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# Some aspects of the linear system for self-dual Yang-Mills fields

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## I. INTRODUCTION

In the past decade rich experience has been accumulated about many two-dimensional systems, which are totally integrable. These totally integrable systems have the following common characteristics: (1) Bianchi-Bäcklund transformations, (2) conservation laws, local and nonlocal, and (3) the corresponding linear system (or the inverse-scattering formulation). For some of these systems, the $S$ matrix has already been constructed as a consequence of those conservation laws.[1]

Recently, the self-dual Yang-Mills equation has been shown to have similar structures. Bäcklund transformations (BT's) with parameters,[2] nonlocal conservation law,[3] and the corresponding linear system.[1,4] Therefore, the self-dual Yang-Mills equation serves as a beautiful example of extending totally integrable systems into four dimensions.

In this paper we shall discuss one way of linearizing the self-dual Yang-Mills equations using the infinite nonlocal currents, the properties of such linear systems, and the solutions to such linear equations in the case of $n$ instantons with $(5n+4)$ parameters. The connection with the linear system of Belavin and Zakharov[5] and the Atiyah-Ward[6] construction for the self-dual Yang-Mills equation are discussed. This connection provides a geometrical interpretation of the infinite number of nonlocal conservation laws.

## II. FORMULATION IN COMPLEX FOUR-DIMENSIONAL EUCLIDEAN SPACE

We use the matrix notation for gauge potentials, etc., defined as

$$A_\mu = \frac{gT^a}{2i} A_\mu^a, \quad \mu = 1,2,3,4, \tag{2.1}$$

where $T^a$ are the generators of the Lie algebra. For SU(2), $T^a = \sigma^a$ are the usual Pauli matrices. $g$ is the coupling constant. Then,

$$F_{\mu\nu} \equiv \frac{gT^a}{2i} F_{\mu\nu}^a \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.2}$$

For SU($N$) gauge theory the gauge potentials $A_\mu^a$ are real, i.e., the matrix $A_\mu$ is traceless and anti-Hermitian.

Following Yang[7] we now consider an analytic continuation of $A_\mu$ into *complex space* where $x_1$, $x_2$, $x_3$, and $x_4$ are complex. The self-duality equations

$$F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{2.3}$$

are then valid also in complex space, in a region containing *real space* where the $x_\mu$ are real. Now consider four new complex variables defined by

$$\begin{aligned} \sqrt{2}y &= x_1 + ix_2, \quad \sqrt{2}\bar{y} = x_1 - ix_2, \\ \sqrt{2}z &= x_3 - ix_4, \quad \sqrt{2}\bar{z} = x_3 + ix_4. \end{aligned} \tag{2.4}$$

It is simple to check that the self-duality equations (2.3) reduce to

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \tag{2.5a}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{2.5b}$$

The equation (2.5a) implies that the potentials $A_y$, $A_z$ ($A_{\bar{y}}, A_{\bar{z}}$) are pure gauges for fixed $\bar{y},\bar{z}$ ($y,z$), i.e., we can find two $N\times N$ complex matrices $D$ and $\bar{D}$ such that

$$\begin{aligned} A_y &= D^{-1}D_{,y}, \quad A_z = D^{-1}D_{,z}, \\ A_{\bar{y}} &= \bar{D}^{-1}\bar{D}_{,\bar{y}}, \quad A_{\bar{z}} = \bar{D}^{-1}\bar{D}_{,\bar{z}}, \end{aligned} \tag{2.6}$$

where $D_{,y} \equiv \partial_y D$, etc. The matrix $D$ ($\bar{D}$) is the phase factor in the complex two-dimensional space of $y$ and $z$ ($\bar{y}$ and $\bar{z}$), and can be written as a path-ordered exponential. The path of integration must lie in the plane $\bar{y},\bar{z}$ = constants ($y,z$ = constants) and is independent of the path chosen in the plane. Since fixing $\bar{y},\bar{z}$ for real $x_\mu$ also fixes $y$ and $z$, we must use complex space. Since $\mathrm{tr}A_\mu = 0$, we have

$$\det D = \det\bar{D} = 1. \tag{2.7}$$

We now define[8] a matrix $J$ by

$$J \equiv D\bar{D}^{-1} . \tag{2.8}$$

Clearly $\det J = 1$. The remaining self-duality equation (2.5b) can be written as

$$(J^{-1}J_{,y})_{,\bar{y}} + (J^{-1}J_{,z})_{,\bar{z}} = 0 .$$

In summary, in this formalism the self-dual equations (2.3) are replaced by the following two sets of matrix equations:

$$B_y \equiv J^{-1}J_{,y}, \quad B_z \equiv J^{-1}J_{,z} \tag{2.9a}$$

and

$$\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z = 0 . \tag{2.9b}$$

Note that gauge transformations are given by

$$D \to DG, \quad \bar{D} \to \bar{D}G \tag{2.10a}$$

and

$$A_\mu \to G^{-1}A_\mu G + G^{-1}\partial_\mu G , \tag{2.10b}$$

where the matrix $G$ satisfies $\det G = 1$. Moreover the matrices $D$ and $\bar{D}$ are determined only up to the transformation

$$D \to \bar{M}(\bar{y},\bar{z})D , \tag{2.11a}$$

$$\bar{D} \to M(y,z)\bar{D} , \tag{2.11b}$$

where $M$ and $\bar{M}$ are $N \times N$ matrices depending on the variables indicated and with unit determinant. The transformation (2.11) has no effect on the gauge potentials. Clearly the matrix $J$ is *gauge invariant* and transforms as

$$J \to \bar{M}JM^{-1} \tag{2.12}$$

under (2.11).

