PHYSICAL REVIEW D VOLUME 24, NUMBER 6 15 SEPTEMBER 1981

Optimized perturbation theory in the Gross-Neveu model

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(Received 26 May 1981)

Perturbative predictions in quantum chromodynamics and other field theories are ambiguous because they depend on the choice of renormalization scheme. "Optimized perturbation theory" has been proposed as the solution to this problem. I apply the method to a simple, soluble model in order to illustrate its formalism and to demonstrate its success.

I. INTRODUCTION

In conventional perturbation theory finite-order results depend on the choice of renormalization scheme (RS). Perturbative results are therefore meaningless without a criterion for choosing the "best" RS. Moreover, the right choice of RS can—and therefore really *must*—depend on which physical quantity one is concerned with. A resolution of this ambiguity has been proposed,[1,2] based on the argument that, *since the exact result is known to be RS independent, the best approximation is the one which is least sensitive to small changes in RS.* The same general idea, dubbed the "principle of minimal sensitivity" (PMS), applies to any situation in which an approximation scheme does not respect the known invariances of the exact result. Several simple examples were used in Ref. 1 to illustrate the soundness of this criterion. None of these examples, however, was a field theory.

In this paper I demonstrate that the PMS criterion works equally well in a field-theoretic context. I study a simple, soluble model in order to be able to compare the finite-order perturbative approximations with a *known* exact result. It is also possible to check the consistency of the formalism of "optimized perturbation theory," which uses the renormalization point μ and the β-function coefficients to parametrize RS dependence.

Specifically, I study perturbative approximations to the four-point function in the large-$\mathfrak{N}$ limit of the Gross-Neveu (GN) model,[3] a $(1+1)$-dimensional model of $\mathfrak{N}$ massless fermion fields interacting through a four-fermion coupling. I have nothing new to say about the physics of the GN model, nor about the $1/\mathfrak{N}$ expansion: I simply use the model as a test-bed for ideas developed for quantum chromodynamics (QCD). The GN model is suitable for my purposes because it is (i) *soluble*, (ii) *renormalizable*, (iii) *massless* (for technical reasons, Ref. 1 treats only massless theories), and (iv) has a *single*, dimensionless bare coupling constant. The model also has many intriguing dynamical similarities to QCD,[3] but I stress that I am not appealing to such analogies here: My main points—the consistency of the RS parametrization of Ref. 1, and the validity of the minimal-sensitivity argument—are independent of the detailed dynamics. It is true that the simple dynamics of the model is responsible for the remarkable accuracy of perturbative results in my examples. However, the potential accuracy is realized *only if one interprets the perturbative approximations correctly*. My point is that the PMS criterion is the most sensible, and successful, interpretation.

At the outset it is necessary to distinguish two separate issues: (i) the comparison of finite-order perturbative approximations with the all-orders ("exact") result, and (ii) the comparison of the all-orders result with the actual ("experimental") result, which includes *nonperturbative* effects as well. The point is this: As in QCD, the true vacuum of the GN model is not the perturbative vacuum, but is displaced by an intrinsically nonperturbative amount $O(e^{-1/g^2})$. This causes dynamical mass generation[3] and is very important at low energies. However, the nonperturbative effects become negligible at high energies where one would want to use perturbation theory. (Like QCD, the model is asymptotically free.[3])

"Optimization" uses the renormalization group to try to approximately reconstruct the all-orders result from a finite number of perturbative coefficients. Since those coefficients contain no information whatsoever about nonperturbative effects, it would not be fair, in this exercise, to compare the optimized results directly with the "experimental" results. The question is, first, how well does finite-order (naive or optimized) perturbation theory approximate the "exact" all-orders result; and, second, how large are the nonperturbative effects? Both issues are interesting, but the first is my primary concern. (I briefly discuss nonperturbative effects in the Appendix.)

In order to be brief, I shall assume that the reader is familiar with Ref. 1, and Secs. II and III of Ref. 3. All statements about the GN model are taken from Ref. 3, even where not explicitly ref-

erenced, and the qualifying phrase "to leading order in $1/\mathfrak{N}$" is to be understood throughout.[4] Section II sets up the framework and explicitly verifies the consistency of the formalism used in Ref. 1. Section III discusses numerical examples, which provide further support for the principle of minimal sensitivity. The conclusions are summarized in Sec. IV.

II. FORMALISM

A. The four-point function

As a prototype example of a physical quantity in the GN model, I shall study the four-point function. Admittedly, this is a somewhat artificial example, only indirectly related to experimental quantities. However, I wish to avoid becoming involved in a discussion of experimental physics in 1+1 dimensions. Normally Green's functions would not be renormalization-group invariants, but in the GN model the fermion field has no anomalous dimension, so the four-point function satisfies a homogeneous RG equation,

$$\left(\mu\frac{\partial}{\partial\mu}\bigg|_g+\beta(g)\frac{\partial}{\partial g}\right)G=0\,, \tag{1}$$

which gives it (for the present purposes, at least) the status of a physical quantity.

