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Role of Sp(12,R) in the harmonic-oscillator quark model

K. C. Bowler and P. J. Corvi
Physics Department, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

A. J. G. Hey and P. D. Jarvis*
Physics Department, University of Southampton, Southampton SO9 5NH, England

R. C. King
Mathematics Department, University of Southampton, Southampton SO9 5NH, England

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We consider the spectrum-generating algebra Sp(12,R) of the harmonic-oscillator quark model and show how it may be used to label the oscillator eigenstates. We give a new and direct method of constructing wave functions of definite symmetry type. We show that Sp(12,R) provides the most appropriate means of classifying the symmetry breaking produced by an anharmonic perturbation and we derive an algebraic mass formula for the $N=2$ multiplets of the model, and indicate how the method may be generalized to arbitrary N. Finally we extend the present successful phenomenological analysis of the baryon spectrum performed by Isgur and Karl, and discuss the possibility that the Δ D35(1925) resonance is evidence for a [$\underline{56}$,1^-] multiplet corresponding to excitation of new gluonic degrees of freedom within baryons.

I. INTRODUCTION

The group theory of the harmonic oscillator has been discussed extensively,[1] especially in connection with its application to nuclear many-body problems[2,3] and to quark models.[4] In an earlier paper,[5] hereafter referred to as I, we indicated briefly the relevance of the so-called spectrum-generating algebra Sp(12, R) of the three-quark harmonic-oscillator model to the problem of classifying and constructing the state vectors, and also to the discussion of the matrix elements of an anharmonic perturbation. Recall[5] that our starting point was the form of the harmonic-oscillator model suggested by Isgur and Karl,[6,7] incorporating anharmonic perturbation, quark mass differences, and some effects of the nonrelativistic reduction of colored-gluon exchange between quarks: The relevant Hamiltonian is

$$H=\sum_i m_i+H_0+H_{\rm hyp} \tag{1.1}$$

with

$$H_0=\sum_{i=1}^{3}\frac{\vec{p}_i^{\,2}}{2m_i}+\sum_{i<j}V_{\rm conf}^{ij}\,, \tag{1.2}$$

where the confining potential $V_{\rm conf}^{ij}$ is written

$$V_{\rm conf}^{ij}=\tfrac{1}{2}K{r_{ij}}^2+U(r_{ij})\,, \tag{1.3}$$

r_{ij} being the magnitude of the separation of quarks i and j, and $U(r_{ij})$ an unknown anharmonic potential. Treating U and the color hyperfine Hamiltonian $H_{\rm hyp}$ perturbatively in the harmonic-oscillator basis, Isgur and Karl obtained a good description of baryon resonances up to about 2 GeV in mass. In this paper we shall only be concerned with the perturbative effect of U. It will again prove convenient for us to introduce the standard three-body coordinates

$$\vec{\rho}=(\vec{r}_1-\vec{r}_2)/\sqrt{2}\,;\quad \vec{\lambda}=(\vec{r}_1+\vec{r}_2-2\vec{r}_3)/\sqrt{6} \tag{1.4}$$

and the corresponding creation and annihilation operators $a_i^\dagger(\rho)$, $a_i^\dagger(\lambda)$, $a_i(\rho)$, and $a_i(\lambda)$ $(i=x,y,z)$. In a spherical basis we define

$$a_\pm^\dagger(\rho)=\mp\frac{1}{\sqrt{2}}[a_x^\dagger(\rho)\pm ia_y^\dagger(\rho)]\,, \tag{1.5}$$

$$a_0^\dagger(\rho)=a_z^\dagger(\rho)\,, \tag{1.6}$$

satisfying the commutation relations

$$[a_\mu,a_{\mu'}^\dagger]=\delta_{\mu\mu'}\quad(\mu,\mu'=+,-,0;\hbar=1) \tag{1.7}$$

with similar expressions for the λ mode.

We noted in I that in the $S=0$ sector, in the absence of anharmonic perturbations, the general excited state of H_0 with given N, orbital angular momentum L, and permutation symmetry P ($=S$, A, M_ρ, or M_λ) may be written

$$|N\rangle\equiv|\psi_{LL_z}^{(P)}\rangle\equiv\hat{\psi}_{LL_z}^{(P)}|0\rangle\,;\quad |0\rangle\equiv|0\rangle_\rho|0\rangle_\lambda \tag{1.8}$$

with $\hat{\psi}_{LL_z}^{(P)}$ an Nth-order monomial of creation operators. In Sec. II of this paper, we show how such excited states may be labeled using the spectrum-generating algebra Sp(12, R). We give details of how the monomials $\hat{\psi}$ may be constructed by a novel and direct method, and we compare our method with other prescriptions in the literature.[4,8-12]

In our previous paper we gave a brief outline of models containing more than just quark degrees of freedom and we discussed the conjecture by Cutkosky and collaborators that the $\Delta D35(1925)$ resonance might be evidence for such gluonic excitations.[11,13,14] Using the parametrization adopted by Isgur and Karl[6,7] in their successful description of baryons with $N \leq 2$, we showed that the mean mass of the nonstrange sector of the $N=3\ [\underline{56}, 1^-]$ multiplet, to which the $\Delta D35(1925)$ is most plausibly assigned, is close to the most recent[15] quoted mass of 1930 ± 20 MeV, thus suggesting a conventional interpretation of the state as a genuine three-quark excitation. In Sec. III we give details of our calculations, employing the state vectors constructed in Sec. II, and we compare our work with earlier papers on the subject[14,16,17] in Sec. V, after having given, in Sec. IV, a group-theoretical analysis of the problem. This is based upon Sp(12, R), whose application to the matrix elements of the anharmonic perturbation U leads to an algebraic mass formula, involving the quadratic Casimir operators of the various subgroup chains, that reproduces the pattern of splitting among the various $N=2$ multiplets first noted by Gromes and Stamatescu,[18] and later by Isgur and Karl,[6,7] in calculations using explicit oscillator wave functions. We indicate how the method can be applied at higher N. Details of some of the group theory involved in the derivation of mass formulas are relegated to two appendices.

II. HARMONIC-OSCILLATOR STATE FUNCTIONS

In this section we develop the formalism leading, for the three-body case, to the unitary degeneracy group U(6) and to the symplectic spectrum-generating group Sp(12, R). These groups provide a means of both labeling and constructing the oscillator state functions of given total orbital angular momentum, parity, and permutation symmetry.

In the case under consideration, involving an $SU(3)_{\text{color}}$ singlet of three totally antisymmetrized quarks, the $SU(6)_{\text{flavor}}$ properties of the oscillator states are completely determined by this permutation symmetry. For the time being, therefore, the $SU(6)_{\text{flavor}}$ and $SU(3)_{\text{color}}$ sectors are ignored. The relevant oscillator modes are associated with the creation and annihilation operators $\vec{a}^\dagger(\rho), \vec{a}^\dagger(\lambda)$, $\vec{a}(\rho)$, and $\vec{a}(\lambda)$ defined in Sec. I. These may be viewed as components of the six-vectors:

$$(a_I) = (a_{ia}) = (\vec{a}(\rho), \vec{a}(\lambda)),$$

$$(a_I^\dagger) = (a_{ia}^\dagger) = (\vec{a}^\dagger(\rho), \vec{a}^\dagger(\lambda))$$

with $i=1, 2, 3$, $a=1, 2$, and $I=1, 2, \ldots, 6$. They satisfy

$$[a_I, a_J] = [a_I^\dagger, a_J^\dagger] = 0\,,$$
$$[a_I, a_J^\dagger] = [a_{ia}, a_{jb}^\dagger] = \delta_{ij}\delta_{ab} = \delta_{IJ} \tag{2.1}$$

for $I, J = 1, 2, \ldots, 6$.

With this notation the oscillator Hamiltonian takes the form

$$H_{\text{osc}} = \omega \sum_{I=1}^{6} \tfrac{1}{2}\{a_I, a_I^\dagger\}\,. \tag{2.2}$$

The degeneracy group is revealed by noting that the 36 bilinear operators

$$E_{IJ} = \tfrac{1}{2}\{a_I^\dagger, a_J\} \tag{2.3}$$

all commute with this Hamiltonian and satisfy the commutation relations:

$$[E_{IJ}, E_{KL}] = \delta_{JK} E_{IL} - \delta_{IL} E_{KJ} \tag{2.4}$$

of the real Lie algebra of GL(6, C) whose complex form is well known as the Lie algebra of U(6). It follows that degenerate oscillator states are associated with the unitary finite-dimensional irreducible representations of U(6). In the canonical labeling scheme based on the structure U(6) $\sim$SU(6)$\times$U(1), one of the state labels is of course the principal quantum number N associated with the U(1) subgroup whose generator $E_{IJ}\delta^{IJ}$ is proportional to the Hamiltonian (2.2).

The spectrum consists of the $N=0$ vacuum state $|0\rangle$, transforming as the representation $\{0\} = \underline{1}$ of U(6); the $N=1$ states $a_I^\dagger|0\rangle$, transforming as $\{1\} = \underline{6}$ of U(6), the $N=2$ states $a_I^\dagger a_J^\dagger|0\rangle$, transforming as $\{2\} = \underline{21}$ of U(6), and so on.[19]

One way to generate this spectrum involves the set of bilinear operators which can be formed from the components a_I and $a_I^\dagger$ of the 12-vector

$$(a_A) = (a_{I\alpha}) = (a_{ia\alpha})$$
$$= (\vec{a}(\rho), \vec{a}(\lambda), \vec{a}^\dagger(\rho), \vec{a}^\dagger(\lambda))$$

with $I=1, 2, \ldots, 6$, $\alpha=1, 2$, and $A=1, 2, \ldots, 12$. These satisfy the commutation rules

$$[a_A, a_B] = [a_{I\alpha}, a_{J\beta}] = \delta_{IJ}\epsilon_{\alpha\beta} = J_{AB} \tag{2.5}$$

for $A, B = 1, 2, \ldots, 12$, with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } J = \delta \times \epsilon\,.$$

The 78 bilinear operators

$$S_{AB} = \tfrac{1}{2}\{a_A, a_B\} \tag{2.6}$$

satisfy the commutation relations

$$[S_{AB}, S_{CD}] = J_{BC}S_{AD} + J_{AD}S_{BC} + J_{AC}S_{BD} + J_{BD}S_{AC} \tag{2.7}$$

of the real Lie algebra of Sp(12, R). The significance of this enlarged group is that the complete Fock space of the oscillator spectrum decomposes into just two infinite-dimensional, unitary, irreducible representations of Sp(12, R): states of even or odd N. Thus the Lie algebra of Sp(12, R) is referred to as a spectrum-generating algebra, analogous for this three-quark case to the algebra Sp($2N$, R) introduced by Goshen and Lipkin[3] for the single N-dimensional oscillator.

In contrast to this, there exists another way in which the physical oscillator states are made manifest as basis states; this time, of finite-dimensional, nonunitary irreducible representations of Sp(12, R). This comes about because both the creation and annihilation operators themselves and, more generally, monomials of them form the basis of such represenations. This follows from the existence of the map

$$S_{AB}:\ a_c \to [S_{AB}, a_C] = J_{AC}a_B + J_{BC}a_A \qquad (2.8)$$

and its generalizations

$$S_{AB}:\ a_C a_D \to [S_{AB}, a_C a_D] = J_{AC}a_B a_D + J_{BC}a_A a_D + J_{AD}a_C a_B + J_{BD}a_C a_A \qquad (2.9)$$

and so on. It is clear, in particular, that the operators 1, a_A, $a_A a_B + a_B a_A$, $a_A a_B a_C + a_B a_C a_A + a_C a_A a_B + a_B a_A a_C + a_C a_B a_A + a_A a_C a_B, \ldots$ form bases of the symmetric representations $\langle 0\rangle = \underline{1}$, $\langle 1\rangle = \underline{12}$, $\langle 2\rangle = \underline{78}, \ldots$ of Sp(12,R).[19] Since the variables 1, $a_I^\dagger$, $a_I^\dagger a_J^\dagger, \ldots$ belong to these bases, the physical oscillator states are indeed associated with finite-dimensional irreducible representations as claimed. The physical states of the representation $\langle N\rangle$ are precisely those for which

$$S_{12} = S_{I1J2}\delta^{IJ} = E_{IJ}\,\delta^{IJ}, \qquad (2.10)$$

acting as in (2.8) and (2.9), has the eigenvalue N.