We conclude this section with a brief discussion of the reality conditions. For an SU($N$) gauge theory the matrices $A_\mu$ must be anti-Hermitian for real $x_\mu$. This is usually achieved by requiring

$$D \doteq (\bar{D}^\dagger)^{-1} \text{ and } GG^\dagger \doteq I , \tag{2.13}$$

where the symbol $\doteq$ is used for equations valid only on real space. However, this is a sufficient condition and not a necessary one. Since we are using complex space, it is often natural and useful[9] to allow complex, i.e., SL($N,C$), gauge transformations even in real space. A necessary and sufficient condition that we can choose a gauge so that the potentials are real is[9] the existence of a matrix $M(y,z)$ depending only on $y$ and $z$ and with a unit determinant such that

$$JM \doteq \text{positive-definite Hermitian matrix.} \tag{2.14}$$

## III. DERIVATION OF INFINITE NUMBER OF NONLOCAL CONSERVATION LAWS AND THE LINEAR SYSTEM

Notice that "gauge potentials" $B_y, B_z$ defined by Eq. (2.9a) are curvatureless and the self-dual equation (2.9b) is a continuitylike equation. They are very similar to the chiral equations.[1] We thus can use a method similar to Ref. 10 to construct infinite nonlocal currents. Consider $B_y$ and $B_z$ of Eq. (2.9b) to be the first conserved currents:

$$V_y^{(1)} \equiv B_y = \partial_{\bar{z}}\chi^{(1)}, \quad V_z^{(1)} \equiv -\partial_{\bar{y}}\chi^{(1)} . \tag{3.1}$$

$\chi^{(1)}$ exists because of Eq. (2.9b). Now we suppose that the $n$th currents exist, i.e.,

$$\partial_{\bar{y}}V_y^{(n)} + \partial_{\bar{z}}V_z^{(n)} = 0 , \tag{3.2}$$

$$V_y^{(n)} = \partial_{\bar{z}}\chi^{(n)}, \quad V_z^{(n)} = -\partial_{\bar{y}}\chi^{(n)} . \tag{3.3}$$

Then the $(n+1)$th currents are

$$V_y^{(n+1)} = \mathfrak{D}_y\chi^{(n)}, \quad V_z^{(n+1)} = \mathfrak{D}_z\chi^{(n)} , \tag{3.4}$$

where $\mathfrak{D}_u \equiv \partial_u + B_u$, $u = y, z$.

Now we show that the $V^{(n+1)}$'s are conserved:

$$\begin{aligned}
\partial_{\bar{y}}V_y^{(n+1)} + \partial_{\bar{z}}V_z^{(n+1)} &= (\partial_{\bar{y}}\mathfrak{D}_y + \partial_{\bar{z}}\mathfrak{D}_z)\chi^{(n)} \text{ from Eq. (3.4)} \\
&= (\mathfrak{D}_y\partial_{\bar{y}} + \mathfrak{D}_z\partial_{\bar{z}})\chi^{(n)} \text{ due to Eq. (2.9b)} \\
&= -\mathfrak{D}_y V_z^{(n)} + \mathfrak{D}_z V_y^{(n)} \text{ using Eq. (3.3)} \\
&= (-\mathfrak{D}_y\mathfrak{D}_z + \mathfrak{D}_z\mathfrak{D}_y)\chi^{(n+1)} \text{ using Eq. (3.4)} \\
&= 0 \text{ due to Eq. (2.9a)} .
\end{aligned}$$

Now we can linearize the self-dual Yang-Mills equation (2.9b), using the nonlocal currents, in the following way. From Eqs. (3.3) and (3.4),

$$\partial_{\bar{z}}\chi^{(n)} = \mathfrak{D}_y\chi^{(n-1)} , \tag{3.5a}$$

$$-\partial_{\bar{y}}\chi^{(n)} = \mathfrak{D}_z\chi^{(n-1)} . \tag{3.5b}$$

Multiplying Eqs. (3.5a) and (3.5b) by $\lambda^n$ ($\lambda$ being a complex paramter), summing over $n$, and defining