The four-point function has the form (see Fig. 1)

$$G(P_1P_2P_3P_4)^{ab}{}_{cd}=-g^2[D(s)\delta^a{}_c\delta^b{}_d-D(u)\delta^a{}_d\delta^b{}_c]\,, \tag{2}$$

where $D(P^2)$ is the σ propagator and $s\equiv-(P_3+P_1)^2$, $u\equiv-(P_4+P_1)^2$, with the convention that all momenta are incoming. I shall study two cases, corresponding to particles 1 and 2 being in antisymmetric or symmetric combinations under the internal symmetry. Conveniently normalized and expressed in terms of the couplant a,

$$a\equiv g^2\mathfrak{N}/\pi\quad(=\lambda/\pi)\,, \tag{3}$$

these are

$$\mathfrak{R}_+(s,u)=\tfrac{1}{2}ia[D(s)+D(u)]\,, \tag{4}$$

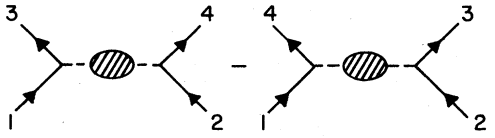

FIG. 1. The fermion four-point function to leading order in $1/\mathfrak{N}$. Particles 1, 2, 3, 4 carry internal-symmetry indices a, b, c, d, respectively. (The minus sign corresponds to the exchange of two identical fermions).

$$\mathfrak{R}_-(s,u)=\frac{2ia}{\ln(u/s)}[D(s)-D(u)]\,. \tag{5}$$

Radiative corrections to the four-point function come only from the σ propagator $D(P^2)$ which itself is simply a sum of fermion bubbles (see Fig. 2). The geometric series is easily summed to give exact (all-orders) results for $\mathfrak{R}_+$ and $\mathfrak{R}_-$. (In the next section we will compare the exact result with the perturbative approximations obtained by keeping only the first few terms of the series.) Explicitly, since the bare σ propagator is just $-i$, one has

$$D(P^2)=-i\{1+[-i\Pi(P^2)]+[-i\Pi(P^2)]^2+\cdots\} \tag{6}$$

$$=-i/[1+i\Pi(P^2)]\,, \tag{7}$$

where $\Pi(P^2)$ is the fermion bubble. In its unrenormalized form

$$\Pi_u(P^2)=-g^2\mathfrak{N}\int\frac{d^2k}{(2\pi)^2}\frac{\mathrm{Tr}[\not k(\not k-\not P)]}{k^2(k-P)^2}$$

$$=\frac{ia}{2}\int_0^1 dy\left[\ln\left(\frac{-\Lambda_{\mathrm{UV}}{}^2}{y(1-y)P^2}\right)-2\right], \tag{8}$$

where Λ_{UV} is an ultraviolet cutoff used to temporarily regulate the unrenormalized theory.

To renormalize the theory one performs a subtraction such that $D(P^2)$ satisfies a particular renormalization condition. A specification of a renormalization condition defines a renormalization scheme (RS). The choice of RS is arbitrary, in the sense that it does not affect the all-orders results for physical quantities. The canonical choice, hereafter called the GN scheme, is to require

$$D(P^2)=-i\quad\text{at } P^2=-\mu^2\,. \tag{9}$$

Any other perturbative RS is equivalent to a renormalization condition of the general form

$$D(P^2)=-i/\zeta\quad\text{at } P^2=-\mu^2\,, \tag{10}$$

where ζ,

$$\zeta=1+w_1a+w_2a^2+\cdots\,, \tag{11}$$

is a (finite) ratio of renormalization constants. Note that $\zeta=1+O(a)$ is required so that $D(P^2)=D_{\mathrm{bare}}+O(a)$, where $D_{\mathrm{bare}}=-i$. [N.B. If the theory is treated nonperturbatively, then other types of renormalization conditions are possible, e.g.,

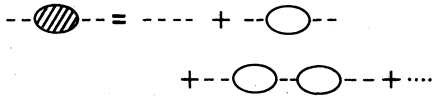

FIG. 2. The σ propagator to leading order in $1/\mathfrak{N}$.

GN Eq. (4.22).[5]] Clearly each RS is characterized by the set of numbers $w_1, w_2, w_3, \ldots$ (and by the choice of μ) and it will shortly become clear how this is equivalent to the RS parametrization used in Ref. 1.

Imposing the renormalization condition (10), the renormalized $\Pi(P^2)$ is given by the subtraction

$$\Pi(P^2) = \Pi_u(P^2) - \Pi_u(-\mu^2) - i(\zeta - 1), \tag{12}$$

and so

$$D(P^2) = -i/[\zeta + \tfrac{1}{2}a\ln(-P^2/\mu^2)]. \tag{13}$$

Substituting into Eqs. (4) and (5), one obtains the exact results for $\mathcal{R}_+$, $\mathcal{R}_-$:

$$\begin{aligned} \mathcal{R}_+^{\text{exact}} &= \tfrac{1}{2}a\left[\frac{1}{\zeta + \frac{1}{2}a\ln(s/\mu^2)} + (s \to u)\right], \\ \mathcal{R}_-^{\text{exact}} &= \frac{2a}{\ln(u/s)}\left[\frac{1}{\zeta + \frac{1}{2}a\ln(s/\mu^2)} - (s \to u)\right]. \end{aligned} \tag{14}$$

