PHYSICAL REVIEW D VOLUME 24, NUMBER 6 15 SEPTEMBER 1981

Brownian motion of a mirror

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(Received 7 November 1980)

Einstein's analysis of the Brownian motion of a mirror in a field of natural radiation played a crucial role, in the first decade of quantum theory, in persuading physicists that the Maxwell description of the electromagnetic field was inadequate, and that light quanta, with discrete momentum as well as energy, have a real existence. Here an alternative analysis is given of this motion. We see that, if the spectral density of the radiation field is given by the Planck spectrum plus the zero-point electromagnetic field, that is, by the spectral density of Planck's second theory, then the motion is correctly described in terms of the Maxwell theory, without any need for a quantum hypothesis. An essential step in this analysis is to include Boyer's correction to the Einstein model, whereby account is taken of the radiative energy loss each time the mirror collides with the walls of the cavity.

I. INTRODUCTION

Einstein's paper of 1909,[1] analyzing the motion of a mirror in the field of radiation inside a cavity at temperature T, has become a classic. It is considered to be[2-5] the first convincing demonstration of a dual corpuscular-undulatory character for the electromagnetic field, and also the first indication that light quanta carry a momentum $\hbar\omega/c$ as well as an energy $\hbar\omega$.

There is some evidence that Einstein later found this result unsatisfactory[6] because he had hoped to prove that the mean-square impulse, $\langle\Delta^2\rangle_{\text{av}}$, on the mirror could be explained entirely by independent collisions with pointlike quanta of light. But his main difficulty at that time was to convince his colleagues that light quanta had any pointlike properties at all. It follows that the main thrust of his article was devoted to the pointlike term in $\langle\Delta^2\rangle_{\text{av}}$. His statement that the other term in $\langle\Delta^2\rangle_{\text{av}}$ is the result furnished by Maxwell's electromagnetic theory seemed so plausible that neither he nor anyone since has calculated this term explicitly.

One purpose of the present article is to fill that gap in Einstein's argument. We shall find that the Maxwell electromagnetic theory does indeed give just the "wavelike" term in $\langle\Delta^2\rangle_{\text{av}}$. It becomes clear why Einstein found this term something of an embarrassment, if we compare his expression for the mean-square impulse on a mirror with the expression he found in 1916[7] for the mean-square impulse on a two level atom. In the latter case, only the pointlike term is present. This result fits much more closely with the assessment Einstein offered at the 1909 Salzburg conference,[8] that the Maxwell equations should be satisfied only for beams of radiation containing large densities of light quanta.

There is further evidence that Einstein and Hopf[9] considered the Maxwell theory inadequate in their article the following year. This article was crucial in determining the outcome, in Einstein's favor, of a long polemic with Planck over the latter's attempts at basing his radiation formula on classical electromagnetism. Einstein[10] and Ehrenfest[11] had pointed out that it was not possible to justify Planck's choice of an entropy function for radiation by considering the equilibrium of a linear resonator in the radiation field, and that to find the distribution of energy over frequency, it would be necessary to study some system which is capable of emitting radiation at several different frequencies. Einstein and Hopf pointed out that a massive particle carrying a linear oscillator constitutes just such a system, owing to the Doppler shift of frequency arising from the particle's motion. By studying the impulse transferred to such a system from the radiation field they were able to show that a mean particle energy of $\frac{1}{2}kT$ required the divergent Rayleigh spectrum rather than the experimentally observed Planck spectrum.

Boyer, however, has pointed out[12] that Einstein and Hopf's analysis overlooked an important part of the electromagnetic interaction, namely that caused by the large accelerations which the particle receives during collisions with the walls of the cavity. He has suggested a correction to Einstein and Hopf's equation of momentum balance, arising from the loss of energy through dipole radiation during such collisions.