$$\chi \equiv \sum_{n=0}^{\infty} \lambda^n \chi^{(n)}$$

we obtain

$$\partial_{\bar{z}}\chi = \lambda\mathfrak{D}_y\chi , \tag{3.6a}$$

$$-\partial_{\bar{y}}\chi = \lambda\mathfrak{D}_z\chi . \tag{3.6b}$$

To show that these equations are indeed a linear system for the self-dual Yang-Mills equation, we need to show that the integrability of $\chi$ from Eqs. (3.6a) and (3.6b) gives Eqs. (2.9a) and (2.9b). Equations (3.6a) and (3.6b) can be rewritten as

$$(\partial_{\bar{z}} - \lambda\partial_y)\chi = \lambda B_y\chi , \tag{3.7a}$$

$$-(\partial_{\bar{y}} + \lambda\partial_z)\chi = \lambda B_z\chi . \tag{3.7b}$$

Differentiating Eq. (3.7a) by $\bar{y}$, and Eq. (3.7b) by $\bar{z}$, we obtain, after some simple manipulations,

$$\lambda(\mathfrak{D}_y\mathfrak{D}_z - \mathfrak{D}_z\mathfrak{D}_y)\chi + (\partial_{\bar{y}}B_y + \partial_{\bar{z}}B_z)\chi = 0\,. \quad (3.8)$$

For Eq. (3.8) to be true for all $\lambda$, we need $\mathfrak{D}_y\mathfrak{D}_z - \mathfrak{D}_z\mathfrak{D}_y = 0$, $\partial_{\bar{y}}B_y + \partial_{\bar{z}}B_z = 0$, which are precisely Eqs. (2.9a) and (2.9b). Therefore, Eqs. (3.7a) and (3.7b) are the linearized equations for Eqs. (2.9a), (2.9b), (3.7a), and (3.7b).

We shall now discuss some properties of the solutions of the linear equations (3.7). Clearly $\chi$ is invariant under gauge transformations. The matrix $\chi$ is not determined uniquely by Eqs. (3.7). To see this nonuniqueness, let $\chi_1$ and $\chi_2$ be two solutions of Eq. (3.7). Then

$$\partial_{\bar{y}}(\chi_1^{-1}\chi_2) = \lambda\chi_1^{-1}(\mathfrak{D}_z\chi_1)\chi_1^{-1}\chi_2 - \lambda\chi_1^{-1}\mathfrak{D}_z\chi_2$$
$$= -\lambda\partial_z(\chi_1^{-1}\chi_2)$$

so that

$$(\partial_{\bar{y}} + \lambda\partial_z)(\chi_1^{-1}\chi_2) = 0\,,$$

and similarly

$$(\partial_{\bar{z}} - \lambda\partial_{\bar{y}})(\chi_1^{-1}\chi_2) = 0\,.$$

Therefore

$$\chi_2 = \chi_1 A(\lambda\bar{z} + y, \lambda\bar{y} - z, \lambda)\,, \quad (3.9)$$

where $A$ is an arbitrary matrix function of the variables $\lambda\bar{z} + y$, $\lambda\bar{y} - z$, and $\lambda$. Taking the trace of Eqs. (3.7) and using $\det J = 1$ (equivalently $\mathrm{Tr}B_y = 0$, etc.) we see that $\det\chi$ is a function of $\lambda\bar{z} + y$, $\lambda\bar{y} - z$, and $\lambda$ only. Thus we can always normalize $\chi$ to satisfy

$$\det\chi = 1\,. \quad (3.10)$$

Note that Eq. (3.10) is consistent with the requirement $\chi(\lambda = 0) = I$, $\chi^{(0)} = I$, which is implicit in our definition of $\chi$.

Another property is that $\chi^{-1}(\lambda)$ and $\chi^\dagger(-1/\bar{\lambda})J$ satisfy the same set of the equations. We can see this by the following arrangements. For $\chi$ satisfying Eqs. (3.7a) and (3.7b), $\chi^{-1}$ satisfy the following corresponding equations:

$$\chi^{-1}(3.7\text{a})\chi^{-1} \to \chi^{-1}[(\partial_{\bar{z}} - \lambda\partial_y)\chi]\chi^{-1} = \lambda\chi^{-1}J^{-1}J_y$$

or

$$-(\partial_{\bar{z}} - \lambda\partial_y)\chi^{-1} = \lambda\chi^{-1}J^{-1}J_y\,. \quad (3.11\text{a})$$

Similarly,

$$\chi^{-1}(3.7\text{b})\chi^{-1} \to \chi^{-1}[(\partial_{\bar{y}} + \lambda\partial_z)\chi]\chi^{-1} = \lambda\chi^{-1}J^{-1}J_z$$

or

$$(\partial_{\bar{y}} + \lambda\partial_z)\chi^{-1} = \lambda\chi^{-1}J^{-1}J_z\,. \quad (3.11\text{b})$$