This generator, besides being the U(1) generator encountered earlier, belongs to the algebra of the subgroup Sp(2, R), with generators

$$S_{\alpha\beta} = S_{I\alpha J\beta}\delta^{IJ}. \qquad (2.11)$$

This group is locally isomorphic to the pseudo-orthogonal group SO(2, 1) as can be seen by noting that $H_1 = \frac{1}{4}(S_{11} + S_{22})$, $H_2 = (i/4)(S_{11} - S_{22})$, and $H_3 = \frac{1}{2}S_{12}$ satisfy

$$[H_i, H_j] = i\epsilon_{ijk}\eta_{kk}H_k, \qquad (2.12)$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The physical states are those components of the SO(2, 1) multiplets of pseudospin $[0] = \underline{1}$, $[\frac{1}{2}] = \underline{2}$, $[1] = \underline{3}, \ldots, [N/2] = \underline{N+1}, \ldots$ with maximum third component 0, $\frac{1}{2}$, 1, ..., $N/2, \ldots$. The corresponding Sp(2, R) multiplets are denoted by $\langle 0\rangle = \underline{1}$, $\langle 1\rangle = \underline{2}$, $\langle 2\rangle = \underline{3}, \ldots \langle N\rangle = N+1, \ldots$.

Just as in the case of a single three-dimensional oscillator, the physical states with fixed pseudospin quantum numbers may be further classified into multiplets of an orthogonal group. In the case of three particles this is the subgroup O(6) of Sp(12, R) with generators

$$O_{IJ} = S_{I\alpha J\beta}\epsilon^{\alpha\beta} = E_{IJ} - E_{JI} \qquad (2.13)$$

satisfying

$$[O_{IJ}, O_{KL}] = \delta_{JK}O_{IL} + \delta_{IL}O_{JK} - \delta_{IK}O_{JL} - \delta_{JL}O_{IK}. \qquad (2.14)$$

The group O(6) contains as a subgroup O(3)×O(2), whose constituents O(3) and O(2) are generated by

$$O_{ij} = O_{iajb}\delta^{ab} \qquad (2.15)$$

and

$$O_{ab} = O_{iajb}\delta^{ij}. \qquad (2.16)$$

The former serve to specify the total orbital angular momentum of the physical states through the familiar generators

$$L_i = i\epsilon_{ijk}O_{jk}, \qquad (2.17)$$

while the latter is the generator of rotations in the two-dimensional space associated with the ρ and λ modes of oscillation. Typically a rotation through θ is given in this space by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It should be stressed, however, that the full group O(2) also includes the reflection

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is particularly important because the permutation group S_3 is a subgroup of O(2) but not of SO(2). This can be seen by noting that the permutations $P(123)$ and $P(12)$ which generate S_3 are given in the two-dimensional, faithful irreducible representation[19] $(2,1) = \underline{M}$ by

$$P(123) = R(2\pi/3), \quad P(12) = \sigma.$$

Thus, to summarize, the subgroup chain to be used in labeling oscillator states is

$$\begin{aligned} \mathrm{Sp}(12, R) &\supset \mathrm{Sp}(2, R)\times \mathrm{O}(6) \\ &\supset \mathrm{Sp}(2, R)\times \mathrm{O}(3)\times \mathrm{O}(2) \\ &\supset \mathrm{U}(1)\times \mathrm{SO}(3)\times S_3. \end{aligned} \qquad (2.18)$$

Its key labels N, L, and P are associated with the irreducible representations of U(1), SO(3), and S_3, respectively.

In order to enumerate at each level, specified by N, the $O(3)\times S_3$ multiplets and hence the $SU(6)_{\text{flavor}}\times O(3)$ multiplets we require the branching rules for various subgroup embeddings. In the case of the continuous groups these are given by simple Young-diagram techniques collected together in an earlier paper[20] and summarized here in Appendix A. The symmetric tensor representation $\langle N\rangle$ of $Sp(12,R)$ reduces on restriction to $Sp(2,R)\times O(6)$ to $\langle\xi/B\rangle\times[\xi/D]$ summed over all Young diagrams ξ of weight N, where the terms ξ/B and ξ/D stand for all possible quotients of the S function ξ, by S functions B and D corresponding to Young diagrams with even column and even row lengths, respectively.[19]

Since it is required that the corresponding physical oscillator states have pseudospin $N/2$ and are associated with the $Sp(2,R)$ representation $\langle N\rangle$, it follows that the only relevant value of ξ is N. Hence, according to the above branching rule, these states belong to the O(6) multiplets $[M]$ with $M=N, N-2, N-4, \ldots$ where the sequence terminates with either 1 or 0. This is in accordance with the result expected through consideration of the degeneracy group U(6), since the totally symmetric tensor representation $\{N\}$ of U(6) yields just these representations $[M]$ of O(6) on restriction to this subgroup.

The branching rule (A2) of Appendix A yields in the case of the representation $[M]$ of O(6) the appropriate multiplets of $O(3)\times O(2)$ when used in conjunction with the modification rules (A4).

The final reduction from O(2) to S_3 has been discussed elsewhere.[21] The scalar, $[0]=\underline{1}$, and pseudoscalar, $[1^2]=[0]^*=\underline{1}^*$, representations of O(2) are of course symmetric and antisymmetric, respectively, under S_3, and thus yield on restriction to this subgroup $(3)=\underline{S}$ and $(1^3)=\underline{A}$. The remaining irreducible representations of O(2) are the doublets $[m]=\underline{2}_m$, labeled by a quantum number m (integer or half-integer in general) such that $R(\theta)$ is mapped to $R(m\theta)$. Here only integer values of m occur and it is easy to deduce that under the restriction from O(2) to S_3

$$[m]\to\begin{cases}\underline{S}+\underline{A} & \text{if } m\equiv 0 \pmod 3,\\ \underline{M} & \text{if } m\equiv 1,2 \pmod 3.\end{cases}\tag{2.19}$$

This allows us to complete the reduction procedure and thereby determine the $SU(6)_{\text{flavor}}\times O(3)$ multiplets at each degeneracy level specified by N. The results for $N=0$, 1, 2, 3, and 4 are displayed in Table I.

Making use of the variables

$$\vec{\xi}=\vec{\rho}+i\vec{\lambda};\quad \vec{\eta}=\vec{\rho}-i\vec{\lambda},\tag{2.20}$$

the physical states are now remarkably easy to construct in terms of auxiliary creation operators[8,10,12] which we define by

TABLE I. $SU(6)_{\text{flavor}}\times O(3)$ multiplet structure of the harmonic-oscillator quark model: levels $N=0$ to $N=4$.

N	O(6)	$O(3)\times O(2)$	$[SU(6), L^P]$
0	$\underline{1}$	$\underline{1}\times\underline{1}$	$[\underline{56},0^+]$
1	$\underline{6}$	$\underline{3}\times\underline{2}_1$	$[\underline{70},1^-]$
2	$\underline{20}$	$(\underline{5}+\underline{1})\times\underline{2}_2$	$[\underline{70},2^+]$, $[\underline{70},0^+]$
		$\underline{3}\times\underline{1}^*$	$[\underline{20},1^+]$
		$\underline{5}\times\underline{1}$	$[\underline{56},2^+]$
	$\underline{1}$	$\underline{1}\times\underline{1}$	$[\underline{56},0^+]$
3	$\underline{50}$	$(\underline{7}+\underline{3})\times\underline{2}_3$	$[\underline{56},3^-]$, $[\underline{20},3^-]$, $[\underline{56},1^-]$, $[\underline{20},1^-]$
		$(\underline{7}+\underline{5}+\underline{3})\times\underline{2}_1$	$[\underline{70},3^-]$, $[\underline{70},2^-]$, $[\underline{70},1^-]$
	$\underline{6}$	$\underline{3}\times\underline{2}_1$	$[\underline{70},1^-]$
4	$\underline{105}$	$(\underline{9}+\underline{5}+\underline{1})\times\underline{2}_4$	$[\underline{70},4^+]$, $[\underline{70},2^+]$, $[\underline{70},0^+]$
		$(\underline{9}+\underline{7}+\underline{5}+\underline{3})\times\underline{2}_2$	$[\underline{70},4^+]$, $[\underline{70},3^+]$, $[\underline{70},2^+]$, $[\underline{70},1^+]$
		$(\underline{9}+\underline{5}+\underline{1})\times\underline{1}$	$[\underline{56},4^+]$, $[\underline{56},2^+]$, $[\underline{56},0^+]$
		$(\underline{7}+\underline{5})\times\underline{1}^*$	$[\underline{20},3^+]$, $[\underline{20},2^+]$
	$\underline{20}$	$(\underline{5}+\underline{1})\times\underline{2}_2$	$[\underline{70},2^+]$, $[\underline{70},0^+]$
		$\underline{3}\times\underline{1}^*$	$[\underline{20},1^+]$
		$\underline{5}\times\underline{1}$	$[\underline{56},2^+]$
	$\underline{1}$	$\underline{1}\times\underline{1}$	$[\underline{56},0^+]$

$$\vec{a}^\dagger(\zeta) = \vec{a}^\dagger(\rho) + i\vec{a}^\dagger(\lambda)\,,$$
$$\vec{a}^\dagger(\eta) = \vec{a}^\dagger(\rho) - i\vec{a}^\dagger(\lambda)\,. \tag{2.21}$$

These are the basis states of two one-dimensional irreducible representations of SO(2). The transformation to this basis thus serves to diagonalize all the rotation matrices $R(\theta)$ including $R(2\pi/3)$. Under the action of the permutations of S_3

$$P(12)\vec{a}^\dagger(\zeta) = -\vec{a}^\dagger(\eta)\,, \quad P(123)\vec{a}^\dagger(\zeta) = e^{-i2\pi/3}\vec{a}^\dagger(\zeta)\,, \tag{2.22}$$
$$P(12)\vec{a}^\dagger(\eta) = -\vec{a}^\dagger(\zeta)\,, \quad P(123)\vec{a}^\dagger(\eta) = e^{i2\pi/3}\vec{a}^\dagger(\eta)\,,$$

so it is trivial to determine the transformation properties of monomials in these operators.

Consider the particular monomials

$$W(\vec{\xi},\vec{\eta}) = (\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\eta))^a(\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\zeta))^b \times(\vec{a}^\dagger(\eta)\cdot\vec{a}^\dagger(\eta))^c a_+^\dagger(\zeta)^p a_+^\dagger(\eta)^q \times[a^\dagger(\zeta)a^\dagger(\eta)]_+^\alpha\,, \tag{2.23}$$

where

$$[a^\dagger(\zeta)a^\dagger(\eta)]_+ = a_+^\dagger(\zeta)a_0^\dagger(\eta) - a_0^\dagger(\zeta)a_+^\dagger(\eta)\,.$$

The corresponding principal quantum number N is clearly $2(a+b+c+\alpha)+(p+q)$, while it is a stretched angular-momentum state with $L_z = L = p+q+\alpha$. The factor

$$\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\eta) = \vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\rho) + \vec{a}^\dagger(\lambda)\cdot\vec{a}^\dagger(\lambda) \tag{2.24}$$

is an O(6) invariant so that the monomial $W(\vec{\xi},\vec{\eta})$ belongs to the representation $[M]$ of O(6) with $M \leq N-2a$. Furthermore, if $m = 2(b-c)+(p-q)$, the corresponding O(2) representation $[m]$ has dimension two with basis $W(\vec{\xi},\vec{\eta})$ and $\overline{W}(\vec{\xi},\vec{\eta})$, where $\overline{W}(\vec{\xi},\vec{\eta}) \equiv W(\vec{\eta},\vec{\xi})$, unless $W(\vec{\xi},\vec{\eta}) = \pm W(\vec{\eta},\vec{\xi})$ in which case $m=0$ and the corresponding representation $[m]=[0]$ or $[0]^*$ is one-dimensional. From these basis states $W(\vec{\xi},\vec{\eta})$ and $\overline{W}(\vec{\xi},\vec{\eta})$ of irreducible representations of SO(2), the basis states of irreducible representations of S_3 are recovered in the form

$$\mathcal{R}W(\vec{\xi},\vec{\eta}) = \tfrac{1}{2}[W(\vec{\xi},\vec{\eta}) + \overline{W}(\vec{\xi},\vec{\eta})]\,,$$
$$\mathcal{I}W(\vec{\xi},\vec{\eta}) = \frac{-i}{2}[W(\vec{\xi},\vec{\eta}) - \overline{W}(\vec{\xi},\vec{\eta})]\,. \tag{2.25}$$

The results depend only upon m (mod 6) and are given in Table II in which S, A, M_ρ, and M_λ signify basis states of the representations $(3)=\underline{S}$, $(1^3)=\underline{A}$, and $(2,1)=\underline{M}$ of S_3, with M_ρ and M_λ transforming under permutations in exactly the same manner as $\vec{\rho}$ and $\vec{\lambda}$.

The particular factor $[a^\dagger(\zeta)a^\dagger(\eta)]_+$ is antisymmetric in the sense that $P(12)[a^\dagger(\zeta)a^\dagger(\eta)]_+ = -[a^\dagger(\zeta)a^\dagger(\eta)]_+$ while $P(123)[a^\dagger(\zeta)a^\dagger(\eta)]_+ = [a^\dagger(\zeta)a^\dagger(\eta)]_+$. It is therefore a basis state of the type A. This factor, moreover, satisfies a syzygylike identity[22,23]

$$[a^\dagger(\zeta)a^\dagger(\eta)]_+[a^\dagger(\zeta)a^\dagger(\eta)]_+ = (\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\zeta))a_+^\dagger(\eta)^2 + (\vec{a}^\dagger(\eta)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\zeta)^2 - 2(\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\zeta)a_+^\dagger(\eta)\,. \tag{2.26}$$

The implication of this and the use of W and $\overline{W}$ in (2.25) is that in constructing all the independent oscillator states for a given value of N it is only necessary to consider those distinct monomials $W(\vec{\xi},\vec{\eta})$ of degree N with $m \geq 0$ and $\alpha = 0$ or 1.