We shall see in this article that a similar correction can be made in Einstein's analysis of the moving mirror. This result confirms the conclusion arrived at by Boyer for the Einstein-Hopf resonator. Both results indicate that it is the blackbody spectrum of Planck's second theory,[13] rather than the Rayleigh spectrum, which is furnished by the Maxwell theory. This blackbody spectrum differs from the original Planck spectrum by the addition of a temperature-independent term, which may be attributed to an energy of $\frac{1}{2}\hbar\omega$ in each cavity mode of oscillation. There is some independent evidence[14-17] for the existence of such a real "zero-point" spectrum, and it has recently been suggested[18-20] that it provides an explanation for the results of the Freedman-Clauser experiment[21] which avoids the need for quantum nonlocality.

According to the results reported here, therefore, Einstein's victory over Planck, at any rate on the basis of black radiation alone, was prematurely awarded. Far from the Maxwell term in $\langle\Delta^2\rangle_{av}$ being expendable, as Einstein hoped, it is rather the pointlike term which may be dispensed with.

Of course, it may be argued that the successes of quantum theory in atomic spectra had already made such discussions obsolete after 1913, and especially after Einstein's new expression for $\langle\Delta^2\rangle_{av}$ in his paper of 1916.[7] It must be admitted that, although Planck's theory has had a modest revival in recent years under the title of *stochastic electrodynamics*,[14-17] it has not been possible to progress far beyond Planck's own efforts[22,23] in atomic spectra. For a recent attempt at the application of stochastic electrodynamics to the hydrogen atom, reference should be made to the article of Marshall and Claverie.[24]

Nevertheless, this reexamination of the Brownian motion of a mirror in a radiation field has, I submit, much besides historical importance. Einstein's formula for $\langle\Delta^2\rangle_{av}$ is still widely quoted in the textbooks as evidence for the complementarity concept applied to light quanta. It is never contrasted with the very different result for the two-level atom, and this in spite of the fact that the latter indicates an exclusively corpuscular character for radiation. Finally, this analysis shows that, while Planck himself did not succeed in his attempt to derive his radiation formula from Maxwell's theory, such a derivation, making use of Planck's hypothesis of natural radiation, is perfectly possible.

II. THE DIFFUSION AND DRIFT COEFFICIENTS

We consider a square mirror, of side a, which is constrained to be parallel to the xy plane and to move along the z axis. This mirror is perfectly reflecting for light in the (angular) frequency range $(\omega, \omega+d\omega)$, and perfectly transmitting outside this range.

In the Einstein model, the motion of the mirror is determined entirely by the radiation pressure on its two faces. Thus the equation of motion is

$$m v_{t+\tau} = m v_t + J , \tag{2.1}$$

where m is its mass, v_t and $v_{t+\tau}$ its velocity at times t and $t+\tau$, and J the impulse obtained by integrating the radiation pressure over the surface of the mirror and between t and $t+\tau$. It will be convenient to introduce the notation

$$J = J_L + J_R , \tag{2.2}$$

where J_L and J_R are the impulses arising from radiation in the $z<0$ and $z>0$ regions, respectively.

Einstein gave the following value for the diffusion coefficient $\bar{J}$:

$$\bar{J} = -P v_t \tau , \tag{2.3}$$

where

$$P = \frac{3a^2}{2c}\left(\rho - \tfrac{1}{3}\omega\frac{d\rho}{d\omega}\right)d\omega , \tag{2.4}$$

and ρ is the spectral density of the cavity radiation corresponding to frequency ω. This expression is valid for nonrelativistic values of v. It is stated without derivation in Einstein's original article, but it can be easily derived from the Doppler-shifted radiation spectrum, which is given in his paper of 1916.[7] There it is shown that the energy density of radiation with angle of incidence θ, and with the appropriate frequency range in the mirror's rest frame, is

$$F(\theta) = \frac{1}{4}\left(\rho + \frac{v}{c}\cos\theta\omega\frac{\partial\rho}{\partial\omega}\right)\left(1 - \frac{3v}{c}\cos\theta\right)d\omega . \tag{2.5}$$

We may deduce that

$$\langle J_L\rangle_{av} = a^2\tau\int_0^{\pi/2} 2\pi F(\theta) 2\cos^2\theta\sin\theta\, d\theta \tag{2.6}$$

and

$$\langle J_R\rangle_{av} = -a^2\tau\int_{\pi/2}^{\pi} 2\pi F(\theta) 2\cos^2\theta\sin\theta\, d\theta , \tag{2.7}$$

from which (2.3) is obtained.