B. Different renormalization schemes

One can obtain the relation between the couplants of different RS's very easily. Since ζ, being a ratio of renormalization constants, is momentum independent, the relation

$$\zeta D(P^2) = D_{\text{GN}}(P^2), \tag{15}$$

obviously true at $P^2 = -\mu^2$, must be true at all P^2. This requires

$$a_{\text{GN}} = \zeta^{-1}a = a[1 - w_1 a + (w_1^2 - w_2)a^2 + \cdots]. \tag{16}$$

It is then simple to find the relation between the β functions of different schemes:

$$\mu\frac{\partial}{\partial\mu}a_{\text{GN}} \equiv \beta_{\text{GN}}(a_{\text{GN}}) = \frac{\beta(a)}{\zeta^2}(\zeta - a\zeta'), \tag{17}$$

where $\zeta' \equiv \partial\zeta/\partial a|_w$. (Note that, since μ is the only massive variable involved in the definition of ζ, the w_i's cannot be μ dependent, for dimensional reasons.) In Ref. 3 it is shown that[6]

$$\beta_{\text{GN}}(a_{\text{GN}}) = -a_{\text{GN}}^2. \tag{18}$$

Therefore,

$$\begin{aligned} \beta(a) &= -a^2/(\zeta - a\zeta') \\ &= -a^2/(1 - w_2a^2 - 2w_3a^3 - 3w_4a^4 - \cdots). \end{aligned} \tag{19}$$

Thus in the notation of Ref. 1, where

$$\beta(a) \equiv -ba^2(1 + ca + c_2a^2 + c_3a^3 + \cdots), \tag{20}$$

the β-function coefficients are given by

$$\begin{aligned} &b = 1, \quad c = 0, \quad c_2 = w_2, \quad c_3 = 2w_3, \\ &c_4 = 3w_4 + w_2^2, \cdots. \end{aligned} \tag{21}$$

(An alternative way of obtaining these results is to repeat the derivation of the β function in Sec. III of Ref. 3 using the general RS.)

Points to note are (i) the coefficients b and c are indeed RS invariant, (ii) the coefficients $c_2, c_3, \ldots$ are related to $w_2, w_3, \ldots$ by a smooth change of variables, and (iii) w_1 cancels out in Eq. (19). The role of w_1 as a RS parameter is taken over by $\tau \equiv \ln(\mu/\tilde{\Lambda})$. As in Ref. 1, the scale parameter $\tilde{\Lambda}$ is defined by writing the integrated β-function equation as

$$\tau \equiv \ln(\mu/\tilde{\Lambda}) = \int_0^a \frac{dx}{\beta(x)} - \int_0^\infty \frac{dx}{\beta^{(2)}(x)}, \tag{22}$$

where $\beta^{(2)}(x)$ is the second-order perturbative approximation to the β function. In this case $\beta^{(2)}(x) = -x^2$, so that

$$\tau = \hat{K}(a) \equiv \frac{1}{a} + \int_0^a dx\left[\frac{1}{\beta(x)} + \frac{1}{x^2}\right]. \tag{23}$$

The $\tilde{\Lambda}$ defined by Eq. (22) is scheme-dependent (because β and a are), but $\tilde{\Lambda}$'s in different RS's are related exactly by the Celmaster-Gonsalves[7] relation, which involves only the one-loop relation between the two schemes. In particular, from Eq. (16) one has

$$\ln(\tilde{\Lambda}/\tilde{\Lambda}_{\text{GN}}) = w_1. \tag{24}$$

This equation, together with Eq. (21), shows explicitly how the labeling of RS's by the parameters $(\tau, c_2, c_3, \ldots)$ comes about.

The fact that physical quantities are independent of RS is expressed by the RG equations

$$\begin{aligned} &\frac{\partial\mathcal{R}}{\partial\tau} = \left(\frac{\partial}{\partial\tau}\bigg|_a + \beta(a)\frac{\partial}{\partial a}\right)\mathcal{R} = 0 \quad (\text{"}j = 1\text{"}), \\ &\frac{\partial\mathcal{R}}{\partial c_j} = \left(\frac{\partial}{\partial c_j}\bigg|_a + \beta_j(a)\frac{\partial}{\partial a}\right)\mathcal{R} = 0 \quad (j = 2, 3, \ldots). \end{aligned} \tag{25}$$

The first of these is the usual RG equation, i.e., Eq. (1) in different notation. The other equations express the fact that $\mathcal{R}$ is independent of all the w_i's in Eq. (11), and applied to $\mathcal{R}_+$ or $\mathcal{R}_-$ they correspond to the requirement

$$a\frac{\partial\zeta}{\partial c_j}\bigg|_a = \beta_j(a)(\zeta - a\zeta'). \tag{26}$$

It is a straightforward mathematical exercise to verify that Eq. (26) is satisfied, using the expression for the β_j functions, $\beta_j \equiv \partial a/\partial c_j$, given in Ref. 1:

$$\beta_j(a) = -\beta(a)\int_0^a \frac{x^{j+2}}{[\hat{\beta}(x)]^2}dx. \tag{27}$$