The $N=2$ and $N=3$ states derived in this way are given explicitly in Tables III and IV. The procedure used in constructing such states of definite angular momentum, parity, and permutation symmetry is thus extremely simple and somewhat more direct than previous procedures. The simplification is in large measure due to the use of the $(\vec{\xi},\vec{\eta})$ basis rather than the $(\vec{\rho},\vec{\lambda})$ basis. The merits of the $(\vec{\xi},\vec{\eta})$ basis can also be seen in the work of Kramers and Moshinsky,[8] which is discussed further by Moshinsky,[10] which involves the subgroup chain

$$U(6)\supset U(3)\times U(2)\supset O(3)\times O(2)\supset SO(3)\times S_3 \tag{2.27}$$

and, more recently, in the work of Bohm[12] using the chain

$$U(6)\supset U(3)\times S_3\supset SO(3)\times S_3\,. \tag{2.28}$$

This contrasts with the complexities associated with the use of the $(\vec{\rho},\vec{\lambda})$ basis which are apparent

TABLE II. Monomials of definite permutation symmetry: $\mathcal{R}W = \frac{1}{2}(W+\overline{W})$ and $\mathcal{I}W = (-i/2)(W-\overline{W})$, where W and $\overline{W}$ are defined in the text.

m (mod 6)	0	1	2	3	4	5
$\mathcal{R}W$	S	M_ρ	M_λ	A	$-M_\lambda$	M_ρ
$\mathcal{I}W$	A	M_λ	M_ρ	S	M_ρ	$-M_\lambda$

TABLE III. State-function monomials at the $N=2$ level.

$[SU(6), L^P]$	$O(3)\times O(2)$	S_3	Monomial
$[\underline{56},0^+]$	$\underline{1}\times\underline{1}$	S	$\mathcal{R}(\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\eta))$
$[\underline{70},0^+]$	$\underline{1}\times\underline{2}_2$	M_ρ	$\mathcal{I}(\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\zeta))$
		M_λ	$\mathcal{R}(\vec{a}^\dagger(\zeta)\cdot\vec{a}^\dagger(\zeta))$
$[\underline{20},1^+]$	$\underline{3}\times\underline{1}^*$	A	$\mathcal{I}([a^\dagger(\zeta)a^\dagger(\eta)]_+)$
$[\underline{56},2^+]$	$\underline{5}\times\underline{1}$	S	$\mathcal{R}(a_+^\dagger(\zeta)a_+^\dagger(\eta))$
$[\underline{70},2^+]$	$\underline{5}\times\underline{2}_2$	M_ρ	$\mathcal{I}(a_+^\dagger(\zeta)^2)$
		M_λ	$\mathcal{R}(a_+^\dagger(\zeta)^2)$

TABLE IV. State-function monomials at the $N=3$ level.

$[SU(6), L^P]$	$O(3)\times O(2)$	S_3	Monomial
$[\underline{56}, 3^-]$	$\underline{7}\times\underline{2}_3$	S	$\mathcal{S}(a_+^\dagger(\xi)^3)$
$[\underline{20}, 3^-]$	$\underline{7}\times\underline{2}_3$	A	$\mathcal{R}(a_+^\dagger(\xi)^3)$
$[\underline{70}, 3^-]$	$\underline{7}\times\underline{2}_1$	M_ρ	$\mathcal{R}(a_+^\dagger(\xi)^2 a_+^\dagger(\eta))$
		M_λ	$\mathcal{S}(a_+^\dagger(\xi)^2 a_+^\dagger(\eta))$
$[\underline{70}, 2^-]$	$\underline{5}\times\underline{2}_1$	M_ρ	$\mathcal{R}(a_+^\dagger(\xi)[a^\dagger(\xi)a^\dagger(\eta)]_+)$
		M_λ	$\mathcal{S}(a_+^\dagger(\xi)[a^\dagger(\xi)a^\dagger(\eta)]_+)$
$[\underline{56}, 1^-]$	$\underline{3}\times\underline{2}_3$	S	$\mathcal{S}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi)a_+^\dagger(\xi))$
$[\underline{20}, 1^-]$	$\underline{3}\times\underline{2}_3$	A	$\mathcal{R}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi)a_+^\dagger(\xi))$
$[\underline{70}, 1^-]$	$\underline{3}\times\underline{2}_1$	M_ρ	$\mathcal{R}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi)a_+^\dagger(\eta))$
		M_λ	$\mathcal{S}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi)a_+^\dagger(\eta))$
$[\underline{70}, 1^-]$	$\underline{3}\times\underline{2}_1$	M_ρ	$\mathcal{R}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta)a_+^\dagger(\xi))$
		M_λ	$\mathcal{S}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta)a_+^\dagger(\xi))$

in the construction procedure of Karl and Obryk[9] based on the reduction

$$U(6)\supset SO(3)\times S_3\,, \tag{2.29}$$

and even more strikingly apparent in that of Horgan[4] based on the subgroup chain

$$U(6)\supset U(3)\times U(2)\supset SO(3)\times S_3\,. \tag{2.30}$$

We shall make use of the subgroup chain

$$Sp(12, R)\supset U(6)\supset O(6)\supset O(3)\times O(2)\supset SO(3)\times S_3 \tag{2.31}$$

which incorporates the group O(6), whose use has been advocated and adopted in this context by Cutkosky and Hendrick,[11] and which appears naturally in (2.18) by virtue of (2.13). However, our scheme is not ideal in that the states obtained directly from (2.33) are not all associated with a unique irreducible representation of O(6). In general, M can take on the values $N-2a$, $N-2a-2$, $N-2a-4, \ldots$ and a more complete labeling scheme involving the specification of M can be obtained merely by orthogonalizing states commencing with the state of lowest value of M which corresponds to the largest value of a for a given N in (2.23).

At the $N=2$ level, there are no ambiguities and this orthogonalization is not necessary, but at the $N=3$ level there are two $L=1$, $P=\underline{M}\,[\underline{70}, 1^-]$ states which may be distinguished by the O(6) labels $[3]=\underline{50}$ and $[1]=\underline{6}$, as can be seen from Table I. The necessary orthogonal combinations of the states given in Table IV are

$$[3]\ [\underline{70}, 1^-]\ \underline{3}\times\underline{2}_1 \begin{cases} M_\rho & \mathcal{R}[(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi))a_+^\dagger(\eta)-\tfrac{1}{2}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\xi)] \\ M_\lambda & \mathcal{S}[(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\xi))a_+^\dagger(\eta)-\tfrac{1}{2}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\xi)]' \end{cases} \tag{2.32}$$

$$[1]\ [\underline{70}, 1^-]\ \underline{3}\times\underline{2}_1 \begin{cases} M_\rho & \mathcal{R}(\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\xi)] \\ M_\lambda & \mathcal{S}[\vec{a}^\dagger(\xi)\cdot\vec{a}^\dagger(\eta))a_+^\dagger(\xi)]\,. \end{cases} \tag{2.33}$$

A similar orthogonalization procedure is required at the $N=4$ level, to distinguish, for example, the two $L=2$, $P=\underline{S}\,[\underline{56}, 2^+]$ states labeled by $M=4$ and $M=2$. This difficulty is also experienced in making use of the states of Karl and Obryk[9] which in this particular case coincide with those given directly by (2.23).

In contrast to this, in the case cited by Horgan[4] of the $N=4$, $L=4$, $P=\underline{M}\,[\underline{70}, 4^+]$ states, the two pairs of states M_ρ and M_λ are again not mutually orthogonal in the scheme of Karl and Obryk. Horgan constructs orthogonal states by diagonalizing a matrix K whose eigenvalues then serve to label the states. However, the mthod used here leads unambiguously to the four states $|N, M, L, L_z = L, m, P\rangle$:

$$|4,4,4,4,4,M_\rho\rangle=\frac{1}{2\sqrt{3}}\{[a_+^\dagger(\rho)]^3 a_+^\dagger(\lambda)-a_+^\dagger(\rho)[a_+^\dagger(\lambda)]^3\}, \tag{2.34}$$

$$|4,4,4,4,4,M_\lambda\rangle=\frac{1}{8\sqrt{3}}\{-[a_+^\dagger(\rho)]^4+6[a_+^\dagger(\rho)]^2[a_+^\dagger(\lambda)]^2 - [a_+^\dagger(\lambda)]^4\},$$

$$|4,4,4,4,2,M_\rho\rangle=\frac{1}{2\sqrt{3}}\{[a_+^\dagger(\rho)]^3 a_+^\dagger(\lambda)+a_+^\dagger(\rho)[a_+^\dagger(\lambda)]^3\}, \tag{2.35}$$

$$|4,4,4,4,2,M_\lambda\rangle=\frac{1}{4\sqrt{3}}\{[a_+^\dagger(\rho)]^4-[a_+^\dagger(\lambda)]^4\},$$

where, as in the next section, (2.21) has been used to express the states in terms of $\vec{a}^\dagger(\rho)$ and $\vec{a}^\dagger(\lambda)$. In this case it is the label m of the O(2) representation $[m]=\underline{2}_m$ which distinguishes the states and guarantees their orthogonality.

For higher values of N, branching multiplicities in the chain (2.31) lead to other labeling ambiguities and the need to further orthogonalize states. For $N \leq 4$, the chain (2.31) does, however, provide a complete labeling scheme.

III. SOME PHENOMENOLOGICAL CONSIDERATIONS

In the absence of hyperfine interactions and neglecting quark-mass differences, the calculation of the first-order energy shift induced by the anharmonic perturbation U is straightforward. We exploit the permutation symmetry of the (flavor-spin) SU(6)×O(3) three-quark states $|\phi\rangle$ to reduce the problem to a calculation for the ρ oscillator alone:

$$\sum_{i<j} \langle \phi | U(r_{ij}) | \phi \rangle = 3\langle \phi | U(\sqrt{2}\rho) | \phi \rangle . \tag{3.1}$$

The ρ-oscillator matrix elements may of course be evaluated by using explicit oscillator wave functions[6,7,16-18] or, more elegantly, by an algebraic procedure which exploits the commutation relations of the creation and annihilation operators. As this latter method does not appear to be widely used we give an example from the $N=2$ level. Table V gives the correctly normalized monomials, constructed by the procedure given in Sec. II, for the five $N=2$ multiplets of the harmonic-oscillator model. The energy shift for the $[\underline{56}, 0^+]$ multiplet is given by

TABLE V. The correctly orthonormalized monomials for the $N=2$ states. Note that only operators with maximal L_z are given.

$\hat{\psi}_{00}^{(S)} = \frac{1}{2\sqrt{3}} \{[\vec{a}^\dagger(\rho)]^2 + [\vec{a}^\dagger(\lambda)]^2\}$	$[\underline{56}, 0^+]$
$\hat{\psi}_{00}^{(M_\rho)} = \frac{1}{\sqrt{3}} \vec{a}^\dagger(\rho) \cdot \vec{a}^\dagger(\lambda)$; $\hat{\psi}_{00}^{(M_\lambda)} = \frac{1}{2\sqrt{3}} \{[\vec{a}^\dagger(\rho)]^2 - [\vec{a}^\dagger(\lambda)]^2\}$	$[\underline{70}, 0^+]$
$\hat{\psi}_{11}^{(A)} = \frac{-1}{\sqrt{2}} [a_+^\dagger(\rho) a_0^\dagger(\lambda) - a_+^\dagger(\lambda) a_0^\dagger(\rho)]$	$[\underline{20}, 1^+]$
$\hat{\psi}_{22}^{(S)} = \frac{1}{2} \{[a_+^\dagger(\rho)]^2 + [a_+^\dagger(\lambda)]^2\}$	$[\underline{56}, 2^+]$
$\hat{\psi}_{22}^{(M_\rho)} = a_+^\dagger(\rho) a_+^\dagger(\lambda)$; $\hat{\psi}_{22}^{(M_\lambda)} = \frac{1}{2} \{[a_+^\dagger(\rho)]^2 - [a_+^\dagger(\lambda)]^2\}$	$[\underline{70}, 2^+]$

$$\Delta E_{[\underline{56},0^+]} = \tfrac{1}{4}\{{}_\rho\langle 0 | \{[\vec{a}^\dagger(\rho)]^2\}^\dagger U(\sqrt{2}\rho)[\vec{a}^\dagger(\rho)]^2 | 0\rangle_\rho {}_\lambda\langle 0 | 0\rangle_\lambda + {}_\rho\langle 0 | \{[\vec{a}^\dagger(\rho)]^2\}^\dagger U(\sqrt{2}\rho) | 0\rangle_\rho {}_\lambda\langle 0 | [\vec{a}^\dagger(\lambda)]^2 | 0\rangle_\lambda$$
$$+ {}_\rho\langle 0 | U(\sqrt{2}\rho)[\vec{a}^\dagger(\rho)]^2 | 0\rangle_\rho {}_\lambda\langle 0 | \{[\vec{a}^\dagger(\lambda)]^2\}^\dagger | 0\rangle_\lambda + {}_\rho\langle 0 | U(\sqrt{2}\rho) | 0\rangle_\rho {}_\lambda\langle 0 | \{[\vec{a}^\dagger(\lambda)]^2\}^\dagger [\vec{a}^\dagger(\lambda)]^2 | 0\rangle_\lambda\} . \tag{3.2}$$