The other quantity considered by Einstein is the drift coefficient

$$\langle\Delta^2\rangle_{av} = \langle (J - \langle J\rangle_{av})^2\rangle = 2\langle (J_L - \langle J_L\rangle_{av})^2\rangle . \tag{2.8}$$

For nonrelativistic velocities, it suffices to calculate this quantity for a stationary mirror. For further discussion of this point, reference should be made to Einstein and Hopf,[9] Einstein,[7] and Boyer.[12] The appropriate expression for the impulse is

$$J_L = 2\int_0^{\tau} d\tau \int_0^a dx \int_0^a dy\, T_{zz}^L , \tag{2.9}$$

where T_{zz}^L is the contribution to the Maxwell stress tensor from cavity radiation incident on the left face of the mirror. To obtain an expression for this tensor component, we must expand the cavity radiation field in plane waves:

$$\vec{E}(\vec{x},t)=\sum_{\vec{k},\lambda}\vec{E}_{\vec{k}\lambda}\cos(\vec{k}\cdot\vec{x}-kct+\theta_{\vec{k}\lambda}), \tag{2.10}$$

with a similar expansion for the magnetic field $\vec{B}$. Here the wave number $\vec{k}$ runs over a discrete set with density $(V/8\pi^3)d^3k$ (V is the volume of the cavity), and the polarization index λ takes one of two values. We shall suppose that the random variables $\theta_{\vec{k}\lambda}$ are independent and uniformly distributed in $(0, 2\pi)$, and that the random variables $\vec{E}_{k\lambda}$ are independent and hence[25-28] normally distributed, with zero mean. Then the stress tensor component is

$$\begin{aligned}T^L_{zz}=\frac{1}{8\pi}\sum_{\vec{k}\lambda}{}'\sum_{\vec{k}'\lambda'}{}'&(E_{z\vec{k}\lambda}E_{z\vec{k}'\lambda'}-E_{x\vec{k}\lambda}E_{x\vec{k}'\lambda'}-E_{y\vec{k}\lambda}E_{y\vec{k}'\lambda'}\\&+B_{z\vec{k}\lambda}B_{z\vec{k}'\lambda'}-B_{x\vec{k}\lambda}B_{x\vec{k}'\lambda'}-B_{y\vec{k}\lambda}B_{y\vec{k}'\lambda'})\\&\times\cos(\vec{k}\cdot\vec{x}-kct+\theta_{\vec{k}\lambda})\cos(\vec{k}'\cdot\vec{x}-k'ct+\theta_{\vec{k}'\lambda'}),\end{aligned} \tag{2.11}$$

where the prime on the summation indicates that it is confined to $\omega<kc<\omega+d\omega$ and $k_z>0$.

Substitution of (2.11) into (2.8) and (2.9) gives, after averaging over $\theta_{\vec{k}\lambda}$,

$$\langle\Delta^2\rangle_{\rm av}=\sum_{\vec{k}\lambda}{}'\sum_{\vec{k}'\lambda'(\vec{k}'\neq\vec{k})}{}' F(\vec{k},\vec{k}')\,\langle(E_{z\vec{k}\lambda}E_{z\vec{k}'\lambda'}-E_{x\vec{k}\lambda}E_{x\vec{k}'\lambda'}-E_{y\vec{k}\lambda}E_{y\vec{k}'\lambda'}+B_{z\vec{k}\lambda}B_{z\vec{k}'\lambda'}-B_{x\vec{k}\lambda}B_{x\vec{k}'\lambda'}-B_{y\vec{k}\lambda}B_{y\vec{k}'\lambda'})\rangle\,, \tag{2.12}$$