One needs to use Eqs. (19) and (20), and the fact that, from Eq. (11),

$$\partial\zeta/\partial c_j\big|_a = \sum_{k=2}^{\infty} a^k(\partial w_k/\partial c_j)\big|_a .$$

Because the exact results for $\mathcal{R}_+$, $\mathcal{R}_-$ in Eq. (14) are *exactly* RS independent, one can evaluate them using a special, convenient choice of RS, e.g., the GN scheme ($\zeta=1$), with $\mu\to\tilde{\Lambda}_{GN}$ [for which $a\to a_{GN}(\mu\to\tilde{\Lambda}_{GN})\to\infty$]. One thereby obtains

$$\mathcal{R}_+^{\text{exact}} = \frac{1}{\ln(s/\tilde{\Lambda}_{GN}{}^2)} + \frac{1}{\ln(u/\tilde{\Lambda}_{GN}{}^2)}\,,$$
$$\mathcal{R}_-^{\text{exact}} = \frac{4}{\ln(u/s)}\left[\frac{1}{\ln(s/\tilde{\Lambda}_{GN}{}^2)} - \frac{1}{\ln(u/\tilde{\Lambda}_{GN}{}^2)}\right]. \tag{28}$$

III. NUMERICAL EXAMPLES

A. Perturbative coefficients and invariants

This section is concerned with the comparison of the all-orders results in Eq. (14), or equivalently, Eq. (28), with low-order perturbative approximations. Expanding Eq. (14) gives perturbation series for $\mathcal{R}_+$, $\mathcal{R}_-$:

$$\mathcal{R}_+ = a(1+r_{+,1}a+r_{+,2}a^2+\cdots)\,,$$
$$\mathcal{R}_- = a^2(1+r_{-,1}a+r_{-,2}a^2+\cdots)\,, \tag{29}$$

with the low-order coefficients given by

$$r_{+,1} = -[w_1+\tfrac{1}{4}\ln(us/\mu^4)]\,,\quad r_{+,2} = r_{+,1}{}^2 - w_2 + \tfrac{1}{16}\ln^2(s/u)$$
$$r_{-,1} = -[2w_1+\tfrac{1}{2}\ln(us/\mu^4)]\,,\quad r_{-,2} = \tfrac{3}{2}w_1\ln(us/\mu^4) + 3w_1{}^2 - 2w_2 + \tfrac{1}{4}[\ln^2(s/\mu^2)+\ln^2(u/\mu^2)+\ln(s/\mu^2)\ln(u/\mu^2)]. \tag{30}$$

This shows explicitly the RS dependence of these coefficients. One can now form the RS invariants ρ_1, ρ_2 for both $\mathcal{R}_+$ and $\mathcal{R}_-$. The invariants are constructed from the coefficients r_1, r_2, ... and the RS parameters τ, c_2, c_3, ... according to formula (5.8) of Ref. 1:

$$\rho_{+,1} \equiv \tau - r_{+,1} = \ln(\mu/\tilde{\Lambda}) + [w_1+\tfrac{1}{4}\ln(us/\mu^4)]$$
$$= \tfrac{1}{4}\ln(us/\tilde{\Lambda}^4) + w_1$$
$$= \tfrac{1}{4}\ln(us/\tilde{\Lambda}_{GN}{}^4)\,, \tag{31}$$

where the last step uses the Celmaster-Gonsalves[7] relation, Eq. (24). (It is convenient to adopt the GN scheme as our "reference" scheme, in the sense of Ref. 1.) Similarly

$$\rho_{-,1} \equiv \tau - \tfrac{1}{2}r_{-,1} = \tfrac{1}{4}\ln(us/\tilde{\Lambda}_{GN}{}^4)\,. \tag{32}$$

For the third-order invariants one obtains

$$\rho_{+,2} \equiv r_{+,2} + c_2 - r_{+,1}{}^2 = \tfrac{1}{16}\ln^2(s/u)\,, \tag{33}$$

$$\rho_{-,2} \equiv r_{-,2} + 2c_2 - \tfrac{3}{4}r_{+,1}{}^2 = \tfrac{1}{16}\ln^2(s/u)\,, \tag{34}$$

using $c_2 = w_2$ from Eq. (21). As expected, all dependence on the RS has canceled in Eqs. (31)–(34).

B. Second-order approximants

The second-order perturbative approximation corresponds to (i) truncating the perturbation series for $\mathcal{R}_{+,-}$ at second order, and (ii) replacing a by its second-order approximation $a^{(2)}$ (defined to be the solution to the β-function equation truncated at second order). Thus,

$$\mathcal{R}^{(2)} = (a^{(2)})^N(1+r_1a^{(2)})\,, \tag{35}$$

where $N=1$, 2 for $\mathcal{R}=\mathcal{R}_+$, $\mathcal{R}_-$, respectively, and with

$$a^{(2)} = 1/\tau = 1/\ln(\mu/\tilde{\Lambda})\,. \tag{36}$$

(Notice that the situation is considerably simplified because the second β-function coefficient c is zero in the GN model.) Since r_1 is related to τ by

$$r_1 = N(\tau-\rho_1) \tag{37}$$

[see Eqs. (31) and (32)] $\mathcal{R}^{(2)}$ can be written as a function of $a^{(2)}$ as

$$\mathcal{R}^{(2)} = (a^{(2)})^N(1+N-N\rho_1a^{(2)})\,. \tag{38}$$

This is like considering $\mathcal{R}^{(2)}$ as a function of renormalization point μ: it is just algebra to convert from $\mu\to\tau\to a^{(2)}$.