Clearly,

$${}_\lambda\langle 0 | [\vec{a}^\dagger(\lambda)]^2 | 0\rangle_\lambda = {}_\lambda\langle 0 | \{[\vec{a}^\dagger(\lambda)]^2\}^\dagger | 0\rangle_\lambda = 0$$

and

$${}_\lambda\langle 0 | 0\rangle_\lambda = 1 ,$$

while by repeated use of the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$ we readily find that

$${}_\lambda\langle 0 | \{[\vec{a}^\dagger(\lambda)]^2\}^\dagger [\vec{a}^\dagger(\lambda)]^2 | 0\rangle_\lambda = 6 ,$$

so that

$$\Delta E_{[\underline{56},0^+]} = \tfrac{3}{2}{}_\rho\langle 0 | U(\sqrt{2}\rho) | 0\rangle_\rho + \tfrac{1}{4}{}_\rho\langle 0 | \{[\vec{a}^\dagger(\rho)]^2\}^\dagger U(\sqrt{2}\rho)[\vec{a}^\dagger(\rho)]^2 | 0\rangle_\rho . \tag{3.3}$$

The right-hand side can be expressed in terms of the Gaussian moments of the potential as noted by Gromes and Stamatescu[18] and by Isgur and Karl.[6,7] Thus, trivially,

$${}_\rho\langle 0 | U(\sqrt{2}\rho) | 0\rangle_\rho = \frac{\alpha^3}{\pi^{3/2}} \int d^3\rho\, U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} , \tag{3.4}$$

where as usual $\alpha = (3Km)^{1/4}$, while, with just a little more work,

$$\tfrac{1}{4}{}_\rho\langle 0 | \{[\vec{a}^\dagger(\rho)]^2\}^\dagger U(\sqrt{2}\rho)[\vec{a}^\dagger(\rho)]^2 | 0\rangle_\rho = \frac{\alpha^7}{\pi^{3/2}} \int d^3\rho\, \rho^4 U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} - \frac{3\alpha^5}{\pi^{3/2}} \int d^3\rho\, \rho^2 U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} + \frac{9\alpha^3}{4\pi^{3/2}} \int d^3\rho\, U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} . \tag{3.5}$$

Isgur and Karl[6,7] define parameters a, b, c as follows:

$$a = \frac{3\alpha^3}{\pi^{3/2}} \int d^3\rho\, U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} , \tag{3.6}$$

$$b = \frac{3\alpha^5}{\pi^{3/2}} \int d^3\rho\, \rho^2 U(\sqrt{2}\rho) e^{-\alpha^2\rho^2} , \tag{3.7}$$

$$c=\frac{3\alpha^7}{\pi^{3/2}}\int d^3\rho\,\rho^4 U(\sqrt{2}\rho)e^{-\alpha^2\rho^2}\ , \tag{3.8}$$

yielding the result

$$\Delta E_{[\underline{56},0^+]}=\tfrac{5}{4}a-b+\tfrac{1}{3}c\ . \tag{3.9}$$

Thus we may write

$$E_{[\underline{56},0^+]}=E_0+2\Omega-\Delta\ , \tag{3.10}$$

where

$$E_0=3m+3\omega+a\ , \tag{3.11}$$

$$\Omega=\omega-\tfrac{1}{2}a+\tfrac{1}{3}b\ , \tag{3.12}$$

$$\Delta=-\tfrac{5}{4}a+\tfrac{5}{3}b-\tfrac{1}{3}c\ . \tag{3.13}$$

The remainder of the $N=2$ results are readily obtained:

$$E_{[\underline{70},0^+]}=E_0+2\Omega-\tfrac{1}{2}\Delta\ , \tag{3.14}$$

$$E_{[\underline{56},2^+]}=E_0+2\Omega-\tfrac{2}{5}\Delta\ , \tag{3.15}$$

$$E_{[\underline{70},2^+]}=E_0+2\Omega-\tfrac{1}{5}\Delta\ , \tag{3.16}$$

$$E_{[\underline{20},1^+]}=E_0+2\Omega\ , \tag{3.17}$$

giving the pattern shown in Fig. 1. We shall show in Sec. IV that this simple splitting pattern has a group-theoretic explanation.

We now consider the corresponding calculation for the eight $N=3$ multiplets. The orthonormalized monomials are given in Table VI. Note that for the two degenerate $[\underline{70},1^-]$ multiplets we have taken the particular *orthogonal* combinations of the monomials that are given by (2.32) and (2.33). As has been noted earlier,[5,16,17] three of the perturbed $N=3$ multiplets depend only on the parameters of the $N\leq 2$ levels. Thus

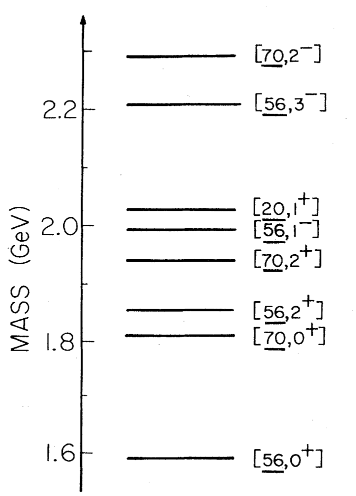


FIG. 1. Splitting pattern caused by the anharmonic perturbation U for the $N=2$ multiplets and some of the $N=3$ multiplets using the parameters of Ref. 7.

$$E_{[\underline{56},1^-]}=E_0+3\Omega-\tfrac{11}{10}\Delta\ , \tag{3.18}$$

$$E_{[\underline{56},3^-]}=E_0+3\Omega-\tfrac{3}{5}\Delta\ , \tag{3.19}$$

$$E_{[\underline{70},2^-]}=E_0+3\Omega-\tfrac{2}{5}\Delta\ . \tag{3.20}$$

The remaining five $N=3$ multiplets depend on a new parameter d, where

$$d=\frac{3\alpha^9}{\pi^{3/2}}\int d^3\rho\,\rho^6 U(\sqrt{2}\rho)e^{-\alpha^2\rho^2}\ . \tag{3.21}$$

If we define the quantity

$$\delta=\tfrac{1}{2}b-\tfrac{2}{5}c+\tfrac{2}{35}d\ , \tag{3.22}$$

the remaining perturbed energy levels at $N=3$ may be written

$$E_{[\underline{70},3^-]}=E_0+3\Omega-\tfrac{7}{10}\Delta+\tfrac{1}{2}\delta\ , \tag{3.23}$$

$$E_{[\underline{20},3^-]}=E_0+3\Omega-\tfrac{2}{5}\Delta+\tfrac{1}{3}\delta\ , \tag{3.24}$$

$$E_{[\underline{20},1^-]}=E_0+3\Omega-\tfrac{2}{5}\Delta+\tfrac{7}{6}\delta\ , \tag{3.25}$$

$$E_{[\underline{70},1^-]}=E_0+3\Omega-\tfrac{29}{40}\Delta+\tfrac{7}{8}\delta \pm\{[\tfrac{7}{8}(\delta-\tfrac{1}{15}\Delta)]^2+\tfrac{1}{45}\Delta^2\}^{1/2} \tag{3.26}$$

with the two (previously degenerate) $[\underline{70},1^-]$ multiplets mixed and split by the perturbation.

After inclusion of the hyperfine interactions, calculated perturbatively to lowest order in H_{hyp}, Isgur and Karl[7] were able to obtain a reasonable phenomenological description of the $N=0$, 1, and 2 levels with $E_0\simeq 1150$ MeV and $\Delta\simeq\Omega\simeq 440$ MeV, in the notation of Ref. 7 (rather than Ref. 6). Using these values, we noted in I that the mean mass of the nonstrange sector of the $[\underline{56},1^-]$, given here by Eq. (3.18), is around 1985 MeV close to the mass of the $\Delta D35$ at 1930 ± 20 MeV. It remains to be seen what effect inclusion of the hyperfine interactions for this multiplet has on the prediction, and this is under investigation, but given the simplicity of the model, the result is remarkably good. Note also that, of the three levels given by Eqs. (3.18)–(3.20), the $[\underline{56},1^-]$ multiplet necessarily lies lowest, as indicated in Fig. 1.

IV. ANHARMONIC SYMMETRY BREAKING

The introduction of anharmonic two-body potentials into the harmonic-oscillator model of the baryons represents a breaking of the symmetry in the U(6) degeneracy group sector. The aim in this section is to amplify this assertion. In particular, we shall show how the first-order mass splittings (3.10)–(3.17) for the $N=2$ level, derived in Sec. III by explicit state-function and operator techniques, can be understood both qualitatively and quantitatively as a mass formula of the Gell-Mann–Okubo type. We also show how these techniques can be usefully applied at the $N=3$ level.

TABLE VI. The correct orthonormalized monomials for the $N=3$ states.

$\hat{\psi}_{33}^{(S)}=\frac{-1}{2\sqrt{6}}\{[a_+^\dagger(\lambda)]^3-3[a_+^\dagger(\rho)]^2a_+^\dagger(\lambda)\}$	$[\underline{56},3^-]$
$\hat{\psi}_{33}^{(A)}=\frac{1}{2\sqrt{6}}\{[a_+^\dagger(\rho)]^3-3[a_+^\dagger(\lambda)]^2a_+^\dagger(\rho)\}$	$[\underline{20},3^-]$
$\hat{\psi}_{33}^{(M_\rho)}=\frac{1}{2\sqrt{2}}\{[a_+^\dagger(\rho)]^3+[a_+^\dagger(\lambda)]^2a_+^\dagger(\rho)\}$ $\hat{\psi}_{33}^{(M_\lambda)}=\frac{1}{2\sqrt{2}}\{[a_+^\dagger(\lambda)]^3+[a_+^\dagger(\rho)]^2a_+^\dagger(\lambda)\}$	$[\underline{70},3^-]$
$\hat{\psi}_{22}^{(M_\rho)}=\frac{-1}{\sqrt{3}}\{a_+^\dagger(\lambda)a_0^\dagger(\rho)-a_+^\dagger(\rho)a_0^\dagger(\lambda)\}a_+^\dagger(\lambda)$ $\hat{\psi}_{22}^{(M_\lambda)}=\frac{1}{\sqrt{3}}\{a_+^\dagger(\lambda)a_0^\dagger(\rho)-a_+^\dagger(\rho)a_0^\dagger(\lambda)\}a_+^\dagger(\rho)$	$[\underline{70},2^-]$
$\hat{\psi}_{11}^{(S)}=\frac{1}{2\sqrt{10}}\{([\vec{a}^\dagger(\rho)]^2-[\vec{a}^\dagger(\lambda)]^2)a_+^\dagger(\lambda)+2[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\rho)\}$	$[\underline{56},1^-]$
$\hat{\psi}_{11}^{(A)}=\frac{1}{2\sqrt{10}}\{([\vec{a}^\dagger(\rho)]^2-[\vec{a}^\dagger(\lambda)]^2)a_+^\dagger(\rho)-2[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\lambda)\}$	$[\underline{20},1^-]$
$\hat{\psi}_{11}^{(M_\rho)}=\frac{1}{4\sqrt{5}}\{[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\rho)-3\vec{a}^\dagger(\lambda)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\rho)+4[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\lambda)\}$ $\hat{\psi}_{11}^{(M_\lambda)}=\frac{1}{4\sqrt{5}}\{[\vec{a}^\dagger(\lambda)\cdot\vec{a}^\dagger(\lambda)-3\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\rho)]a_+^\dagger(\lambda)+4[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\rho)\}$	$[\underline{70},1^-]$
$\hat{\psi}_{11}^{(M_\rho)}=\frac{1}{4}\{[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\rho)+\vec{a}^\dagger(\lambda)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\rho)\}$ $\hat{\psi}_{11}^{(M_\lambda)}=\frac{1}{4}\{[\vec{a}^\dagger(\rho)\cdot\vec{a}^\dagger(\rho)+\vec{a}^\dagger(\lambda)\cdot\vec{a}^\dagger(\lambda)]a_+^\dagger(\lambda)\}$	$[\underline{70},1^-]$

To bring about this understanding it is necessary to consider not just the degeneracy group U(6) but the spectrum-generating group Sp(12, R). This follows from the fact that the anharmonic perturbation is a function of all twelve components of the vector (a_A) through the dependence of the potential upon $\vec{\rho}$ and $\vec{\lambda}$ and hence upon $\vec{a}(\rho)$, $\vec{a}^\dagger(\rho)$, $\vec{a}(\lambda)$, and $\vec{a}^\dagger(\lambda)$. It is assumed that this potential may be cast in the form

$$V(\vec{\rho},\vec{\lambda})=U(\sqrt{2}\vec{\rho})+U\left(-\frac{1}{\sqrt{2}}\vec{\rho}+(\tfrac{3}{2})^{1/2}\vec{\lambda}\right)+U\left(-\frac{1}{\sqrt{2}}\vec{\rho}-(\tfrac{3}{2})^{1/2}\vec{\lambda}\right), \qquad (4.1)$$

where

$$U(\sqrt{2}\vec{\rho})=\sum_n \beta_{2n}(\vec{\rho}\cdot\vec{\rho})^n \qquad (4.2)$$

and β_{2n} are arbitrary coefficients independent of $\vec{\rho}$.

