# Renormalization of $g\phi^3$ theory in a six-dimensional conformally flat space-time

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(Received 7 April 1981)

Using dimensional regularization and 't Hooft's renormalization scheme, $\phi^3$ theory is renormalized at the one-loop level in a six-dimensional conformally flat space-time. The calculation is done by treating the scalar curvature as an interaction term. Using the renormalization group the effective conformal coupling $\xi(\kappa)$ is calculated and it is shown that in the UV limit $\xi(\kappa)\to 6/25$ as $\kappa\to\infty$, instead of the conformal value of 1/5.

There has recently been a great deal of interest in the renormalization of interacting field theories in curved space-time. Renormalization-group methods are extremely useful in studying renormalization and have recently been applied to $\phi^4$ theory in curved space-time.[1,2] In this paper I study $\phi^3$ theory in a six-dimensional conformally flat space-time. This theory is of interest because it is asymptotically free.[3,4]

The renormalization of massless $(\phi^3)_6$ theory in spherical space-time has been considered by Drummond.[5] However, Drummond and I use different renormalization schemes. Drummond takes $\xi_0=\xi(n)$, whereas I use the more general renormalization scheme of 't Hooft. As a result I, unlike Drummond, get nonconformally invariant counterterms. As discussed by Birrell and Davies[6] these two schemes are inequivalent and the correct one can only be determined by experiment.

I will restrict myself to a conformally flat $n$-dimensional space-time with metric

$$ds^2=\Omega^2(\eta)\left[d\eta^2-\sum_{i=1}^{n-1}(dx^i)^2\right]. \tag{1}$$

The scalar field $\phi$ has Lagrangian density

$$\mathcal{L}=\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\nu\phi\partial_\mu\phi-\tfrac{1}{2}(m_R{}^2+\xi_R R)\phi^2 -\tfrac{1}{2}(\delta m^2+\delta\xi R)\phi^2-\frac{1}{3!}g_B\phi^3\right] \tag{2}$$

with

$$Z(g,n)=1+\sum_{\nu=1}^{\infty}\frac{c_\nu(g)}{(n-6)^\nu} =1+\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{c_{\nu r}g^r}{(n-6)^\nu}, \tag{3}$$

$$m_0{}^2Z=m^2+\sum_{\nu=1}^{\infty}\frac{m^2b_\nu(g)}{(n-6)^\nu} =m^2+\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{b_{\nu r}g^r}{(n-6)^\nu}, \tag{4}$$

$$\xi_0Z=\xi+\sum_{\nu=1}^{\infty}\frac{d_\nu(\xi,g)}{(n-6)^\nu} =\xi+\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{d_{\nu r}(\xi)g^r}{(n-6)^\nu}, \tag{5}$$

and

$$g_0Z^{3/2}=\mu^{3-n/2}\left[g+g\sum_{\nu=1}^{\infty}\frac{a_\nu(g)}{(n-6)^\nu}\right] =\mu^{3-n/2}\left[g+g\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{a_{\nu r}g^r}{(n-6)^\nu}\right]. \tag{6}$$

Momentum-space representations for the Feynman propagator in a conformally flat space-time have been developed by Birrell[7,8] and by Bunch and Panangaden.[9] In this paper I will use Birrell's representation. Birrell defines a propagator $g_F(x,x')$ as

$$g_F(x,x')=[\Omega(x)]^{(n-2)/2}G_F(x,x')[\Omega(x')]^{(n-2)/2}, \tag{7}$$

where $G_F(x,x')$ is the Feynman propagator, and then shows that

$$g_F(q,q';k)=\sum_{i=1}^{3}g_F^{(i)}(q,q';k), \tag{8}$$

where

$$g_F^{(1)}(q,q';k)=\delta(q-q')\tilde{G}_F(q;k), \tag{9}$$

$$g_F^{(2)}(q,q';k)=-\tilde{G}_F(q;k)\tilde{V}(q,q')\tilde{G}_F(q';k), \tag{10}$$

and

$$g_F^{(3)}(q,q';k)=\tilde{G}_F(q;k)\tilde{G}_F(q';k)\times\int_{-\infty}^{\infty}\tilde{G}_F(p;k)\tilde{V}(q-p)t_k(p,q')dp. \tag{11}$$

Here $G_F(p_0;|\vec{p}|)=(p_0{}^2-|\vec{p}|^2-m_-{}^2+i\epsilon)^{-1}$, $t_k$ satisfies a Lippmann-Schwinger-type equation

$$t_k(p,p')=V(p,p')-\int_{-\infty}^{\infty}\tilde{G}_F(q;k)\tilde{V}(p,q)t_k(q,p')dq,$$

$\tilde{V}(p,q)$ is the Fourier transform of

$$V(\eta)\equiv[m_-{}^2-m^2\Omega^3(\eta)]-\left\{\xi-\left[\frac{n-2}{4(n-1)}\right]\right\}\Omega^3(\eta)R(\eta), \tag{12}$$

and

$$m_-=m\Omega^{3/2}(\eta=-\infty). \tag{13}$$

Although it is perfectly possible to calculate all the counterterms using Birrell's method, it is easier to treat the scalar curvature as an interaction term; this allows one to do the calculation using flat-space-time propagators.