The optimum second-order result corresponds to the stationary point of $\mathcal{R}^{(2)}$, which from Eq. (38) is given by

$$\bar{a}^{(2)} = 1/\rho_1\,,\quad \mathcal{R}^{(2)}_{\text{opt}} = 1/\rho_1{}^N\,. \tag{39}$$

One can also, of course, obtain the above results by mechanically substituting into the formulas in Ref. 1, which greatly simplify when $c=0$. In fact, because $c=0$, the optimization condition corresponds to $\bar{r}_1=0$, i.e., the PMS criterion becomes equivalent to requiring the next-to-leading order correction term to vanish (the "fastest apparent convergence" criterion).

From the expressions for $\rho_{+,1}$, $\rho_{-,1}$ in Eqs. (31) and (32), one finds the optimized second-order predictions for $\mathcal{R}_+$ and $\mathcal{R}_-$ to be

$$\mathcal{R}^{(2)}_{+,\text{opt}} = 1/[\tfrac{1}{4}\ln(us/\tilde{\Lambda}_{GN}{}^4)]\,,$$
$$\mathcal{R}^{(2)}_{-,\text{opt}} = 1/[\tfrac{1}{4}\ln(us/\tilde{\Lambda}_{GN}{}^4)]^2\,. \tag{40}$$

Note that I have chosen to write the predictions in terms of $\tilde{\Lambda}_{GN}$ as the free parameter of the theory. I could equally well express the results in terms

of the $\tilde{\Lambda}$ of some other scheme by using the Celmaster-Gonsalves relation: the content of the predictions would be unchanged, although the algebra would be less compact.

Comparing Eq. (40) with the exact expressions in Eq. (28), and noting that the "optimized" result corresponds to *maximizing* $\mathcal{R}_{+,-}^{(2)}$, one can see that there are Rayleigh-Ritz–type inequalities

$$\mathcal{R}_{+}^{(2)} \leq \mathcal{R}_{+}^{\text{exact}} , \quad \mathcal{R}_{-}^{(2)} \leq \mathcal{R}_{-}^{\text{exact}} . \tag{41}$$

[One may assume $\ln(s/\tilde{\Lambda}_{\text{GN}}{}^2)$, $\ln(u/\tilde{\Lambda}_{\text{GN}}{}^2) > 0$ because the use of finite-order perturbation theory presupposes $s, u \gg \tilde{\Lambda}^2$.] Therefore, in these examples, the PMS criterion does yield the *optimum* result in the strictest sense of minimizing the error $|\mathcal{R}^{\text{exact}} - \mathcal{R}^{(2)}|$. Note that for $s = u$ the optimized result is actually exact. If the physical, kinematic variables u, s are made progressively more unequal, then perturbation theory becomes steadily worse. This is expected: for $u \gg s$ ($\gg \Lambda^2$) perturbative coefficients become dominated by large $\ln(u/s)$ terms, making finite-order results less trustworthy. At some point one would need to switch to an approximation scheme which resummed these large logarithms.

As illustrations I show in Fig. 3 the numerical results for a case in which the physical variables u, s have values such that $\sqrt{su} = (10\tilde{\Lambda}_{\text{GN}})^2$, and $u/s = 10$. I have deliberately chosen a situation rather unfavorable for perturbation theory: the mean energy is not very high (so the effective couplant is rather large; $\bar{a} = 1/\ln 10 \simeq 0.43$), and the two scales u and s have a sizable ratio. Nevertheless, the second-order result is surprisingly good, when evaluated at the optimum point. One can consider other examples with different values of the kinematic variables u, s. In particular, it is easy to see what happens if one varies u/s keeping $\sqrt{su}$ fixed: The second-order result remains the same, but, as u/s departs from unity, the approximation increasingly underestimates the exact result (see Fig. 3).

C. Third-order approximants

Third-order results depend on RS through the choice of renormalization point μ and also through the fact that the third β-function coefficient c_2 is different in different schemes. Again it is convenient to change variables $\mu \to \tau \to a^{(3)}$, where the relation between τ and a (shorthand for $a^{(3)}$ in this subsection) is given by

$$\tau = \hat{K}^{(3)}(a) = \frac{1}{a} + c_2 \int_0^a \frac{dx}{(1 + c_2 x^2)} \tag{42}$$

$$= \frac{1}{a} + \sqrt{c_2}\tan^{-1}(\sqrt{c_2}\,a) \quad \text{for } c_2 \geq 0 \tag{43}$$

$$= \frac{1}{a} - \tfrac{1}{2}(|c_2|)^{1/2} \ln\left|\frac{1 + (|c_2|)^{1/2} a}{1 - (|c_2|)^{1/2} a}\right| \quad \text{for } c_2 < 0 .$$