where

$$F(\vec{k},\vec{k}')=\frac{1}{16\pi^2}\int_0^\tau dt\int_0^\tau dt'\int_0^a dx\int_0^a dx'\int_0^a dy\int_0^a dy'\cos[\vec{k}\cdot(\vec{x}-\vec{x}')-kc(t-t')]\cos[\vec{k}'(\vec{x}-\vec{x}')-k'c(t-t')] \tag{2.13}$$

$$=\frac{2}{\pi^2}\left[\frac{\sin^2\frac{1}{2}a(k_x-k_x')}{(k_x-k_x')^2}\frac{\sin^2\frac{1}{2}a(k_y-k_y')}{(k_y-k_y')^2}\frac{\sin^2\frac{1}{2}c\tau(k-k')}{c^2(k-k')^2}+(\vec{k}\to-\vec{k},k\to-k)\right]. \tag{2.14}$$

To find the ensemble average in (2.12), we need the following averages for natural radiation[29]:

$$\sum_\lambda\langle E_{i\vec{k}\lambda}E_{j\vec{k}\lambda}\rangle=\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{8\pi^3c^3\rho}{\omega^2V}=\sum_\lambda\langle B_{i\vec{k}\lambda}B_{j\vec{k}\lambda}\rangle\,, \tag{2.15}$$

$$\sum_\lambda\langle E_{i\vec{k}\lambda}B_{j\vec{k}\lambda}\rangle\epsilon_{ijl}=\frac{2k_l}{k}\frac{8\pi^3c^3\rho}{\omega^2V}\,. \tag{2.16}$$

Then, after summation over λ, λ', the ensemble average in (2.12) takes the value

$$2\left(\frac{8\pi^3c^3\rho}{\omega^2V}\right)^2\left(\frac{kk'+k_zk_z'-k_xk_x'-k_yk_y'}{kk'}\right)^2. \tag{2.17}$$

Provided the bandwidth $d\omega$ of the mirror is large compared with both c/a and $1/\tau$, we may approximate (2.14) with the narrow-line approximation:

$$F(\vec{k},\vec{k}')=\frac{\pi a^2\tau}{4c}\,\delta(k_x-k_x')\delta(k_y-k_y')\delta(k-k')\,. \tag{2.18}$$

Then, replacing the integrations over $\vec{k}$ and $\vec{k}'$ by integrations,

$$\langle\Delta^2\rangle_{\rm av}=\frac{\pi a^2\tau}{2c}\left(\frac{c^3\rho}{\omega^2}\right)^2\int' d\vec{k}\int' d\vec{k}'\,\delta(k_x-k_x')\delta(k_y-k_y')\delta(k-k')\left(\frac{kk'+k_zk_z'-k_xk_x'-k_yk_y'}{kk'}\right)^2 \tag{2.19}$$

$$=\frac{\pi^2c^2\rho^2a^2\tau}{\omega^2}d\omega\,. \tag{2.20}$$

III. THE BLACKBODY SPECTRUM

Both the drift coefficient $\bar{J}$, given in (2.3), and the diffusion coefficient $\langle\Delta^2\rangle_{\rm av}$, given in (2.20), are simple functions of ρ, and are proportional to the product $f\tau$, where f is the area of the mirror. The simplicity of these expressions owes much to our having used the geometrical optical limit, in which diffraction effects at the mirror's edge are neglected. Also the proportionality to $f\tau$ is a consequence of the way in which the various plane-wave components of the radiation field interfere. For this property to hold, it is essential that we make the narrow-line approximation of (2.18), which means that both τ and a/c must be large compared with the mirror's reciprocal bandwidth.