The justification for this form, quite apart from the requirement that it be totally symmetric and composed of two-body contributions, is twofold. Firstly, a very large class of potentials may be expected to have an expansion of the form (4.2), which is consistent with a perturbation scheme based on the dominant harmonic term having $n=1$. Secondly, without involving nonlinear realizations, the Sp(12, R) algebra (2.7) of the operators (2.6) is associated with a Fock space in which only multinomials, bilinear in a_A, have a well-defined action.

It follows that

$$V(\vec{\rho},\vec{\lambda})=\sum_n \beta_{2n}V^{(2n)}, \qquad (4.3)$$

where at each order n, the perturbation $V^{(2n)}$ is realized as a homogeneous polynomial in $\vec{\rho}$ and $\vec{\lambda}$, and hence in a_A, of degree $2n$. The strengths of each order are governed by the values of the distinct coupling constants β_{2n}. The symmetry of (4.1) and the form of (4.2) ensure, furthermore, that each term $V^{(2n)}$ transforms as a component of the totally symmetric tensor representation $\langle 2n\rangle$ of Sp(12, R).

It is necessary, in order to achieve a quantitative understanding of the level splitting, to determine the transformation properties of the various terms $V^{(2n)}$, with respect to the subgroups of Sp(12, R) discussed in the previous sections. By construction, $V^{(2n)}$ is both an O(3) and an S_3 singlet, transforming as $[0]=\underline{1}$ and $(3)=\underline{S}$, respectively. With regard to its $O(\underline{2})$ properties, it is convenient to make use once more of the $(\vec{\xi},\vec{\eta})$ basis (2.20). In terms of these vectors,

$$V^{(2n)}=[(\vec{\rho}\cdot\vec{\rho})]^n+\left[\left(-\tfrac{1}{2}\vec{\rho}+\frac{\sqrt{3}}{2}\vec{\lambda}\right)\cdot\left(-\tfrac{1}{2}\vec{\rho}+\frac{\sqrt{3}}{2}\vec{\lambda}\right)\right]^n+\left[\left(-\tfrac{1}{2}\vec{\rho}-\frac{\sqrt{3}}{2}\vec{\lambda}\right)\cdot\left(-\tfrac{1}{2}\vec{\rho}-\frac{\sqrt{3}}{2}\vec{\lambda}\right)\right]^n$$
$$=\frac{1}{4^n}\{[(\vec{\xi}+\vec{\eta})\cdot(\vec{\xi}+\vec{\eta})]^n+[(\omega^{-1}\vec{\xi}+\omega\vec{\eta})\cdot(\omega^{-1}\vec{\xi}+\omega\vec{\eta})]^n+[(\omega\vec{\xi}+\omega^{-1}\vec{\eta})\cdot(\omega\vec{\xi}+\omega^{-1}\vec{\eta})]^n\}$$
$$=\frac{1}{4^n}[(\vec{\xi}\cdot\vec{\xi}+2\vec{\xi}\cdot\vec{\eta}+\vec{\eta}\cdot\vec{\eta})^n+(\omega\vec{\xi}\cdot\vec{\xi}+2\vec{\xi}\cdot\vec{\eta}+\omega^{-1}\vec{\eta}\cdot\vec{\eta})^n+(\omega^{-1}\vec{\xi}\cdot\vec{\xi}+2\vec{\xi}\cdot\vec{\eta}+\omega\vec{\eta}\cdot\vec{\eta})^n], \quad (4.4)$$

where $\omega=e^{-i2\pi/3}$. Hence,

$$V^{(2)}=\tfrac{3}{2}\vec{\xi}\cdot\vec{\eta}, \quad (4.5)$$

$$V^{(4)}=\tfrac{3}{8}[2(\vec{\xi}\cdot\vec{\eta})^2+(\vec{\xi}\cdot\vec{\xi})(\vec{\eta}\cdot\vec{\eta})], \quad (4.6)$$

$$V^{(6)}=\tfrac{3}{64}[8(\vec{\xi}\cdot\vec{\eta})^3+12(\vec{\xi}\cdot\vec{\eta})(\vec{\xi}\cdot\vec{\xi})(\vec{\eta}\cdot\vec{\eta})+(\vec{\xi}\cdot\vec{\xi})^3+(\vec{\eta}\cdot\vec{\eta})^3], \quad (4.7)$$

etc.

The first of these terms is just the familiar harmonic term

$$V^{(2)}=\tfrac{3}{2}(\vec{\rho}\cdot\vec{\rho}+\vec{\lambda}\cdot\vec{\lambda}), \quad (4.8)$$

which is both an O(2) and an O(6) singlet. However $V^{(4)}$ is not an O(6) singlet, although it is a linear combination of O(2) singlets. It is in fact a linear combination of terms transforming as O(3)×O(2)-singlet members of the O(6) representations $[4]=\underline{105}$ and $[0]=\underline{1}$. The term $V^{(6)}$ is not even an O(2) singlet, involving as it does a term transforming as the S_3-singlet S state of the O(2) representation $[6]=\underline{2}_6$.

In order to identify the Sp(2, R) content of $V^{(2n)}$ it is merely necessary to expand $\vec{\xi}$ and $\vec{\eta}$, in terms of the annihilation and creation operators $a_A=a_{i_\alpha}$, distinguished by $\alpha=1$ and 2, respectively. Since $\vec{\rho}=(1/\sqrt{2}\alpha)[\vec{a}(\rho)+\vec{a}^\dagger(\rho)]$ and $\vec{\lambda}=(1/\sqrt{2}\alpha)[\vec{a}(\lambda)+\vec{a}^\dagger(\lambda)]$ the expansion of $V^{(2n)}$ yields a monomial of degree $2n$ which is totally symmetric under permutations of the indices α. It follows that $V^{(2n)}$ transforms under Sp(2, R) as a sum of components of the symmetric representation $\langle 2n\rangle$. In the terminology appropriate to the locally isomorphic group SO(2, 1) this corresponds to the statement that $V^{(2n)}$ has pseudospin $[n]=\underline{2n+1}$.

It should be noted that not every component of $V^{(2n)}$ contributes to the values of the energy levels. Quite apart from the requirement used in constructing $V^{(2n)}$ that it be an O(3)×S_3 singlet, the only effective component must also be a U(1) singlet. This ensures that the third component of the pseudospin of the operator is zero. Any other value merely gives a vanishing contribution to the matrix elements $\langle N|V^{(2n)}|N\rangle$. In terms of the monomial constituting $V^{(2n)}$ this condition corresponds to the fact that the effective component, besides being symmetric under the interchange of creation and annihilation operators, is of the same degree in these operators taken separately. In the case $n=1$, for example, this implies that the effective part of $V^{(2)}$, given in (4.8), is simply proportional to the Hamiltonian (2.2).

The fact that the operators $V^{(2n)}$ are not, for each value of n, associated with a single irreducible representation of O(6) makes it convenient to consider other subgroups of Sp(12, R). These include the group Sp(6, R) with generators:

$$S_{PQ}=S_{i\alpha,j\beta}=S_{PaQb}\,\delta^{ab}=S_{ia\alpha,jb\beta}\,\delta^{ab} \quad (4.9)$$

with $P, Q=1, 2, \ldots, 6$. This appears in the labeling chain:

$$\mathrm{Sp}(12,R)\supset\mathrm{Sp}(6,R)\times\mathrm{O}(2)\supset\mathrm{Sp}(2,R)\times\mathrm{O}(3)\times\mathrm{O}(2)\supset U(1)\times\mathrm{SO}(3)\times S_3 \quad (4.10)$$

which is an alternative to the chain (2.18) of Sec. II. That (4.10) is useful in dealing with $V^{(2n)}$ is a consequence of this operator transforming as an S_3 symmetric, O(3) singlet, pseudospin $\underline{2n+1}$, component of the Sp(12, R) representation $\langle 2n\rangle$. The branching rules of Appendix A then ensure that this component necessarily belongs to the irreducible representation $\langle 2n\rangle$ of Sp(6, R).

Having established the transformation properties of the terms in the perturbation expansion (4.3) of the potential, it is necessary to discuss their role in determining the breaking of the degeneracy at each level N. This involves calculating

$$\Delta E^{(2n)}=\langle N|V^{(2n)}|N\rangle$$

which, thanks to the Wigner-Eckart theorem, factorizes into the product of a reduced matrix element, signified by $\langle\,||2n||\,\rangle$, and an appropriate Clebsch-Gordan coefficient. There are many approaches to such calculations. Here we outline a method discussed in detail elsewhere[24] which leads to an algebraic formula of the Gell-Mann Okubo-type, together with a very simple method of checking the results.

At each level N the relevant operators are those coupling to the Sp(12, R) product:

$$\langle N\rangle\times\langle N\rangle=\langle 2N\rangle+\langle 2N-2\rangle+\langle 2N-4\rangle+\langle 2N-1,1\rangle+\langle 2N-3,1\rangle+\cdots. \quad (4.11)$$

The operators transforming as $\langle 0\rangle=\underline{1}$ and $\langle 2\rangle=\underline{78}$, corresponding to $n=0$ and $n=1$, produce no splitting, since the former gives an overall shift in energy $\Delta E^{(0)}=\langle\,||0||\rangle$, and the latter is just a harmonic term giving $\Delta E^{(2)}=C_1\{1\}\langle\,||2||\rangle$, where $C_1\{1\}$, the first-order Casimir of U(1), is equal to N. At the $N=0$ and 1 levels there are no further contributions as can be seen from (4.11). However, the $N=2$ level is split by the single operator $V^{(4)}$, transforming as $\langle 4\rangle=\underline{1365}$ of Sp(12, R), while the $N=3$ level is split by the operators $V^{(4)}$ *and* $V^{(6)}$, transforming as $\langle 4\rangle=\underline{1365}$ and $\langle 6\rangle=\underline{12\,376}$ of Sp(12, R).

Concentrating on the $n=2$, lowest-order anharmonic term, this is labeled with respect to the groups Sp(12, R), Sp(6, R), Sp(2, R), U(1), O(3), O(2), S_3 by

$$\langle 4\rangle,\quad \langle 4\rangle,\quad \langle 4\rangle,\quad [0],\quad \{0\},\quad [0],\quad (3)$$
$$=\underline{1365},\ \underline{126},\ \underline{5},\ \underline{1},\ \underline{1},\ \underline{1},\ \underline{S}. \tag{4.12}$$

It is a straightforward task to construct the corresponding tensor operator in the enveloping algebra of Sp(12, R). It is a symmetrized second-order product of generators. An arbitrary component of $\langle 4\rangle=\underline{1365}$ is simply

$$X_{ABCD}=\{S_{AB},S_{CD}\}+\{S_{AC},S_{BD}\}+\{S_{AD},S_{BC}\}. \tag{4.13}$$

The projection onto the $(\underline{126},\underline{5},\underline{1},\underline{1},\underline{1})$ state is described in Appendix B. The resulting operator can be expanded in terms of the set of quadratic Casimir operators of Sp(12, R) and its subgroups. The algebraic formula involves several different labeling chains including those containing O(6), U(6), and Sp(2, R) already mentioned. The embedding diagram required is given in Fig. 2.

Thirteen different subgroups are involved and Table VII defines these subgroups by specifying their generators explicitly. In addition, the expansion of $V^{(4)}$ as a component of X_{ABCD} involves at least one of the operators

$$\Sigma=\tfrac{1}{2}\delta^{ij}\delta^{kl}\delta^{ab}\delta^{cd}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\{S_{ia\alpha,kc\gamma},S_{jd\delta,lb\beta}\} \tag{4.14}$$

and

$$\Sigma'=\tfrac{1}{2}\delta^{ij}\delta^{kl}\delta^{ab}\delta^{cd}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\{S_{ia\alpha,kc\gamma},S_{lb\delta,jd\beta}\}. \tag{4.15}$$

These are invariants of the groups Sp(4, R)×O(3) and Sp(6, R)×O(2), respectively, in that they commute with the generators of these groups. However, they do not belong to the enveloping algebra of these groups and are thus not Casimir operators.