Now we take Hermitian conjugates of Eqs. (3.7a) and (3.7b):

$$(3.7\text{a})^\dagger \to (\partial_z - \bar{\lambda}\partial_{\bar{y}})\chi^\dagger = \bar{\lambda}\chi^\dagger J_{,\bar{y}}J^{-1}\,,$$

where $J^\dagger = J$ is used, or

$$[(\partial_z - \lambda\partial_{\bar{y}})\chi^\dagger]J - \bar{\lambda}\chi^\dagger J_{,z} = 0\,,$$

or

$$(\partial_z - \bar{\lambda}\partial_{\bar{y}})(\chi^\dagger J) - (\chi^\dagger J)J^{-1}J_{,z} = 0\,. \quad (3.12\text{a})$$

Similarly,

$$(3.7\text{b})^\dagger \to -(-\bar{\lambda}\partial_{\bar{z}} - \partial_y)(\chi^\dagger J) - (\chi^{-1}J)J^{-1}J_{,y} = 0\,. \quad (3.12\text{b})$$

Comparing Eq. (3.11a) with Eq. (3.10b), and Eq. (3.11b) with Eq. (3.10a), we see that $\chi^{-1}(\lambda)$ and $\chi^\dagger(-1/\bar{\lambda})J$ satisfy the same equations. From Eq. (3.9), we have

$$\chi^\dagger(-1/\bar{\lambda})J = A(\lambda\bar{z} + y, \lambda\bar{y} - z, \lambda)\chi^{-1}(\lambda)\,. \quad (3.13)$$

## IV. SOLUTION TO THE LINEAR SYSTEM FOR THE $n$ INSTANTON WITH $(5n+4)$ PARAMETERS AND SOLUTIONS IN THE $A_1$ *ANSATZ* OF THE $R$-GAUGE FORMULATION

The $R$ gauge was discussed in detail in Refs. 7 and 8. For completeness we briefly discuss it again. In the $R$ gauge for SU(2) gauge theory, the matrices $D$ and $\bar{D}$ are chosen to be lower and upper triangular, respectively,

$$D = \frac{1}{\sqrt{\phi}}\begin{pmatrix} 1 & 0 \\ \rho & \phi \end{pmatrix}, \quad \bar{D} = \frac{1}{\sqrt{\phi}}\begin{pmatrix} \phi & -\bar{\rho} \\ 0 & 1 \end{pmatrix}, \quad (4.1)$$

and

$$J \equiv D\bar{D}^{-1} = \frac{1}{\phi}\begin{pmatrix} 1 & \rho \\ \rho & \phi^2 + \rho\bar{\rho} \end{pmatrix}. \quad (4.2)$$

The self-dual equations (2.9b) become

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\ln\phi + \frac{\rho_{,y}\bar{\rho}_{,\bar{y}} + \rho_{,z}\bar{\rho}_{,\bar{z}}}{\phi^2} = 0\,, \quad (4.3\text{a})$$

$$\left(\frac{\rho_{,y}}{\phi^2}\right)_{,\bar{y}} + \left(\frac{\rho_{,z}}{\phi^2}\right)_{,\bar{z}} = 0\,,$$

and (4.3b)

$$\left(\frac{\bar{\rho}_{,\bar{y}}}{\phi^2}\right)_{,y} + \left(\frac{\bar{\rho}_{,\bar{z}}}{\phi^2}\right)_{,z} = 0\,.$$

In the Atiyah-Ward $A_1$ *Ansatz*, which also coincides with the Corrigan-Fairlie-'t Hooft-Wilczek *Ansatz*,[11]

$$\rho_{,y} = \phi_{,\bar{z}}\,, \quad \rho_{,z} = -\phi_{,\bar{y}}\,,$$
$$\bar{\rho}_{,\bar{y}} = \phi_{,z}\,, \quad \bar{\rho}_{,\bar{z}} = -\phi_{,y}\,, \quad (4.4)$$

which automatically satisfies two of the self-dual equations (4.3b), and the third self-dual equation (4.3a) becomes

$$\frac{1}{\phi}\Box\phi=0\,. \tag{4.5}$$

Now for Eqs. (2.9a), (3.7a), and (3.7b) we put $J$ in the $A_1$ *Ansatz*, i.e.,

$$J^{-1}J_{,y}=\begin{bmatrix}(\bar\rho/\phi)_{,\bar z} & (\bar\rho^2/\phi)_{,\bar z}+\bar\rho_{,y}\\ -(1/\phi)_{,\bar z} & -(\bar\rho/\phi)_{,\bar z}\end{bmatrix},$$

and similarly for $J^{-1}J_{,z}$. It turned out that we could solve for $\chi$,

$$\chi=I+\frac{1}{\phi}\begin{bmatrix}\lambda\bar\rho & \lambda\bar\rho\theta+(\theta-\bar\rho)\phi\\ -\lambda & \lambda\phi\end{bmatrix}, \tag{4.6}$$

where $\theta$ has the expansion $\theta=\sum_0^\infty\lambda^n\theta^{(n)}$ and satisfies

$$(\theta-\bar\rho)_{,\bar z}=\lambda\theta_{,y}\,, \tag{4.7a}$$

$$(\theta-\bar\rho)_{,\bar y}=-\lambda\theta_{,z}\,. \tag{4.7b}$$

The $(5n+4)$-paramter instanton solution is the multi-instanton solution for Eq. (4.5), i.e.,

$$\phi=1+\sum_{i=1}^{n}c_i/R_i^{\,2}\,, \tag{4.8a}$$

where $R_i=y_i\bar y_i+z_i\bar z_i$, $y_i\equiv y-y_i^{(0)}, z_i\equiv z-z_i^{(0)}$, $y_i^{(0)}$ and $z_i^{(0)}$ are complex numbers, and $c_i$ are real numbers.