One can examine $\mathcal{R}_{+,-}^{(3)}$ as a function of RS (i.e., of a, c_2) as follows. First fix the physical variables and calculate the invariants ρ_1, ρ_2. Choose values for a and c_2, and evaluate τ from the above equations. Then evaluate r_1, r_2 from

$$\begin{aligned} r_1 &= N(\tau - \rho_1) , \\ r_2 &= \rho_2 - N c_2 + \tfrac{1}{2} N(N+1)(\tau - \rho_1)^2 \end{aligned} \tag{44}$$

[which follow from the definitions of ρ_1, ρ_2; see Eq. (5.11) of Ref. 1]. One can then construct $\mathcal{R}^{(3)}(a, c_2)$ from

$$\mathcal{R}^{(3)} \equiv a^N (1 + r_1 a + r_2 a^2) . \tag{45}$$

Repeating the procedure for various values of a,

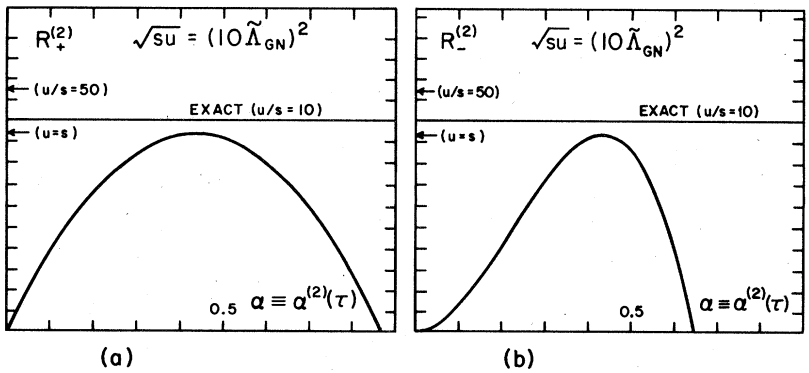

FIG. 3. Second-order perturbative approximations to $\mathcal{R}_+$ and $\mathcal{R}_-$ as functions of RS. [See Eq. (38) with $N = 1, 2$, respectively.] The stationary point of these curves represents the optimum result. In the example shown here, the kinematic variables s, u are given by $\sqrt{su} = (10\tilde{\Lambda}_{\text{GN}})^2$, $u/s = 10$: The exact result is then represented by the horizontal line. (The approximant depends only on $\sqrt{su}$, while the exact result also varies slightly with u/s: Exact results for two other cases, $u = s$ and $u/s = 50$, are also indicated.)

c_2, one builds up a picture of the scheme dependence of $\mathcal{R}^{(3)}$.

The curves in Fig. 4 have been obtained in this manner. The example shown is that used earlier, i.e., $\sqrt{su} = (10\tilde{\Lambda}_{GN})^2$, $u/s = 10$. Figure 4(a) for the $N = 1$ case is directly comparable to Fig. 7 of Ref. 1. The figures show that the surface $\mathcal{R}^{(3)}(a, c_2)$ has a flat region around a saddle point *and that the flat region is where the function best approximates the exact result.*

The optimized result corresponds to the value of $\mathcal{R}^{(3)}$ at the stationary point, whose exact position can be located by solving the optimization equations [Eqs. (5.13) and (5.14) of Ref. 1]. This is fairly straightforward to do numerically—especially so because $c = 0$. The results, expressed in terms of the percentage error, are given in the last column of Table I (which also quotes results for the case where $u/s = 100$).

Table I presents a comparison of the optimized results (PMS) with two other suggested ways of dealing with the RS ambiguity. The "canonical" scheme follows the spirit of "momentum subtraction",[7,8] and corresponds to the GN scheme with the "intuitive" choice of $\mu^2 = \sqrt{su}$. (Of course, in QCD the identification of a "natural" scheme is much less clear-cut.[7,8]) The FAC (fastest apparent convergence) results correspond to absorbing all the correction terms into the couplant[9], i.e., adjusting the RS such that $r_1 = r_2 = 0$. Like the PMS criterion, this recognizes that the RS can be adjusted according to the quantity being calculated. It often gives somewhat similar results, a property that is enhanced here because $c = 0$. However, the FAC criterion can go rather disastrously wrong in other examples.[1]

The cases considered in Table I are for $\mathcal{R}_+$ and $\mathcal{R}_-$ at an average energy $\sqrt{su} = (10\tilde{\Lambda}_{GN})^2$. When $u/s = 1$, all three methods (canonical, FAC, PMS) give the same result, which is, in fact, exact.[10] For larger u/s (= 10, 100) perturbation theory becomes worse in each method, as expected. Nevertheless, the PMS results are consistently better—often substantially better—than the results of the other methods. In fact, in view of the size of the effective couplant $\bar{a} \simeq 0.5$, the PMS criterion is doing a remarkable job of anticipating the higher-order behavior of the series.

IV. CONCLUSIONS

Optimized perturbation theory is remarkably successful in the GN model—in that low orders give very accurate approximations to the all-orders result.[11] (I continue to leave aside the separate question of nonperturbative effects; see the Appendix.) There are two reasons for this success. One is that perturbation theory is intrinsically well behaved in the GN model—so one can *potentially* obtain very accurate results from low orders. The second reason is that the PMS criterion realizes this potential by making best use of the information contained in the low-order results.