The notation, definitions, and eigenvalues of the quadratic Casimir operators and of Σ and Σ' are given, along with the method of computation of the eigenvalues, in Appendices A and B. Suffice to say that typically the quadratic Casimir operator of Sp(12, R) is given by

$$C_2=J^{DB}J^{CA}S_{AD}S_{BC}\,. \tag{4.16}$$

The eigenvalues $C_2\langle 12\rangle$ may be evaluated, for example, by making use of the finite-dimensional representations discussed in Sec. II and defined by (2.8) and its generalizations. This implies that

$$C_2:\ a_Pa_Q\ldots\to J^{DB}J^{CA}[S_{AD},[S_{BC},a_Pa_Q\ldots]]$$
$$=C_2\langle 12\rangle a_Pa_Q\ldots. \tag{4.17}$$

Since the results depend only on the commutation

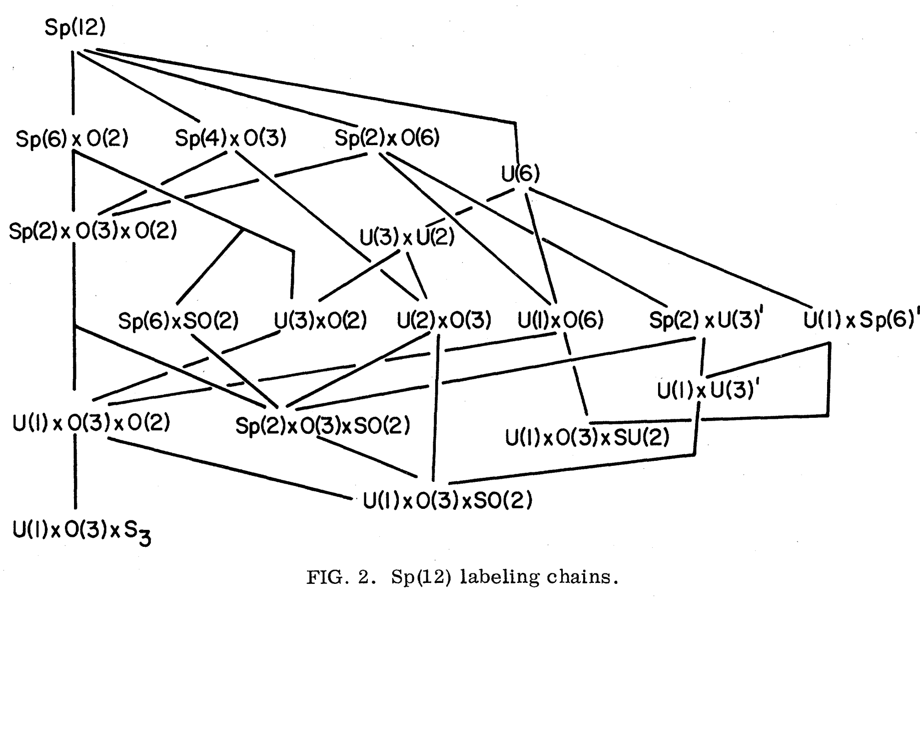


FIG. 2. Sp(12) labeling chains.

TABLE VII. Subgroup generators. Index notation as in text ($A\equiv ia\alpha$, $I\equiv ia$, $P\equiv i\alpha$, $U\equiv a\alpha$) and metric tensor $J_{IJ}=\delta_{ij}\epsilon_{ab}$.

Group	Generators
U(1)	$E_{IJ}\delta^{IJ}$
U(2)	$E_{iajb}\delta^{ij}$
U(3)	$E_{iajb}\delta^{ab}$
U(6)	$E_{IJ}=S_{I2J1}$
Sp(2)	$S_{I\alpha J\beta}\,\delta^{IJ}$
Sp(4)	$S_{UiVj}\delta^{ij}$
Sp(6)	$S_{PaQb}\delta^{ab}$
O(2)	$O_{iajb}\delta^{ij}$
O(3)	$O_{iajb}\delta^{ab}$
O(6)	$O_{IJ}=S_{I\alpha J\beta}\epsilon^{\alpha\beta}=E_{IJ}-E_{JI}$
U(3)′	$\frac{1}{2}(O_{i1j1}+O_{i2j2}+iO_{i1j2}-iO_{i2j1})$
Sp(6)′	$J_{IM}\,E_{MJ}+J_{JM}\,E_{MI}$

relations, they are identical with the finite-dimensional compact case yielding

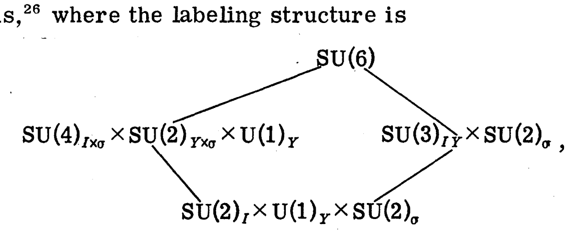
$$C_2\langle 12\rangle = 2\sum_r \lambda_r(\lambda_r+14-2r)$$

in the representation $\langle\bar{\lambda}\rangle=\langle\lambda_1, \lambda_2, \ldots\rangle$, as given elsewhere.[24,25]

In terms of these eigenvalues, the resulting mass formula is

$$\Delta E^{(4)}=\langle\,||4||\,\rangle(3C_2\{1\}+6C_2\{2\}+12C_2\{3\}-4C_2\{3'\} - \tfrac{1}{2}C_2\langle 2\rangle - C_2\langle 4\rangle - C_2\langle 6\rangle + 3C_2\langle 6'\rangle - C_2\langle 12\rangle - C_2[2]-3C_2[3]+\Sigma)\,. \quad (4.18)$$

Several aspects of this formula should be noted. Firstly, the use of overcomplete, noncommuting labeling chains is familiar from similar studies of symmetry breaking in nonrelativistic SU(6) models,[26] where the labeling structure is



where Wigner's $SU(4)_{I\times\sigma}$ and the familiar $SU(2)_{Y\times\sigma}$ are used to place different isospin and hypercharge submultiplets into a larger multiplet. The overcompleteness means here that for Sp(12, R), just as for SU(6), the formula is only useful for states which are associated with a unique irreducible representation of each subgroup. Thus, for example, at the $N=3$ level the formula fails for the $[\underline{70}, 1^-]$ states which may be diagonalized with respect to O(6), as in Table I, but not simultaneously with respect to Sp(6, R). This cannot be avoided and is a result of the proliferation of subgroup chains and labels, necessitated by the nonmaximal nature of the embedding of the physical symmetry group $U(1)\times SO(3)\times S_3$ in Sp(12, R).

The validity of (4.18) is easy to verify once it is realized that the expansion of $V^{(4)}$ in the form of components of (4.13) can only involve quadratic Casimir operators and Σ. This is done by expanding $V^{(4)}$ in terms of the complete set of 14 operators with arbitrary coefficients. These are then fixed by noting that $\langle N|\,V^{(4)}\,|N\rangle$ is necessarily zero for all the states bearing the Sp(12, R), Sp(6, R), Sp(2, R) labels $\langle 1\rangle, \langle 1\rangle, \langle 1\rangle$; $\langle 1^2\rangle, \langle 2\rangle, \langle 2\rangle$; $\langle 1^2\rangle, \langle 2\rangle, \langle 0\rangle$; $\langle 1^2\rangle, \langle 1^2\rangle, \langle 2\rangle$; $\langle 1^2\rangle, \langle 1^2\rangle, \langle 0\rangle$; $\langle 2\rangle, \langle 1^2\rangle, \langle 2\rangle$; $\langle 2\rangle, \langle 1^2\rangle, \langle 0\rangle$; and $\langle 2\rangle, \langle 2\rangle, \langle 0\rangle$. Even the last seven of these sets of labels provides enough information, through 18 conditions, to fix and check the coefficients.

Notice that the formula for $\Delta E^{(4)}$ has therefore been derived in two different ways. The first method involved the explicit construction of $V^{(4)}$ and its reexpression in terms of quadratic Casimir operators. The second approach determined the coefficients of the Casimir operators by taking matrix elements of $V^{(4)}$ between specific states of the $N=0$, 1, and 2 levels. The equivalence of these two approaches demonstrates explicitly that the reduced matrix element $\langle\,||\;||\,\rangle$ is indeed independent of the state label N. This is characteristic of a spectrum-generating algebra.

Returning to the application of (4.18) the physical eigenstates for the $N=2$ and $N=3$ levels are identified, along with all the appropriate subgroup labels and operator eigenvalues, in Tables VIII and IX. The splittings for $N=2$ are precisely those of Sec. III with

$$\langle\,||4||\,\rangle = -\tfrac{1}{80}\Delta = \tfrac{1}{16}(\tfrac{1}{4}a - \tfrac{1}{3}b + \tfrac{1}{15}c)\,, \quad (4.19)$$

where a, b, and c are the familiar Isgur and Karl parameters (3.6)–(3.8). The total effect of the anharmonic perturbation on the $N=2$ states is given by

$$\Delta E = \Delta E^{(0)} + \Delta E^{(2)} + \Delta E^{(4)}, \quad (4.20)$$

where

$$\Delta E^{(0)} = \langle\,||0||\,\rangle\,,$$
$$\Delta E^{(2)} = C_1\{1\}\langle\,||2||\,\rangle\,,$$

and $\Delta E^{(4)}$ is given by (4.18). The correspondence between the parameters of Isgur and Karl and the reduced matrix elements is

TABLE VIII. Subgroup representation labels and operator eigenvalues for $N=2$.

$[SU(6), L^P]$ Subgroup H	$[\underline{20}, 1^+]$ Representation	$C_2(H)$	$[\underline{70}, 2^+]$ Representation	$C_2(H)$	$[\underline{56}, 2^+]$ Representation	$C_2(H)$	$[\underline{70}, 0^+]$ Representation	$C_2(H)$	$[\underline{56}, 0^+]$ Representation	$C_2(H)$
U(1)	$\{2\}=\underline{1}_2$	4	$\{2\}=\underline{1}_2$	4	$\{2\}=\underline{1}_2$	4	$\{2\}=\underline{1}_2$	4	$\{2\}=\underline{1}_2$	4
U(2)	$\{1^2\}=\underline{\bar{1}}$	2	$\{2\}=\underline{3}$	6	$\{2\}=\underline{3}$	6	$\{2\}=\underline{3}$	6	$\{2\}=\underline{3}$	6
U(3)	$\{1^2\}=\underline{\bar{3}}$	4	$\{2\}=\underline{6}$	8	$\{2\}=\underline{6}$	8	$\{2\}=\underline{6}$	8	$\{2\}=\underline{6}$	8
U(3)′	$\{2,1\}=\underline{8}$	6	$\{2\}=\underline{6}$	8	$\{2,1\}=\underline{8}$	6	$\{2\}=\underline{6}$	8	$\{0\}=\underline{1}$	0
U(6)	$\{2\}=\underline{21}$	14	$\{2\}=\underline{21}$	14	$\{2\}=\underline{21}$	14	$\{2\}=\underline{21}$	14	$\{2\}=\underline{21}$	14
Sp(2)	$\langle 2\rangle=\underline{3}$	16	$\langle 2\rangle=\underline{3}$	16	$\langle 2\rangle=\underline{3}$	16	$\langle 2\rangle=\underline{3}$	16	$\langle 2\rangle=\underline{3}$	16
Sp(4)	$\langle 1^2\rangle=\underline{5}$	16	$\langle 2\rangle=\underline{10}$	24	$\langle 2\rangle=\underline{10}$	24	$\langle 2\rangle=\underline{10}$	24	$\langle 2\rangle=\underline{10}$	24
Sp(6)	$\langle 1^2\rangle=\underline{14}$	24	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32
Sp(6)′	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32	$\langle 2\rangle=\underline{21}$	32
Sp(12)	$\langle 2\rangle=\underline{78}$	56	$\langle 2\rangle=\underline{78}$	56	$\langle 2\rangle=\underline{78}$	56	$\langle 2\rangle=\underline{78}$	56	$\langle 2\rangle=\underline{78}$	56
O(2)	$[0]^*=\underline{1}^*$	0	$[2]=\underline{2}_2$	8	$[0]=\underline{1}$	0	$[2]=\underline{2}_2$	8	$[0]=\underline{1}$	0
O(3)	$[1]=\underline{3}$	4	$[2]=\underline{5}$	12	$[2]=\underline{5}$	12	$[0]=\underline{1}$	0	$[0]=\underline{1}$	0
O(6)	$[2]=\underline{20}$	24	$[2]=\underline{20}$	24	$[2]=\underline{20}$	24	$[2]=\underline{20}$	24	$[0]=\underline{1}$	0
Σ	−28		−28		−28		−40		−40	
$\Delta E^{(4)}$	0		16		32		40		80	
(in units of $\langle \Vert 4 \Vert \rangle$)										

TABLE IX. Subgroup representation labels and operator eigenvalues for $N=3$.