First we can solve the $A_1$ *Ansatz* [Eq. (4.4)],

$$\rho=-\sum_{i=1}^{n}\frac{c_i^{\,2}}{R_i^{\,2}}\frac{y_i}{\bar z_i}\,, \tag{4.8b}$$

$$\bar\rho=-\sum_{i=1}^{n}\frac{c_i^{\,2}}{R_i^{\,2}}\frac{\bar y_i}{z_i}\doteq\rho^*\,. \tag{4.8c}$$

Substituting $\bar\rho$ of Eq. (4.8c) into Eqs. (4.5a) and (4.5b), we obtain

$$\theta=\sum_{i=1}^{n}\frac{c_i}{R_i}\frac{\bar y_i}{\lambda\bar y_i-z_i}. \tag{4.9}$$

Note that for these solutions $\det\chi=1$, $\chi(\lambda=0)=I$. It is also interesting to note that the $\chi$ has poles in $\lambda$ with locations depending upon coordinates. The infinite number of conserved quantities $\chi^{(n)}$ is precisely the coefficient of $\lambda^n$ in the power-series expansion of $\chi(\lambda)$.

## V. CONNECTION WITH THE BELAVIN-ZAKHAROV LINEAR SYSTEM

Now we want to express Eqs. (3.7a) and (3.7b) in terms of the potentials $A_u$, $u=y,\bar y,z,\bar z$, which are expressed in $D,\bar D$ matrices in Eq. (2.6). From Eq. (2.8) $J\equiv D\bar D^{-1}$, so

$$\begin{aligned}J^{-1}J_{,y}&=\bar D(D^{-1}D_{y}-\bar D^{-1}\bar D_{,y})\bar D^{-1}\\&=\bar D(A_y-\bar D^{-1}D_{,y})\bar D^{-1}\,,\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\partial_y\chi&=\partial_y(\bar D\bar D^{-1}\chi)\\&=\bar D\bar D^{-1}(\partial_y\bar D)\bar D^{-1}\chi+\bar D\partial_y(\bar D^{-1}\chi)\,,\end{aligned} \tag{5.2}$$

$$\begin{aligned}\partial_{\bar z}\chi&=\partial_{\bar z}(\bar D\bar D^{-1}\chi)\\&=\bar D\bar D^{-1}(\partial_{\bar z}\bar D)\bar D^{-1}\chi+\bar D\partial_{\bar z}(\bar D^{-1}\chi)\\&=\bar DA_{\bar z}(\bar D^{-1}\chi)+\bar D\partial_{\bar z}(\bar D^{-1}\chi)\,.\end{aligned} \tag{5.3}$$

Substituting Eq. (5.1), (5.2), and (5.3) into the linear equation (3.7a), $(\partial_{\bar z}-\lambda\partial_z)\chi=\lambda J^{-1}J_{,y}\chi$, we obtain

$$A_{\bar z}(\bar D^{-1}\chi)+\partial_{\bar z}(\bar D^{-1}\chi)-\lambda\partial_y(\bar D^{-1}\chi)=\lambda A_y(\bar D^{-1}\chi)\,. \tag{5.4}$$

Now, defining

$$\psi\equiv D^{-1}\chi\,,$$

Eq. (5.4) becomes

$$(\lambda A_y-A_{\bar z})\psi=(-\lambda\partial_y+\partial_{\bar z})\psi\,. \tag{5.6a}$$

For the other linear equation (3.7b), we obtain

$$(\lambda A_z+A_{\bar y})\psi=-(\lambda\partial_z+\partial_{\bar y})\psi\,. \tag{5.6b}$$

These are the linear equations of Belavin and Zakharov.[5]

Note that $\psi$ is not gauge invariant, i.e., $\psi\to G^{-1}\psi$ under the gauge transformation (2.10). As with $\chi$, Eqs. (3.9) and (3.12), we can derive the corresponding relations to $\psi$:

$$\psi_1^{-1}\psi_2=B(\lambda\bar z+y,\lambda\bar y-z,\lambda)\,, \tag{5.7}$$

where $\psi_1$ and $\psi_2$ are both solutions to Eqs. (5.6a) and (5.6b), and $B$ is an arbitrary matrix function of the variables indicated. After operations similar to Eqs. (3.10a), (3.10b), (3.11a), and (3.11b), we find that $\psi^{-1}(\lambda)$ and $\psi^\dagger(-1/\bar\lambda)$ satisfy the same set of equations. Therefore, from Eq. (5.7), we obtain

$$\psi^\dagger(-1/\bar\lambda)\psi(\lambda)=B(\lambda\bar z+y,\lambda\bar y-z,\lambda)\,. \tag{5.8}$$