The point is this: "Optimization" is essentially a precise formulation of the old idea[12] that the renormalization group[13] can resum the ln(energy/μ) terms in a perturbation series. This idea is especially powerful in the GN model, because all the essential physics is contained in the lowest-order, fermion-bubble diagram: Higher orders are basically repetitions which ensure that the

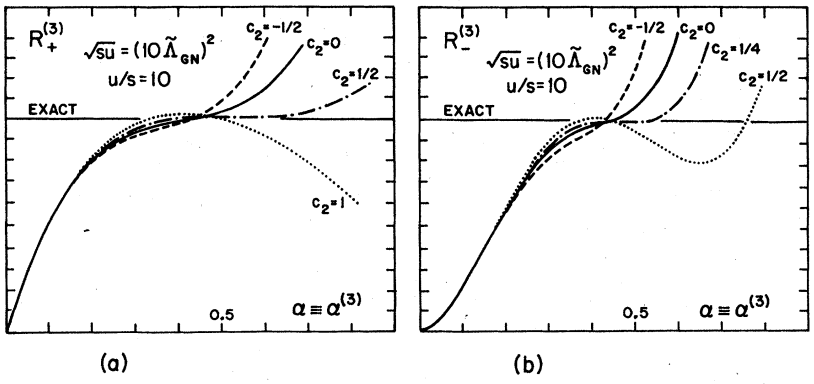

FIG. 4. Third-order perturbative approximations to $\mathcal{R}_+$ and $\mathcal{R}_-$ as functions of RS. (See Sec. III C with $N = 1, 2$, respectively.) At fixed values of the *physical* variables [$\sqrt{su} = (10\tilde{\Lambda}_{GN})^2$, $u/s = 10$] the third-order approximants are functions of two RS parameters a, c_2, while the exact result is a constant. The approximations are most nearly constant in the vicinity of their saddle points, i.e., near $a = 0.45$, $c_2 = \frac{1}{2}$ for $\mathcal{R}_+$ and near $a = 0.45$, $c_2 = \frac{1}{4}$ for $\mathcal{R}_-$ (for the precise values, see Table I). As expected by the PMS argument, this is where the approximations are most accurate.

TABLE I. Percentage errors of third-order perturbative approximants in various schemes. Canonical refers to use of the GN scheme with $\mu^2=\sqrt{su}$. FAC corresponds to choosing the RS such that $r_1=r_2=0$. PMS refers to the optimized result, in the sense of Ref. 1. (In the latter cases the values of the RS parameters a, c_2 are shown: the canonical scheme corresponds to the fixed values $a=0.434$, $c_2=0$.)

$\sqrt{su}=(10\tilde{\Lambda}_{GN})^2$		Canonical	FAC		PMS	
$\mathcal{R}^{(3)}_{++}$	$u/s=1$		$a=0.434$ $c_2=0$		Exact	
$\mathcal{R}^{(3)}_{+}$	$u/s=10$	−0.39%	$a=0.465$ $c_2=0.331$	0.30%	$a=0.464$ $c_2=0.497$	0.22%
	$u/s=100$	−6.25%	$a=0.634$ $c_2=1.325$	9.55%	$a=0.605$ $c_2=1.994$	5.60%
$\mathcal{R}^{(3)}_{-}$	$u/s=10$	−0.39%	$a=0.449$ $c_2=0.166$	0.036%	$a=0.449$ $c_2=0.221$	0.016%
	$u/s=100$	−6.25%	$a=0.504$ $c_2=0.663$	0.83%	$a=0.505$ $c_2=0.882$	0.32%

full result is RS invariant. [For the $s=u$ case this is literally true, and so the second-order result is actually exact, when optimized. For $s\neq u$ there are $\ln(s/u)$ terms in addition to $\ln(\sqrt{su}/\mu^2)$ terms, so the optimized result is not perfect.]

The GN model is extreme in this respect, but it is not a false guide. Although in more realistic theories higher-order terms do contain new physics, they also consist largely of repetitions of lower-order physics. With the wrong choice of RS, these recurrences, which grow combinatorically, can easily dominate the numerical size of high-order terms. Thanks to the renormalization group, one can identify a RS in which these trivial repetitions are most suppressed: They are not needed if the approximant already satisfies the RG-equations (i.e., is stationary with respect to small variations in RS).

It is a striking feature of perturbative approximations that they do have a stationary point, when considered as functions of RS. The qualitative behavior seen in Figs. 3 and 4 is common to all massless, renormalizable field theories.[1] The PMS argument says that the value of the approximant at its stationary point is what the (otherwise ambiguous) approximate result actually *means*. Just *how* accurate the approximate result will be is plainly a model-dependent question. The examples discussed here say nothing about the accuracy of perturbative approximations in QCD: They *do* tell us something about what perturbative approximations *mean*—and they strongly support the PMS interpretation.

ACKNOWLEDGMENTS

I am very grateful to Cosmas Zachos for interesting me in the Gross-Neveu model, and for many useful discussions. I would also like to thank both Cosmas Zachos and Francis Halzen for valuable comments on the original manuscript. This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contract No. DE-AC02 76ER00881-213.