$[SU(6), L^P]$ Subgroup H	$[\underline{70}, 2^-]$ Representation	$C_2(H)$	$[\underline{20}, 3^-]$ $[\underline{56}, 3^-]$ Representation	$C_2(H)$	$[\underline{70}, 3^-]$ Representation	$C_2(H)$	$[\underline{20}, 1^-]$ $[\underline{56}, 1^-]$ Representation	$C_2(H)$	$[\underline{70}, 1^-]$ Representation	$C_2(H)$	$[\underline{70}, 1^-]$ Representation	$C_2(H)$
U(1)	$\{3\}=\underline{1}_3$	9	$\{3\}=\underline{1}_3$	9	$\{3\}=\underline{1}_3$	9	$\{3\}=\underline{1}_3$	9	$\{3\}=\underline{1}_3$	9	$\{3\}=\underline{1}_3$	9
U(2)	$\{2,1\}=\underline{2}'$	6	$\{3\}=\underline{4}$	12	$\{3\}=\underline{4}$	12	$\{3\}=\underline{4}$	12	$\{3\}=\underline{4}$	12	$\{3\}=\underline{4}$	12
									$\{2,1\}=\underline{2}'$	6	$\{2,1\}=\underline{2}'$	6
U(3)	$\{2,1\}=\underline{8}$	9	$\{3\}=\underline{10}$	15	$\{3\}=\underline{10}$	15	$\{3\}=\underline{10}$	15	$\{3\}=\underline{10}$	15	$\{2,1\}=\underline{8}$	9
U(3)′	$\{3,1\}=\underline{15}$	11	$\{3\}=\underline{10}$	15	$\{3,1\}=\underline{15}$	11	$\{3\}=\underline{10}$	15	$\{3,1\}=\underline{15}$	11	$\{3,1\}=\underline{15}$	11
									$\{1\}=\underline{3}$	3	$\{1\}=\underline{3}$	3
U(6)	$\{3\}=\underline{56}$	24	$\{3\}=\underline{56}$	24	$\{3\}=\underline{56}$	24	$\{3\}=\underline{56}$	24	$\{3\}=\underline{56}$	24	$\{3\}=\underline{56}$	24
Sp(2)	$\langle 3\rangle=\underline{4}$	30	$\langle 3\rangle=\underline{4}$	30	$\langle 3\rangle=\underline{4}$	30	$\langle 3\rangle=\underline{4}$	30	$\langle 3\rangle=\underline{4}$	30	$\langle 3\rangle=\underline{4}$	30
Sp(4)	$\langle 2,1\rangle=\underline{16}$	30	$\langle 3\rangle=\underline{20}$	42	$\langle 3\rangle=\underline{20}$	42	$\langle 3\rangle=\underline{20}$	42	$\langle 3\rangle=\underline{20}$	42	$\langle 3\rangle=\underline{20}$	42
									$\langle 2,1\rangle=\underline{16}$	30	$\langle 2,1\rangle=\underline{16}$	30
Sp(6)	$\langle 2,1\rangle=\underline{64}$	42	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 2,1\rangle=\underline{64}$	42
Sp(6)′	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54	$\langle 3\rangle=\underline{56}$	54
Sp(12)	$\langle 3\rangle=\underline{364}$	90	$\langle 3\rangle=\underline{364}$	90	$\langle 3\rangle=\underline{364}$	90	$\langle 3\rangle=\underline{364}$	90	$\langle 3\rangle=\underline{364}$	90	$\langle 3\rangle=\underline{364}$	90
O(2)	$[1]=\underline{2}_1$	2	$[3]=\underline{2}_3$	18	$[1]=\underline{2}_1$	2	$[3]=\underline{2}_3$	18	$[1]=\underline{2}_1$	2	$[1]=\underline{2}_1$	2
O(3)	$[2]=\underline{5}$	12	$[3]=\underline{7}$	24	$[3]=\underline{7}$	24	$[1]=\underline{3}$	4	$[1]=\underline{3}$	4	$[1]=\underline{3}$	4
O(6)	$[3]=\underline{50}$	42	$[3]=\underline{50}$	42	$[3]=\underline{50}$	42	$[3]=\underline{50}$	42	$[3]=\underline{50}$	42	$[3]=\underline{50}$	42
									$[1]=\underline{6}$	10	$[1]=\underline{6}$	10
Σ	−42		−42		−42		−62		−62		−62	
$\Delta E^{(4)}$	32		48		80		88					
(in units of $\langle \Vert 4 \Vert \rangle$)												

$$\langle \| 0 \| \rangle \equiv a \equiv E_0 - (3m + 3\omega) ,$$

$$\langle \| 2 \| \rangle \equiv -\tfrac{1}{2}a + \tfrac{1}{3}b \equiv \Omega - \omega ,$$

and $\langle \| 4 \| \rangle$ is given by (4.19).

As we established earlier, in the $N=3$ case, there are two anharmonic reduced matrix elements $\langle \| 4 \| \rangle$ and $\langle \| 6 \| \rangle$. In fact, as was demonstrated explicitly in Sec. III, for some of these states the Clebsch-Gordan coefficient multiplying $\langle \| 6 \| \rangle$ vanishes, so that the level splittings are again given by (4.18) in terms of $\langle \| 4 \| \rangle$ alone. One of the zeros has the same origin as that appropriate to the $N=2$ state $[\underline{20}, 1^+]$ for which $\Delta E^{(4)} = 0$ by virtue, as explained above, of this state carrying the Sp(6, R) label $\langle 1^2 \rangle = \underline{14}$. The same argument implies that $\Delta E^{(6)} = 0$ for the $N=3$ state $[\underline{70}, 2^-]$, which carries the Sp(6, R) label $\langle 2, 1\rangle = \underline{64}$ and therefore decouples from $V^{(6)}$ which transforms as a component of the Sp(6, R) representation $\langle 6 \rangle = \underline{462}$.

Other zeros owe their origin to the Sp(6, $R)_\rho$ subgroup of Sp(12, R) not previously used in this or the preceding sections. However, the fact that, for the physical states, $\langle \phi | V(\vec{\rho}, \vec{\lambda}) | \phi \rangle = 3\langle \phi | U(\sqrt{2}\rho) \phi \rangle$ implies that matrix elements may be calculated merely by looking at the expectation values of $(\vec{\rho} \cdot \vec{\rho})^n$. In the case $n=3$ it is clear that the Sp(6, $R)_\rho$ representation associated with $V^{(6)}$ is $\langle 6 \rangle = \underline{462}$ and that at the $N=3$ level, only those states with maximal Sp(6, $R)_\rho$ assignment $\langle 3 \rangle = \underline{56}$, may couple to $V^{(6)}$. For the states constructed in the previous section in terms of $\vec{a}^\dagger(\zeta)$ and $\vec{a}^\dagger(\eta)$ it is only necessary to examine the leading power in $\vec{a}^\dagger(\rho)$. Any factor $[a^\dagger(\zeta) a^\dagger(\eta)]_+$ leads to the total Sp(6, R) and Sp(6, $R)_\rho$ being nonmaximal: the state $[\underline{70}, 2^-]$ is of this type. Of the remaining monomials, $\mathcal{R}(\zeta^{1/2(N+m)}\eta^{1/2(N-m)})$ has leading power ρ^N and is therefore associated with maximal Sp(6, R), whereas $\mathcal{I}(\zeta^{1/2(N+m)}\eta^{1/2(N-m)})$ has leading power ρ^{N-1} and is therefore nonmaximal. States of this latter form at the $N=3$ level are the $[\underline{56}, 1^-]$ and $[\underline{56}, 3^-]$ states. Thus the $[\underline{70}, 2^-]$, $[\underline{56}, 1^-]$, and $[\underline{56}, 3^-]$ states decouple from $V^{(6)}$. The level splittings produced by $V^{(4)}$ can be calculated from (4.18)–(4.20) and the results are indicated in Tables VIII and IX. They agree with the explicit state-function and operator calculations of Sec. III.

V. CONCLUSIONS

Dalitz, Horgan, and Reinders[17] have looked in detail at the question of the assignment of the $\Delta D35(1930)$ to the $N=3$ $[\underline{56}, 1^-]$. Instead of just looking at mean masses of multiplets in the harmonic-oscillator quark model, they attempted to do better than that, and obtained a sum rule relating the mass of the $\Delta D35$ of the $[\underline{56}, 1^-]$ to masses of known Δ states, which they assigned to the $N=2$ and $N=0$ $\underline{56}$ multiplets. Specifically, they give the result

$$M(\Delta D35) = \tfrac{3}{5} M(\Delta F37) + \tfrac{1}{15} M(\Delta P31) + \tfrac{5}{6} M(\Delta P33^*) - \tfrac{1}{2} M(\Delta P33) \qquad (5.1)$$

relating the masses of the $N=3$ $[\underline{56}, 1^-]$, the $N=2$ $[\underline{56}, 2^+]$ and $[\underline{56}, 0^+]$, and the $N=0$ $[\underline{56}, 0^+]$ multiplets. Identifying the $\Delta F37(1930)$ and $\Delta P31(1940)$ as belonging to the $N=2$ $[\underline{56}, 2^+]$ and the $\Delta P33(1690)$ as belonging to the $N=2$ $[\underline{56}, 0^+]$, they predict

$$M(\Delta D35) = 2088 \pm 25 \text{ MeV} , \qquad (5.2)$$

some 150 MeV higher than the candidate $\Delta D35(1930)$. The sum rule (5.1) is derived by a spin average over Δ's within the $N=2$ band $\underline{56}$'s, and the right-hand side is actually independent of the magnitude of the spin-orbit effects which they consider. In general, however, spin-orbit forces will be expected to mix the $N=3$ band $\Delta D35$'s of the $[\underline{56}, 1^-]$ and $[\underline{70}, 2^-]$: Dalitz, Horgan, and Reinders estimate that such mixing will be small. At first sight, therefore, it seems that this sum rule provides a better, and more specific, test of the assignment of the $\Delta D35$ to the $[\underline{56}, 1^-]$ than does our less ambitious procedure of estimating merely the nonstrange mean mass using the parameters of the Isgur-Karl model. However, the whole analysis of Ref. 17 is dependent on the neglect of spin-tensor forces. Such tensor forces can mix Δ states of the same J within the $N=2$ band, and also, of course, they can mix the $\Delta D35$'s of the $[\underline{56}, 1^-]$, $[\underline{70}, 2^-]$, $[\underline{56}, 3^-]$, and $[\underline{70}, 3^-]$ $N=3$ multiplets. Since the analysis of Isgur and Karl suggests that spin-tensor forces are indeed important in determining the masses and mixing of the individual states of SU(6) multiplets, the status of the above prediction (5.1) for the $\Delta D35$ is obscure. In fact, the detailed predictions of the Isgur-Karl model for the $N=2$ states[6] indicate that the $\Delta P31(1940)$, classified by Dalitz, Horgan, and Reinders as a pure $[\underline{56}, 2^+]$ state, is actually an almost complete mixture of $[\underline{56}, 2^+]$ and $[\underline{70}, 0^+]$ basis states.

In view of all these uncertainties, it seems better to retreat to an examination of the zeroth-order nonstrange mean masses of the $N=3$ multiplets for a first indication of whether or not an assignment of the $\Delta D35$ to the $N=3$ $[\underline{56}, 1^-]$ is at all plausible. In this respect, therefore, our analysis of Sec. III is more akin to the earlier analysis of Horgan,[16] who discussed such mean masses in the context of his (flavor) SU(6) mass fits. He predicted the central mass value of this multiplet to be around 2080 ± 50 MeV, about 100-MeV higher than our value for the nonstrange sector. Given the fundamental differences of approach of the Isgur-Karl Hamiltonian, which includes a flavor-SU(6)-independent anharmonic perturbation U together with spin-tensor interactions, and of Horgan, who introduces flavor-SU(6)-*dependent* anharmonic perturbations and does *not* include tensor forces, the two estimates are surprisingly close. In fact, the algebraic structure of our results for the $[\underline{56}, 1^-]$, $[\underline{56}, 3^-]$, and $[\underline{70}, 2^-]$ multiplets may be obtained from Table I of Ref. 16 by identifying the parameter a_4 with the (in principle, independent) parameter b_4.[27] Phenomenologically, Horgan found the values[16]

$$a_4 = 2000 \text{ MeV}$$

and

$$b_4 = 2100 \text{ MeV},$$

thus lending support to Isgur and Karl's (and our) less general treatment of anharmonic perturbations.

In contrast to these approaches, all based on the nonrelativistic harmonic-oscillator quark model, Cutkosky and Hendrick[11,14] investigated the status of the $[\underline{56}, 1^-]$ in a quark model based on the string picture of confinement. In the three-quark version of their model, they concluded that the mean mass of the lowest $[\underline{56}, 1^-]$ state was some 200 MeV too high for the $\Delta D35(1930)$ to be accommodated in this multiplet. They obtained qualitative agreement for the mean positions of other SU(6) multiplets, but no attempt was made to make detailed fits including hyperfine splitting. In terms of this model, therefore, the $\Delta D35(1930)$ appears to be a good candidate for a new type of baryon, in which some gluonic degrees of freedom are excited.

