one constant, call it f_0, not fixed by this nonlinear analysis.

The actual solution of Eq. (14) is by no means trivial (to this author), and an approximate development will surely be needed. But this depends critically on the nature of the input functions $A(y)$, $C(y)$, and on the moment properties of W, and cannot be specified in advance. There may well be special circumstances for which a simple quadrature can be effected; e.g., upon the variable change

$$f(y)=g(y)\exp\left[-\frac{1}{2}\int_{y_0}^{y}dy'A(y')\right],$$

there results the corresponding equation for g,

$$\left(\frac{g'}{g}\right)^2+\left(\frac{\gamma}{\beta}-\frac{1}{4}\right)A^2=\frac{C(y)}{g(y)}\exp\left[+\frac{1}{2}\int_{y_0}^{y}dy'A(y')\right], \tag{16}$$

and a solution can be written (or developed) for (or near) the special case $\gamma/\beta=\frac{1}{4}$. Whether or not this condition is even approximately satisfied can be determined, at some y_0, from the first of Eqs. (15). Alternately, the vanishing of $C(y)$, or the approximate constancy of $A(y)$, would be sufficient to generate a solution. Once f is known, to within the constant factor f_0, the function U can be expressed as

$$U(y)=\left(\frac{f_0}{\alpha}\right)\left(\frac{f(y)}{f_0}\right)\alpha Z(y) \tag{17}$$

and is given to within the constant (f_0/α). At any y, say y_0, W can then be inferred from the scattering data,

$$W(b^2f_0)=\left(\frac{\alpha}{f_0}\right)\frac{L(y_0,b)}{(\alpha Z_0)}, \tag{18}$$

but in order to specify $W(x)$, a choice must be made for f_0 as well as the parameter α/f_0.

Knowledge of $L(y,b)$ at two nearby values of rapidity would permit a differential equation to be constructed for $W(x)$, in terms of the known dependence of f/f_0 and αZ; but, again, one will find $W(x)$ dependent upon two constants which must be specified independently. Of course, an alternate way of extracting the b dependence of W is to calculate the ratio $W(b^2f)/W(b_0^2f)$, where b_0 denotes some fixed-impact parameter, so that

$$W(b^2f)=W(b_0^2f)\,L(y,b)/L(y,b_0).$$

At a fixed y, say y_0, one then has an expression for $W(b^2f_0)$ given in terms of the constant $W(b_0^2f_0)$, which again means that $W(x)$ is known only if the two constants f_0 and $W(b_0^2f_0)$ are specified.

If the scattering data is given as a continuous function of rapidity, then an independent way of approaching the problem is to write

$$f(y)=\frac{\alpha^2}{\beta}\frac{1}{(\alpha Z)^2}\int d^2b\,L^2(y,b) \tag{19}$$

or

$$\frac{f(y)}{f_0}=\left(\frac{\alpha Z_0}{\alpha Z(y)}\right)^2\frac{\int d^2b\,L^2(y,b)}{\int d^2b\,L^2(y_0,b)},$$

with $\alpha Z(y)$ given by the multiplicity data. To within f_0, Eq. (19) fixes $f(y)$; but, as expressed by Eq. (18), $W(x)$ still requires the specification of two parameters.

In fact, if $L(y,b)$ is known as a continuous function of y, as assumed for Eq. (19), then a test of the ansatz itself is given by the formula

$$f(y)=\left(\frac{\alpha_n}{\beta_n}\right)^{1/n-1}\left[\frac{\int d^2b\,L^n(y,b)}{[\alpha Z(y)]^n}\right]^{1/n-1}, \tag{20}$$

with $\beta_n=\int d^2C\,W^n(C^2)$ and α constant. For each value of n, the $f(y)$ of Eq. (20) is given as a constant multiplying an experimentally determined function of rapidity; and the validity of the general ansatz $L(y,b)=U(y)\,W(b^2f(y))$ demands that, to within a multiplicative, n-dependent constant, the functional form of the rapidity dependence must be independent of n.

In summary, the situation is somewhat complicated when experimental data is used to determine the functions $U(y)$ and $f(y)$; but nevertheless, within certain constraints dictated by approximate solutions to Eq. (16) or to the experimental knowledge of the scattering eikonal as a continuous function of rapidity, comparisons can be made between measured, differential-cross-section determinations of $L(y,b)$ at different energies, and the predictions made by the GMA of Eq. (1), which correlates multiplicity and scattering information in a very definite way.

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