In the $A_1$ *Ansatz*, from Eqs. (4.1) and (4.6) for $\bar D$ and $\chi$, respectively, we can calculate

$$\psi=\bar D^{-1}\chi=\frac{1}{\sqrt\phi}\begin{bmatrix}1 & \theta\\ -\lambda & \phi-\lambda\theta\end{bmatrix}. \tag{5.9}$$

For the $n$-instanton solution, $\phi$ and $\theta$ are given by Eqs. (4.8) and (4.9), respectively. So $\psi$, as a function of $\lambda$, has poles depending on coordinates and it is well behaved at $\lambda=0$.

In the $A_1$ *Ansatz*, using Eq. (5.9), Eq. (5.8) becomes

$$\begin{aligned}\psi^\dagger(-1/\bar\lambda)\psi(\lambda)&=\begin{bmatrix}0 & \lambda^{-1}\\ -\lambda^{-1} & \phi+\frac{1}{\lambda}\bar\theta(-1/\bar\lambda)-\lambda\theta(\lambda)\end{bmatrix}\\&=\begin{bmatrix}0 & \lambda^{-1}\\ -\lambda & 1-\frac{\lambda c_i}{(y+\lambda\bar z)(z-\lambda\bar y)}\end{bmatrix}\end{aligned} \tag{5.10}$$

for the one-instanton case of Eqs. (4.8) and (4.9).

Now we compare our $\psi$ of Eqs. (4.8) and (5.9), with the one-instanton $\psi$ given in Ref. 5 by Belavin and Zakharov:

$$\psi^{\rm BZ}_{1(\lambda)}=\frac{1}{(2R^2+1)^{1/2}}\times\begin{bmatrix}(R^2+1)+\lambda\bar{y}\bar{z}+yz/\lambda & \lambda\bar{y}^2-z^2/\lambda\\ -\lambda\bar{z}^2+y^2/\lambda & (R^2+1)-\lambda\bar{y}\bar{z}-yz/\lambda\end{bmatrix}. \qquad (5.11)$$

It has only fixed poles in $\lambda$. It diverges both at $\lambda=0$ and $\infty$, and thus is unsuitable for expansion either in power series of $\lambda^n$ or $\lambda^{-n}$ to obtain the infinite conservation laws. Note also that

$$[\psi^{\rm BZ}(\lambda)]^{-1}=[\psi^{\rm BZ}(-1/\bar{\lambda})]^{\dagger}, \qquad (5.12)$$

which corresponds to a special choice of $B=1$ in Eq. (5.7).

## VI. GEOMETRIC CONSTRUCTION OF THE LINEAR SYSTEM

It has been shown by Ward[6] that self-dual Yang-Mills fields in (compactified) Euclidean space can be reformulated in terms of certain gauge fields defined on the complex projective space CP.[3] Mathematically this means that self-dual or anti-self-dual gauge fields correspond in a one-to-one fashion to certain vector bundles over $\mathrm{CP}^3$, which has been studied by mathematicians. In this section we shall describe some results[6,12] of this approach and its relation to our earlier discussion.

In Sec. II, the self-dual Yang-Mills equations are are viewed as two curvatureless conditions $f_{yz}=0$ and $f_{\bar{y}\bar{z}}=0$ on the $yz$ and $\bar{y}\bar{z}$ planes, plus a third constraint equation $f_{y\bar{y}}+f_{z\bar{z}}=0$, see Eqs. (2.5a) and (2.5b). Actually there are an infinite number of such planes. All those planes passing through a given point can be characterized by a free complex parameter $\lambda$ such that the three equations are encompassed in one equation[13]

$$F_{(y-\lambda\bar{z})(z+\lambda\bar{y})}=0\,. \qquad (6.1)$$

Since Eq. (6.1) holds for all values of $\lambda$, the $\lambda^0$ term gives $f_{yz}=0$, the $\lambda^2$ term gives $f_{\bar{z}\bar{y}}=0$, and the $\lambda^1$ term gives $f_{y\bar{y}}+f_{z\bar{z}}=0$. From Eq. (6.1), which can be interpreted as curvatureless in the complex variables $y-\lambda\bar{z}$ and $z+\lambda\bar{y}$, the potentials must be the following form:

$$A_{(y-\lambda\bar{z})}=\psi\partial_{(y-\lambda\bar{z})}\psi^{-1}\,, \qquad (6.2a)$$

$$A_{(z+\lambda\bar{y})}=\psi\partial_{(z+\lambda\bar{y})}\psi^{-1}\,, \qquad (6.2b)$$

or

$$(\lambda A_y-A_{\bar{z}})\psi=-(\lambda\partial_y-\partial_{\bar{z}})\psi\,,$$

$$(\lambda A_z+A_{\bar{y}})\psi=-(\lambda_z+\partial_{\bar{y}})\psi\,.$$

Since the gauge potentials are traceless, $\det\psi=1$. These are precisely the linearized equations of Belavin and Zakharov.[5] This observation is also the basis of Ward's formulation.