Along with the basic equation of motion (2.1), Einstein made the assumption, following the argument of his original Brownian motion paper,[30] that the average kinetic energy of the mirror should take the constant value of $\frac{1}{2}kT$ in a cavity containing black radiation at temperature T. He deduced the relation

$$\langle \Delta^2 \rangle_{\text{av}} = 2kTP\tau \,. \tag{3.1}$$

If we substitute the values of $\langle \Delta^2 \rangle_{\text{av}}$ and P obtained in the previous section, we find that

$$\rho - \tfrac{1}{3}\,\omega \frac{d\rho}{d\omega} = \frac{\pi^2 c^3}{3\omega^2 kT}\,\rho^2 \,, \tag{3.2}$$

and, with the additional assumption that ρ is zero when ω is zero, this leads to the unique solution

$$\rho = \frac{\omega^2 kT}{\pi^2 c^3} \,, \tag{3.3}$$

which is the Rayleigh spectrum.

Between 1905 and 1911, Einstein, along with Jeans and Ehrenfest,[31] devoted a great deal of energy to proving that Planck's essentially classical treatment of blackbody radiation could lead consistently only to the Rayleigh spectrum. Their criticism centered on Planck's use of the *ad hoc* entropy function for radiation,

$$S(\rho) = k \int \frac{V\omega^2}{\pi^2 c^3}\left[\left(\frac{\pi^2 c^3 \rho}{\hbar\omega^3} + 1\right) \ln\left(\frac{\pi^2 c^3 \rho}{\hbar\omega^3} + 1\right) - \frac{\pi^2 c^3 \rho}{\hbar\omega^3} \ln\left(\frac{\pi^2 c^3 \rho}{\hbar\omega^3}\right)\right] d\omega \,. \tag{3.4}$$

It was with this entropy function, along with the thermodynamic relation

$$\frac{1}{V}\frac{\delta S}{\delta\rho} = \frac{1}{T} \,, \tag{3.5}$$

that Planck first obtained his blackbody spectrum.

In his efforts to justify (3.4), Planck used Boltzmann's method of "complexion counting" with a cell size, in phase space, of $2\pi\hbar$. However, Einstein and Ehrenfest claimed that such a procedure implied discontinuity in the emission and absorption of light, thereby undermining Planck's other assumption that the processes are governed by the Maxwell theory.

Einstein and Hopf's paper,[9] in 1911, was crucial. They proved that Planck's hypothesis of natural radiation, applied consistently to a certain classical system, leads to the Rayleigh spectrum with no more reference to statistical mechanics than that a massive particle, free of external force fields, has an average translational kinetic energy of $\frac{3}{2}kT$. They bypassed all previous discussion of the entropy function because the classical system they discussed, the moving linear oscillator, had the property of emitting and absorbing radiation over a wide frequency band, in view of the Doppler shift caused by its motion.

Einstein and Hopf obtained the following values of $\bar{J}$ and $\langle \Delta^2 \rangle_{\text{av}}$ for their moving oscillator:

$$\bar{J} = -\frac{6\pi c\sigma}{5\omega}\left(\rho - \frac{1}{3}\frac{d\rho}{d\omega}\right) v\tau \,, \tag{3.6}$$

$$\langle \Delta^2 \rangle_{\text{av}} = \frac{4\pi^3 c^4 \sigma}{5\omega^3}\,\rho^2 \tau \,. \tag{3.7}$$

In these expressions, σ is a parameter related to the natural linewidth of the oscillator. They become identical with (2.3) and (2.20), if we make the replacement

$$\sigma \to \frac{5\omega a^2 d\omega}{4\pi c^2} \,. \tag{3.8}$$

Hence the Einstein-Hopf particle gave rise to the same equation, (3.2), for ρ as the moving mirror studied here. We therefore seem to be led inevitably back to the Rayleigh spectrum. The results of Sec. II above simply seem to confirm the conclusions reached by Einstein and Hopf.