Following MacFarlane and Woo,[4] I use massless propagators and mass insertions. The diagrams which contribute at the one-loop level are shown in Fig. 1. The new diagrams are 1(e) and its counterterm 1(f). The triangle in 1(e) represents an insertion of $V(q_0 - q_0')$ and the cross in diagram 1(f) represents the scalar curvature counterterm.

The contribution from diagram 1(c) is the same as in flat space-time[4] except that $m^2$ is replaced by $m_-^2$. Thus from diagram 1(c) I get

$$N_2(p) = \frac{m_-^2 g^2}{(4\pi)^{n/2}} \Gamma(3 - \tfrac{1}{2}n) \left(\frac{-p^2}{\mu^2}\right)^{n/2-3} B(\tfrac{1}{2}n - 1, \tfrac{1}{2}n - 2) . \tag{14}$$

The rule for inserting $V$ into a line which carries momentum $q$ is to insert $V(q_0 - q_0')$, multiply by $\exp[i(q_0 - q_0')\eta]$, and integrate over $q_0'$. Thus from diagram 1(e) I get

$$S_2(p) = \frac{ig^2\mu^{6-n}}{(2\pi)^n} \int \frac{\hat{V}(q_0 - q_0') e^{i(q_0 - q_0')\eta}}{q'^2 q^2 (p-q)^2} \, dq_0' \, d^n q , \tag{15}$$

where $q'^2 = q_0'^2 - |\vec{q}|^2$. Changing variables to $k_0 = q_0 - q_0'$ gives

$$S_2(p) = \frac{ig^2\mu^{6-n}}{(2\pi)^n} \int dk_0 e^{ik_0\eta} \hat{V}(k_0) \int d^n q \frac{1}{(q-k)^2 q^2 (p-q)^2} , \tag{16}$$

where $k = (k_0, 0, 0, 0)$.

The integral over $q$ is logarithmically divergent and thus the residue does not depend on $k$. Evaluating the integral gives

$$S_2(p) = -\frac{1}{2} \frac{g^2}{(4\pi)^{n/2}} \left(\frac{-p^2}{\mu^2}\right)^{n/2-3} \Gamma(3 - \tfrac{1}{2}n) V(\eta) . \tag{17}$$

Thus

$$\begin{aligned} N_2(p) + S_2(p) &= -\frac{-g^2}{(4\pi)^{n/2}} \left(\frac{-p^2}{\mu^2}\right)^{n/2-3} [m_-^2 - V(\eta)] \frac{1}{n-6} \\ &= \frac{-g^2}{(4\pi)^{n/2}} \left(\frac{-p^2}{\mu^2}\right)^{n/2-3} \frac{\Omega^3(\eta)}{n-6} \left[ m^2 + \left(\xi - \frac{n-2}{4(n-1)}\right) R(\eta) \right] . \end{aligned} \tag{18}$$

The poles can be canceled by the counterterm graphs 1(d) and 1(f). In diagram 1(d) the mark indicates $-\frac{1}{2} b_{12} [\Omega^3(\eta) m^2 g^2 \phi^2 / (n-6)]$ and in diagram 1(f) the mark indicates $-\frac{1}{2} d_{12} [\Omega^3(\eta) R(\eta) g^2 \phi^2 / (n-6)]$. The poles cancel if I pick $b_{12} = 1/(4\pi)^3$ and $d_{12} = (\xi - \frac{1}{5})/(4\pi)^3$. It is easy to convince oneself, by writing out the full expression for the $S$ matrix including all factors of $\sqrt{-g}$, that all the terms have the same factor of $\Omega(\eta)$. The coefficient $b_{12}$ agrees with the flat-space calculation by MacFarlane and Woo,[4] as of course do the coefficients $a_{12}$ and $c_{12}$.

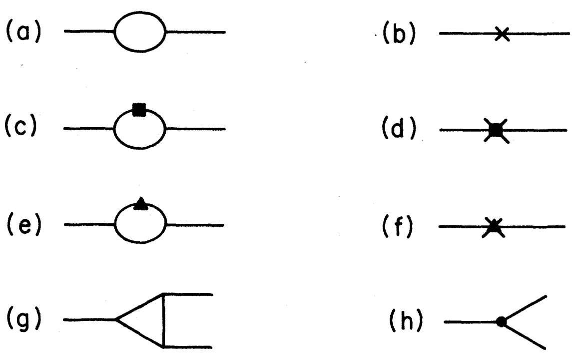


FIG. 1. (a) Self-energy. (b) Wave-function counterterm. (c) Self-energy with mass insertion. (d) Mass counterterm. (e) Self-energy with $V(q)$ insertion. (f) Scalar curvature counterterm. (g) Vertex. (h) Vertex counterterm.