It is convenient ot represent the points in the (complex) Euclidean space by a $2\times2$ matrix

$$x=x_4+i\sigma\cdot x=\begin{bmatrix}x_4+ix_3 & x_2+ix_1\\ -x_2+ix_1 & x_4-ix_3\end{bmatrix}=\sqrt{2}\,i\begin{bmatrix}z & \bar{y}\\ y & -\bar{z}\end{bmatrix}. \qquad (6.3)$$

Note that $\det x=x_\mu x_\mu$. Let us consider two spinors $\omega$ and $\pi$,

$$\omega=\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\text{ and }\pi=\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}. \qquad (6.4)$$

Then it can be shown that the equation

$$\omega=x\pi\,, \qquad (6.5)$$

two linear equations for $x_\mu$, describe the so-called $\beta$ plane, i.e., if $\xi_\mu$ and $\eta_\mu$ are only two vectors in this plane then $\xi_\mu\xi_\mu=\eta_\mu\eta_\mu=0$ and the tensor $\xi_\mu\eta_\nu-\xi_\gamma\eta_\mu$ is self-dual. Clearly $(\omega,\pi)$ and $(c\omega,c\pi)$ for any nonzero complex number $c$ describes the same $\beta$ plane. Thus the space of $\beta$ planes is the complex projective space $\mathrm{CP}^3$. Each $\beta$ plane contains a unique *real* point given by

$$x^{\alpha\alpha'}=\frac{\pi^\alpha\bar{\omega}^{\alpha'}+\bar{\pi}^\alpha\omega^{\alpha'}}{\pi_\lambda\bar{\pi}^\lambda},\quad \alpha,\ \alpha'=1,\ 2 \qquad (6.6)$$

where $\bar{\pi}^\alpha\equiv(\pi_\alpha)^*\equiv(\epsilon_{\alpha\beta}\pi^\beta)^*$, $\epsilon_{11}=\epsilon_{22}=0$, and $\epsilon_{12}=-\epsilon_{21}=1$. The $\beta$ planes through a given point $x$ are specified by $(\pi_1,\pi_2)$, and since $(\pi_1,\pi_2)$ and $(c\pi_1,c\pi_2)$ describe the same $\beta$ plane, the space is just $\mathrm{CP}^1$ or the Riemann sphere. Let us parametrize the $\beta$ planes by $\lambda=-\pi_2/\pi_1$. Ward used this correspondence between $\mathrm{CP}^3$ and $R^4$ to show the equivalence between self-dual gauge fields in $R^4$ (more precisely on $S^4$) and certain "analytic vector bundles" over $\mathrm{CP}^3$. The $\mathrm{CP}^3$ gauge potentials (or the connection) are obtained from the $R^4$ potentials by a "change of coordinates" (or pullback) via Eq. (6.6). Then the self-duality of the $R^4$ gauge fields can be shown to be equivalent to the vanishing of certain components of the corresponding $\mathrm{CP}^3$ gauge fields. The vector bundle over $\mathrm{CP}^3$ is characterized by transition function corresponding to suitable coordinate patches on $\mathrm{CP}^3$. Note the $R^4$ gauge field is self-dual if and only if the components of the field tensor vanishes along the $\beta$ planes [this statement is just Eq. (6.1)]. The $CP^3$ "transition function" is then given in terms of the "integrable phase factor in the $\beta$ planes." This transition function $g(\omega,\pi) = g(\omega_1,\omega_2,\pi_1,\pi_2)$ is a $2\times 2$ matrix [for SU(2) theory] for which (i) $g$ is a homogeneous function of degree zero of $\omega_1,\omega_2,\pi_1$, and $\pi_2$, analytic in a suitable region; (ii) $\det g=1$; (iii) the function $g(x\pi,\pi)$, which by homogeniety is a function of $\lambda\bar{z}+y$, $\lambda\bar{y}-z$, and $\lambda(=-\pi_2/\pi_1)$ only, is analytic in a suitable annular region in $\lambda$; (iv) $g$ can be "split"

$$g(x\pi,\pi)=\psi_\infty(x,\lambda)^{-1}\psi_0(x,\lambda)\,, \tag{6.7}$$

where $\psi_\infty$ and $\psi_0$ are analytic in suitable regions containing $\lambda=\infty$ and 0, respectively. In addition $g$ has some "reality properties" corresponding to $A_\mu^a=$ real. On the other hand given a matrix $g$ with the above properties the $R^4$ gauge potentials are given by[5,14]