In his analysis of the mirror, Einstein kept the classical expression for $\bar{J}$, and argued that, in order to make Eq. (3.1) consistent with the Planck spectrum, the mean-square impulse would have to be

$$\langle \Delta^2 \rangle_{\text{av}} = \frac{\pi^2 c^2 a^2 \tau d\omega}{\omega^2}\left(\rho^2 + \frac{\hbar\omega^3 \rho}{\pi^2 c^3}\right). \tag{3.9}$$

Without doing the classical calculation of the previous section, he recognized, "on dimensional grounds," that such a calculation would give, up to a constant multiple, the first term in (3.9). We have now confirmed this dimensional argument and found that the constant multiple is, in fact, one.

It was, however, the second term in (3.9) that received Einstein's main attention. This term, which dominates the first term at high frequencies, he interpreted as collisions of the mirror with independent particlelike light quanta, each carrying $\hbar\omega$ and momentum $\hbar\omega/c$. Such a description of the electromagnetic field is, of course, very different from that afforded by Maxwell's equations, and accepted by Planck, and most other contemporaries, in 1909. The discovery of this term was, therefore, significant in persuading physicists of the need for discontinuous processes in the electromagnetic field.

But, as Boyer[12] has shown in his discussion of the Einstein-Hopf result, such a conclusion may have been premature. Equation (3.1) is derived from (2.1) by making use of the equipartition assumption

$$\tfrac{1}{2}m\langle v_t^2 \rangle_{\text{av}} = \tfrac{1}{2}m\langle v_{t+\tau}^2 \rangle_{\text{av}} = \tfrac{1}{2}kT \,. \tag{3.10}$$

However, all proofs of this result consider mechanical systems with a finite, though sometimes very large, number of degrees of freedom, in which the total energy, a function of a finite number of variables, is conserved. Now all collisions, whether between molecules and molecules, molecules and Brownian particles, or Brownian particles and cavity walls involve interactions which are ultimately electromagnetic. So all such collisions result in electromagnetic radiation and energy loss. Taking radiative energy loss into account, Eq. (3.10) is obviously not satisfied. For example, just after a collision with a wall, a particle's average speed will be less than it is just before its collision with the next wall.

To derive a version of (3.1) which takes account of radiative energy loss at the walls,[32] we consider an ensemble of mirrors, and denote by Q the average rate of such energy loss. Then, from (2.1) and (2.3),

$$\tau Q=\tfrac{1}{2}m\,\langle(v_t-m^{-1}Pv_t\tau+m^{-1}\Delta)^2\rangle_{\rm av}-\tfrac{1}{2}m\langle v_t^{\,2}\rangle_{\rm av}\,. \tag{3.11}$$

We now substitute from (2.4) and (2.20), neglecting the terms in τ^2, and obtain

$$Q=\frac{\pi^2c^2a^2}{2m\omega^2}d\omega\left[\rho^2-\frac{3\omega^2}{\pi^2c^3}\,m\langle v_t^{\,2}\rangle_{\rm av}\left(\rho-\tfrac{1}{3}\omega\frac{d\rho}{d\omega}\right)\right]\,. \tag{3.12}$$

Because of the factor m^{-1} in Q, we may still replace $m\langle v_t^{\,2}\rangle_{\rm av}$ by kT on the right-hand side of (3.12), even though (3.10) is no longer strictly satisfied. A more exact analysis would take account of the variation of the average kinetic energy of the mirror as it crosses the box, but this replacement remains valid in the limit of large m. Hence

$$Q=\frac{\pi^2c^2a^2}{2m\omega^2}d\omega\left[\rho^2-\frac{3kT\omega^2}{\pi^2c^3}\left(\rho-\tfrac{1}{3}\omega\frac{d\rho}{d\omega}\right)\right]. \tag{3.13}$$

To obtain an equation corresponding to (3.2), it is necessary to know Q. A detailed analysis of the electromagnetic interaction between the mirror and the wall is out of the question, but, provided we make an additional assumption proposed by Boyer, such an analysis is not necessary.