I have gone through the calculations of $b_{12}$ and $d_{12}$ in some detail in order to illustrate the exact nature of the cancellations. In the calculation of $N_2$ and $S_2$, one gets terms involving $m_-^2$ and $V(\eta)$, but the counterterms involve $m^2$ and $R(\eta)$. Thus in order for the theory to be renormalizable, the coefficients of the $m_-^2$ and the $V(\eta)$ terms must be the same.

The appearance of nonconformal counterterms is due to the use of a nonconformally invariant renormalization scheme (see the discussion of Birrell and Davies of this point). In contrast Drummond uses a conformally invariant renormalization scheme and thus has no conformally noninvariant counterterms at the one-loop level.

I am currently extending this calculation to the two-loop level where one must deal with overlapping divergences and where one gets state-dependent infinities which must cancel in order for the theory to be renormalizable.

The curved-space-time renormalization-group equation[1,2] is

$$\left[\kappa \frac{\partial}{\partial \kappa} - \beta(g) \frac{\partial}{\partial g} + (1 + \gamma_m) m \frac{\partial}{\partial m} + \gamma_\xi \xi_R \frac{\partial}{\partial \xi_R} - D_\Gamma + \gamma_\Gamma(g)\right] \times \Gamma_R(\kappa p_0, g_R, m_R, \xi, \mu) = 0 \tag{19}$$

with

$$\gamma_\xi(g_R)=\lim_{n\to 6}\mu\frac{\partial}{\partial\mu}\ln Z(g_B(n)\mu^{(n-6)},n).$$

All the other terms have their usual meanings.

In order to calculate the $\beta$ and $\gamma$ functions it is necessary to relate the coefficients in the Laurent series for $Z$, $m_0 Z$, $g_0 Z$, and $\xi_0 Z$ to the coefficients in the Laurent series for $m_0$, $g_0$, and $\xi_0$. Expanding $\xi_0$ as

$$\xi_0=\xi+\sum_{\nu=1}^{\infty}\frac{h_\nu(g)}{(n-6)^\nu} \tag{20}$$

gives $h_1=(d_{12}-\xi c_{12})g^2$. In terms of $h_1$,

$$\gamma_\xi(g)=\frac{1}{2}\frac{dh_1}{dg}. \tag{21}$$

The effective coupling $\xi(\kappa)$ satisfies[1]

$$\kappa\frac{d\xi(\kappa)}{d\kappa}=-\gamma_\xi(g(\kappa))\xi(\kappa) \tag{22}$$

with $\xi(1)=\xi$. The theory is asymptotically free with

$$g(\kappa)=g(1)\left[\frac{\alpha}{g(1)^2\ln\kappa+\alpha}\right]^{1/2}, \tag{23}$$

where $\alpha=2(4\pi)^3/3$, so in the UV region I only need to calculate $\xi(\kappa)$ to $O(g^2)$. From Eq. (21) $\gamma_\xi(g(\kappa))=[5/6(4\pi)^3](\xi-\frac{6}{25})g^2(\kappa)$. Using this along with Eqs. (22) and (23) gives

$$\xi(\kappa)=\tfrac{6}{25}\xi(1)\{\xi(1)-[g^2(1)\ln\kappa+\alpha]^{-5/9}[\xi(1)-\tfrac{6}{25}]\}^{-1} \tag{24}$$

and thus as $\kappa\to\infty$, $\xi(\kappa)\to\frac{6}{25}$.

In six dimensions the conformally invariant value for $\xi$ is $\frac{1}{5}$, so in the UV region the theory is not conformally invariant even if $m=0$. This is in sharp contrast to the IR behavior of $\xi(\kappa)$ for $\phi^4$, since for $\phi^4$ in four dimensions $\xi(\kappa)\to\frac{1}{6}$ as $\kappa\to 0$.

The difference lies in the behavior of $Z$. For $\phi^4$ at the one-loop level $Z=1$ so $\xi(\kappa)$ has an IR fixed point at $\xi=\frac{1}{6}$. However, for $\phi^3$ there are nontrivial contributions to $Z$ at the one-loop level so the UV fixed point of $\xi(\kappa)$ is shifted from $\frac{1}{5}$ to $\frac{6}{25}$.

This allows one to guess at the result for four-dimensional QCD in a Robertson-Walker universe. Since the vertex structure of $\phi^3$ is the same as quantum chromodynamics (QCD) (except for the four-gluon vertex) and since for QCD there are nontrivial contributions to $Z$ at the one-loop level, it is reasonable to expect that the UV fixed point of $\xi(\kappa)$ will differ from the conformal values. This means that in the early universe (assuming that it is a Robertson-Walker universe) QCD will not be conformally invariant even when the particle energies are much higher than their mass. This will lead to particle creation. A good discussion of the relationship between conformal-symmetry breaking and particle production can be found in Birrell and Davies[6] and the references within.

I would like to thank R. Rauch for helpful conversations and M. Dresden for continual help and encouragement. This work was partially supported by NSF Contract No. PHY-79-06376.

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