APPENDIX: NONPERTURBATIVE EFFECTS

The "exact" results quoted in Eq. (28) have singularities at $u=\tilde{\Lambda}_{GN}{}^2$ and $s=\tilde{\Lambda}_{GN}{}^2$, and therefore cannot be physical. In fact, as mentioned in the Introduction, there are nonperturbative effects at low energies which smooth out the singularities. Dynamical mass generation modifies the results in Eq. (28), replacing $\ln(s/\tilde{\Lambda}_{GN}{}^2)$ by $B(s,M^2)$, and $\ln(u/\tilde{\Lambda}_{GN}{}^2)$ by $B(u,M^2)$, where[3]

$$B(s,M^2)=\left(\frac{s+4M^2}{s}\right)^{1/2}\ln\left(\frac{(s+4M^2)^{1/2}+\sqrt{s}}{(s+4M^2)^{1/2}-\sqrt{s}}\right). \tag{A1}$$

The dynamically generated fermion mass M is not predicted absolutely by the theory, but its relation to $\tilde{\Lambda}$, the free parameter of the theory, is calculable. In fact, since $B(s,M^2)\to\ln(s/M^2)$ as $s\to\infty$, one finds that $M=\tilde{\Lambda}_{GN}$. (N.B. In terms of the couplant, $M=\mu\exp[-1/a_{GN}(\mu)]$, so M is intrinsically nonperturbative.[3]) The function $B(s,M^2)$ is essentially the fermion bubble with a massive fermion. It is monotonic, beginning at $B(0,M^2)=2$ and tending to its asymptotic value $\ln(s/M^2)$ from above, with $O(M^2/s)$ corrections.

For large values of s, therefore, the nonperturbative effects are suppressed by a power of M^2/s. *Formally*, they are negligible compared to the error involved in truncating the perturbation series $O((1/\ln s/M^2)^{\text{power}})$. However, in practice the situation may be the reverse. Perturba-

tion theory, when optimized, converges so fast in the GN model that I chose rather low energies for illustrative purposes, in order to make the truncation error clearly visible. However, at those energies the nonperturbative effects are still very important, so the examples are physically unrealistic, although this does not matter for my purposes. For larger s the nonperturbative effects become smaller much faster than the perturbative truncation error.

It is interesting to note that the nonperturbative effects are simply accounted for by doing perturbation theory as if the Lagrangian had contained a fermion mass term. Possibly by using constituent quark masses (presuming the constituent-current difference to be due to nonperturbative effects), and perhaps a phenomenological gluon mass,[14] one can account for some of the nonperturbative effects in QCD.

Also note that, even though $\mathcal{R}_+$ and $\mathcal{R}_-$ are not analytic in the couplant, they have convergent perturbation series. The series does not, however, converge to quite the right answer—it misses the $O(e^{-1/a})$ nonperturbative terms. Contrary to popular myth, the Dyson argument[15] does *not* prove the divergence of QED or QCD perturbation series: it does prove that perturbation theory is not the whole story.

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[1]P. M. Stevenson, Phys. Rev. D 23, 2916 (1981).

[2]P. M. Stevenson, Phys. Lett. 100B, 61 (1981); in Perturbative Quantum Chromodynamics, Tallahassee, 1981, edited by D. W. Duke and J. F. Owens (AIP, New York, to be published).

[3]D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

[4]I assume $\mathfrak{N}$ to be enormous, and ignore all corrections suppressed by $1/\mathfrak{N}$. Since $\mathfrak{N}$ is not renormalized, the use of the $1/\mathfrak{N}$ expansion does not affect the RS-dependence properties of perturbation theory.

[5]These nonperturbative RS's are related to the GN scheme only by *singular* transformations. See Y. Frishman, R. Horsley, and U. Wolff, Nucl. Phys. B183, 509 (1981), and references therein.

[6]Note that Eq. (3.7) of Ref. 3 has been misprinted, and should read $\beta(g)=-\lambda g/2\pi$, $\gamma_0(g)=-\lambda/2\pi$. The conversion of notation is given, through Eq. (3), by $\beta(a)=(2g\mathfrak{N}/\pi)\beta(g)$.

[7]W. Celmaster and R. J. Gonsalves, Phys. Rev. D 20, 1420 (1979).

[8]W. Celmaster and D. Sivers, Phys. Rev. D 23, 227 (1981).

[9]G. Grunberg, Phys. Lett. 95B, 70 (1980); see also S. Wolfram, Caltech Report No. 68–690, 1978 (unpublished).

[10]It is a simple exercise to check that, since the invariants $\rho_{+,2}$ and $\rho_{-,2}$ vanish for $s=u$, the optimization equations are satisfied by $\bar{c}_2=\bar{r}_1=\bar{r}_2=0$. Hence, as at second order, they yield the exact result in this special case.

[11]This discussion assumes that, as in QCD, one is doing perturbation theory mechanically, without further knowledge of the theory. One must forget that the full structure of the GN model is actually known.

[12]See, for example, M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); H. D. Politzer, Phys. Rep. 14C, 129 (1974).

[13]E. C. G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26, 449 (1953).

[14]G. Parisi and R. Petronzio, Phys. Lett. 94B, 51 (1980).

[15]F. J. Dyson, Phys. Rev. 85, 631 (1952).