What conclusions can we come to? It is certainly true that the Isgur-Karl Hamiltonian has had more success than any other quark model in fitting the enormous amount of baryon data available, for both positive- and negative-parity states, to the $N=2$ and $N=1$ oscillator bands, respectively. In view of this, it seems entirely reasonable to take this model as the most reliable guide to the spectrum, and examine the model's predictions for the $N=3$ states. In the approximation of neglecting hyperfine interactions, taking Isgur and Karl's parameters determined from their fit of the $N=0$, 1, and 2 levels we predict a mean mass of the nonstrange sector of the $[\underline{56}, 1^-]$ only 55 MeV above the quoted mass, 1930 ± 20 MeV, for the $\Delta D35$.[15] While it is clear that hyperfine interactions will mix and shift the masses of the Δ's of the $N=3$ band, it seems to us impressive that such a constrained and simple model as that described here can get so close to the mass of the $\Delta D35$, with no "fine tuning" of the three parameters E_0, Ω, and Δ. We therefore conclude that the $\Delta D35(1930)$ does not constitute unambiguous evidence for some new degree of freedom inside baryons.

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APPENDIX A: BRANCHING RULES AND CASIMIR INVARIANTS

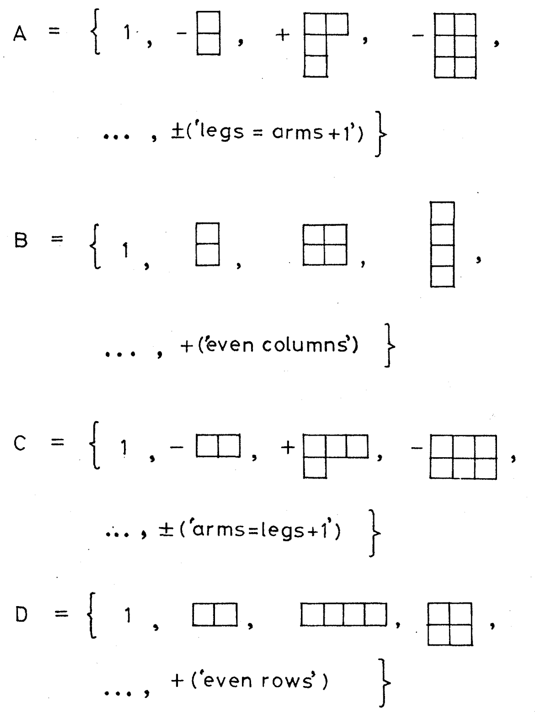
The treatment of tensor representations $\{\lambda\}$ of U(n) by Young-diagram techniques is familiar from many texts[1,19] and will not be repeated here. Irreducible tensor representations $[\lambda]$ and $\langle\lambda\rangle$ of O(n) and Sp(n), respectively, correspond to Young diagrams with $\leq [n/2]$ rows [where n is even for Sp(n)]. Here and in Appendix B we drop the distinction between the various real forms of a given complex algebra, since the results for finite-dimensional representations are the same. The tensors are traceless with respect to the appropriate (symmetric or antisymmetric) metric tensor. In the case of O(n), there are also associated pseudotensor representations $[\lambda]^*$: If the $[n/2]$th row length is nonzero, for n even, then $[\lambda]^*$ is equivalent to $[\lambda]$. The following branching rules for the symplectic and orthogonal groups have been derived[20]

$$\mathrm{Sp}(st) \supset \mathrm{Sp}(s) \times \mathrm{O}(t): \quad \langle\lambda\rangle = \sum_{\zeta} \langle\zeta/B\rangle \times [((\lambda/A) \circ \zeta)/D], \tag{A1}$$

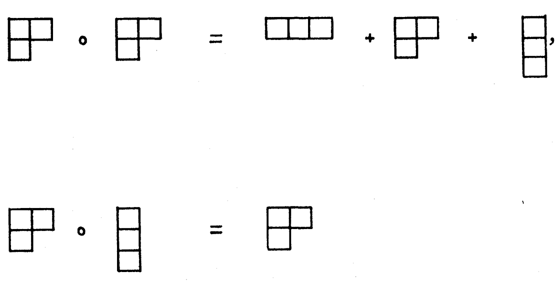
$$\mathrm{O}(st) \supset \mathrm{O}(s) \times \mathrm{O}(t): \quad [\lambda] = \sum_{\zeta} [\zeta/D] \times [((\lambda/C) \circ \zeta)/D], \tag{A2}$$

$$\mathrm{O}(st) \supset \mathrm{Sp}(s) \times \mathrm{Sp}(t): \quad [\lambda] = \sum_{\zeta} \langle\zeta/B\rangle \times \langle((\lambda/C) \circ \zeta)/B\rangle. \tag{A3}$$

Here, the notation $/A$, $/B$, $/C$, and $/D$ signifies division by all admissible elements of the following infinite collections (in Young-diagram notation):



For a given term μ in (λ/A or C), ζ runs over all diagrams with the same number of boxes as μ, and the product $\mu \circ \zeta$ is the Kronecker product of the appropriate permutation group representations whose evaluation corresponds to the generalization of the familiar rules



for the permutation group S_3. In these reductions, nonstandard Young diagrams may arise. For these diagrams, there are modification rules whereby a continuous "boundary hook" h (a line of boxes in contact on the surface of the diagram, starting with the first box in the last row) of a certain length is to be removed. If h, beginning in column 1, row p, and ending in column x, can be removed to reveal a standard Young diagram, then the contribution of the original diagram is taken to be

$$[\lambda]=(-1)^{x-1}[\lambda-h]^*, \quad \text{length } |h|=2p-n, \tag{A4}$$

$$\langle\lambda\rangle=(-1)^{x}\langle\lambda-h\rangle, \quad \text{length } |h|=2p-n-2\,. \tag{A5}$$

The eigenvalues of the quadratic Casimir invariants of an irreducible tensor representation $\{\lambda\}$, $[\lambda]$, or $\langle\lambda\rangle$, of U(n), O(n), or Sp(n), are easily given in terms of the Young-diagram row lengths λ_r. With the generators normalized as in Sec. II, the Casimir $C_2[\mathrm{O}(n)]=\delta^{IL}O_{IJ}\,\delta^{JK}O_{KL}$, etc., and the eigenvalues are[24,25]

$$\mathrm{U}(n):\quad C_2\{n\}=\sum_r \lambda_r(\lambda_r+n+1-2r), \tag{A6}$$

$$\mathrm{O}(n):\quad C_2[n]=2\sum_r \lambda_r(\lambda_r+n-2r), \tag{A7}$$

$$\mathrm{Sp}(n):\quad C_2\langle n\rangle=2\sum_r \lambda_r(\lambda_r+n+2-2r)\,. \tag{A8}$$

APPENDIX B: DERIVATION OF ALGEBRAIC MASS FORMULAS

As explained in Sec. IV, the derivation involves the explicit construction of the tensor operator corresponding to $V^{(4)}$ in terms of a bilinear expression in the generators. An arbitrary component of $\langle 4\rangle \sim \underline{1365}$ was given as

$$X_{ABCD} \propto \{S_{AB}, S_{CD}\}+\{S_{AC}, S_{BD}\}+\{S_{AD}, S_{BC}\}\,. \tag{B1}$$

In the Sp(6)×O(2) basis, representing $A=i a\alpha$ by Pa the $\underline{126}\times\underline{1}$ submultiplet is clearly

$$Y_{PQRS}\propto (X_{PaQbRcSd}+X_{PaRbQcSd}+X_{PaSbQcRd})\delta^{ab}\delta^{cd} \tag{B2}$$

and similarly, in the Sp(2)×O(3) basis of Sp(6), replacing P by $i\alpha$, the $\underline{5}\times\underline{1}$ submultiplet is

$$Z_{\alpha\beta\gamma\delta}\propto (Y_{i\alpha j\beta k\gamma l\delta}+Y_{i\alpha j\gamma k\beta l\delta}+Y_{i\alpha j\delta k\beta l\gamma})\delta^{ij}\delta^{kl}\,. \tag{B3}$$

The $H_3=0$ component, and thus the desired tensor operator, is

$$V^{(4)}\propto Z_{1122}\,. \tag{B4}$$

Using these definitions, we can now rewrite $V^{(4)}$ in terms of various combinations $\{S_{AB}, S_{CD}\}$, each of which is a U(1)×O(3)×O(2) invariant. There are fourteen independent ways of using the tensors δ^{ij}, δ^{ab}, and $\epsilon^{\alpha\beta}$, ensuring $\alpha=1,2$ equally often, to make such invariants. We shall not enumerate them, but in Table VII we identify explicitly the generators of the thirteen subgroups of the embedding diagram in terms of whose Casimir operators, together with one of the invariants Σ or Σ', the fourteen tensor invariants can be expressed. In this table the normalization is as in Appendix A, so the Casimir eigenvalues can be computed once the subgroup representations are known. The result of this algebra is the mass formula quoted in Sec. IV.

It is of some interest to consider in more detail the invariants Σ and Σ', given by (4.14) and (4.15): Σ is explicitly an $\mathrm{Sp}(4,R)\times \mathrm{O}(3)$ invariant, while Σ' is an $\mathrm{Sp}(6,R)\times \mathrm{O}(2)$ invariant, and simple manipulation of the definitions (the prototype of all the rearrangements necessary for the formula) shows that

$$\Sigma - \Sigma' = C_2[\mathrm{O}(6)] + C_2[\mathrm{O}(3)] - 4C_2\{\mathrm{U}(3)'\}. \quad \text{(B5)}$$

We shall treat Σ and Σ' together, by assuming that the symmetry is $\mathrm{Sp}(s)\times\mathrm{O}(t)$, and denote this generic form as Λ.

Like C_2, Λ is a bilinear operator $\{S, S'\}$ and must be evaluated on tensor operator states T by means of double commutators $[S,[S',T]] + [S',[S,T]]$. We define

$$\Lambda = \tfrac{1}{2} J^{\alpha\beta} J^{\gamma\delta} G^{IJ} G^{KL} \{S_{I\alpha K\gamma}, S_{J\delta L\beta}\} \quad \text{(B6)}$$

for $\alpha, \beta = 1, 2, \ldots, s$, and $I, J, \ldots = 1, 2, \ldots, t$ where G and J are the symmetric and antisymmetric metrics of $\mathrm{O}(t)$ and $\mathrm{Sp}(s)$, respectively. We find, for symmetric tensors $X_{I\alpha}, X_{I\alpha J\beta}, \ldots$,

$$\begin{aligned}
[\Lambda, X_{I\alpha}] &= -2(s+t)X_{I\alpha}, \\
[\Lambda, X_{I\alpha J\beta}] &= -4(s+t)X_{I\alpha J\beta} - 4G_{IJ}X_{L\alpha M\beta}G^{LM} \\
&\quad + 4J_{\alpha\beta}X_{I\gamma J\delta}J^{\gamma\delta}, \\
[\Lambda, X_{I\alpha J\beta K\gamma}] &= -6(s+t)X_{I\alpha J\beta K\gamma} \\
&\quad - 4G_{IJ}X_{L\alpha M\beta K\gamma}G^{LM} - 4G_{JK}X_{I\alpha L\beta M\gamma}G^{LM} \\
&\quad - 4G_{IK}X_{L\alpha J\beta M\gamma}G^{LM} + 4J_{\alpha\beta}X_{I\sigma J\tau K\gamma}J^{\sigma\tau} \\
&\quad + 4J_{\beta\gamma}X_{I\alpha J\sigma K\tau}J^{\sigma\tau} + 4J_{\alpha\gamma}X_{I\sigma J\beta K\tau}J^{\sigma\tau}
\end{aligned} \quad \text{(B7)}$$

and so on. If we consider only the stretched states [for example, in the O(6) case $(\vec{a}^{\dagger}(\zeta)\cdot\vec{a}^{\dagger}(\eta))^{a}(a_{+}^{\dagger}(\zeta))^{\max}$] neither the $J^{\sigma\tau}$ traces, nor the G^{LM} traces [other than those corresponding to $(\vec{a}^{\dagger}(\zeta)\cdot\vec{a}^{\dagger}(\eta))^{a}$ in the original tensor], can contribute, and we have explicitly

$$\begin{aligned}
[\Lambda, X_{I\alpha J\beta}] &= -4(s+t)X_{I\alpha J\beta}, \\
[\Lambda, X_{I\alpha J\beta K\gamma}] &= -6(s+t)X_{I\alpha J\beta K\gamma}, \\
[\Lambda, G^{IJ}X_{I\alpha J\beta K\gamma}] &= -(6s+10t+8)G^{IJ}X_{I\alpha J\beta K\gamma}.
\end{aligned} \quad \text{(B8)}$$

These eigenvalues are set out for $N=2$ and 3 in Tables VIII and IX.

*Present address: Department of Physics, University of Tasmania, Hobart Tasmania 7001, Australia.

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