$$\lambda A_y - A_{\bar{z}} = \psi_\infty(\lambda\partial_y - \partial_{\bar{z}})\psi_\infty^{-1} = \psi_0(\lambda\partial_y - \partial_{\bar{z}})\psi_0^{-1}\,, \tag{6.8a}$$

$$\lambda A_z + A_{\bar{y}} = \psi_\infty(\lambda\partial_z + \partial_{\bar{y}})\psi_\infty^{-1} = \psi_0(\lambda\partial_z + \partial_{\bar{y}})\psi_0^{-1}\,, \tag{6.8b}$$

where $\psi_\infty$ and $\psi_0$ are given by (6.7). The second inequality in Eqs. (6.8a) and (6.8b) follows from an argument similar to that used for Eq. (3.9). A version of Liouville's theorem can be used to show that the right-hand sides of Eqs. (6.8a) and (6.8b) are linear in $\lambda$, which can therefore be used to define the potentials as in (6.8a) and (6.8b). The self-duality then follows from the consistency of these equations. The factorization (6.7) of $g$ is not unique. However, using the Liouville theorem it can be shown that this ambiguity is precisely the gauge transformation of the potentials defined by (6.8). Finally, two transition matrices $g$ and $g'=g_L g g_R$, where the matrices $g_L$ and $g_R$ are functions of $\lambda\bar{z}+y$, $\lambda\bar{y}-z$, and $\lambda$, and analytic around $\lambda=\infty(\pi_1=0)$ and $\lambda=0(\pi_1=0)$, respectively, are "equivalent," i.e., they give gauge equivalent potentials.

We can now relate the Atiyah Ward approach to the discussion of previous sections. Let us choose $\pi=(1,\pi_2/\pi_1)\equiv(1,-\lambda)$ for $\lambda\neq\infty$ or $\pi_1\neq 0$, then the $\beta$ planes are given by

$$dx\pi=0\,,$$

i.e.,

$$\begin{pmatrix} dz & d\bar{y} \\ dy & -d\bar{z}\end{pmatrix}\begin{pmatrix}1\\ \lambda\end{pmatrix}=\begin{pmatrix} dz-\lambda d\bar{y}\\ dy+\lambda d\bar{z}\end{pmatrix}=0\,. \tag{6.9}$$

Note that for $\pi_2\neq 0$, we need to take $\pi=(\pi_1/\pi_2,1) = (-1/\lambda,1)$. Thus the planes $y,z=$ const and $\bar{y},\bar{z}=$ const are the $\beta$ planes corresponding to $\lambda=0$ and $\infty$, respectively. From this observation and Eq. (2.8) we see that, apart from the matrices $M$ and $\bar{M}$ [see Eq. (2.11)] $D$ and $\bar{D}$ can be identified with $\psi_\infty(\lambda=\infty)$ and $\psi_0(\lambda=0)$, respectively. Furthermore, if we ignore the analyticity (in $\lambda$) properties of $\psi_0$ and $\psi_\infty$, then the Eqs. (6.8) are just the linear equations of Belavin and Zakharov.[5] The solutions of Belavin and Zakharov [see Eq. (5.11)] and also those of Arinshtein[13] do not have the analyticity properties of either $\psi_0$ or $\psi_\infty$. However the solution given in Eq. (5.9) has the required analyticity property of $\psi_0$ and then $\psi^\dagger(-1/\bar{\lambda})^{-1}$ has the analyticity property of $\psi_\infty$, and the matrix $\psi^\dagger(-1/\bar{\lambda})\times\psi(\lambda)$ given by Eq. (5.10) is equivalent to the transition matrix of the $A_1$ *Ansatz* of Atiyah and Ward.[5]

## VII. DISCUSSION

A great deal of results, both classical and quantum, have been obtained in the study of two-dimensional exactly integrable systems[15] using inverse scattering methods. As we have seen, the self-dual Yang-Mills field possesses many of these features. The connection with the linear system of Belavin and Zakharov[5] and the Atiyah-Ward[6] construction for the self-dual Yang-Mills equation are discussed. This connection provides a geometrical interpretation of the infinite number of nonlocal conservation laws.

One of the goals of the classical inverse-scattering method is the construction of solutions of the equation of motion. For the self-dual [SU(2)] Yang-Mills fields, Atiyah and Ward[5] proposed a series of *Ansätze*, a specific form of the transition matrix, for the construction of instantons. Though the Atiyah-Ward *Ansätze* has not been very useful for the construction of instantons, these *Ansätze* have recently been used to construct monopole solutions.[9] General instantons solutions for arbitrary gauge groups have been constructed by Atiyah *et al.*,[16] using a somewhat different method. Another result, which is important in the study of the quantum theory of the inverse scattering technique, is the construction of action-angle variables. However, it remains to be seen whether any analogous result can also be obtained for the self-dual gauge fields.


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