We begin by considering the case of zero temperature, but we no longer suppose that the spectrum in that case is zero. Instead we take the zero-point spectrum of Planck's second theory, which has been revived more recently under the name of *stochastic electrodynamics*,[14-17]

$$\rho_0(\omega)=\frac{\hbar\omega^3}{\pi^2c^3}\,. \tag{3.14}$$

This spectrum gives $P=0$ and $\langle\Delta^2\rangle_{\rm av}\neq 0$. An unconfined mirror is therefore accelerated by the zero-point field to arbitrarily high energies, which seems entirely natural in view of the requirement of Lorentz invariance for the velocity distribution. (For more on the Lorentz invariance properties of this spectrum see Refs. 12 and 28.) We therefore obtain

$$Q_0=\frac{\pi^2c^2a^2}{2m\omega^2}d\omega\,\rho_0^{\,2}\,. \tag{3.15}$$

Boyer's assumption is, quite simply, that the radiative energy loss at the walls is the same at all temperatures, that is,

$$Q=Q_0\,. \tag{3.16}$$

In my view, Boyer's attempt at justifying this assumption [Ref. 12, the paragraph between Eqs. (26) and (27)] is not entirely convincing, and it will require further investigation. His main achievement was to see that Q is not zero, and that this additional assumption results in the Planck spectrum. This is seen by substituting (3.16) in (3.13),

$$\rho-\tfrac{1}{3}\omega\frac{d\rho}{d\omega}=\frac{\pi^2c^3}{3kT\omega^2}\,(\rho^2-\rho_0^{\,2})\,. \tag{3.17}$$

With the condition that ρ vanishes when ω is zero, the solution is

$$\rho=\frac{\hbar\omega^3}{2\pi^2c^3}\coth\frac{\hbar\omega}{2kT} \tag{3.18}$$

which is the sum of the zero-point and the Planck spectra. It is the view of stochastic electrodynamics that radiation detectors register only the difference between this and the zero-point spectrum, so, with the aid of Boyer's assumption (3.16), we have obtained a derivation of the Planck spectrum based entirely on Maxwell's electrodynamics.

IV. SUMMARY AND DISCUSSION

Planck's first attempts, in the years 1893–1900,[33] to derive the spectrum of black radiation were based on his study of a linear oscillator in equilibrium with such radiation. He hoped to replace Boltzmann's hypothesis of molecular chaos by a similar one, that of "natural radiation," applicable to the electromagnetic field, and hence to construct an entropy function directly, following Boltzmann's H-theorem method for gases.

This method failed and so Planck "as an act of desperation"[33] was forced to adopt a more *ad hoc* approach. This was to guess the form of the entropy function (3.4), and hence obtain the probability W of the equilibrium state by inverting the Boltzmann relation to give

$$W = \exp(S/k) . \tag{4.1}$$

He then sought to justify the resulting probability by a modified form of Boltzmann's "complexion counting," which he could do only with the introduction of the finite energy quantum $\hbar\omega$.

Planck's critics, among whom Einstein, during the first decade, was the most severe,[31] pointed out that Planck's entropy function was inconsistent both with classical statistical mechanics and with Maxwell's electrodynamics, but, right up to the Solvay conference in 1911, Planck insisted that his theory was based on the latter and that the former could not be applied to a system with an infinite number of degrees of freedom.

Einstein's view prevailed eventually over that of Planck, and his analysis of the moving mirror, together with his analysis of the moving oscillator (Einstein and Hopf) were of crucial importance in gaining him that victory.

According to the analysis presented here, and originated by Boyer, this victory was prematurely awarded. Planck was correct in insisting that Maxwell electrodynamics gave an adequate basis for deducing the spectrum of black radiation. He was also correct in denying the relevance of classical statistical mechanics, or at least of the equipartition law, to a system with an infinite number of degrees of freedom.

Electromagnetic radiation in a cavity is, of course, a system of the latter type, but, perhaps even more importantly, it is, according to stochastic electrodynamics, an *open system*. This means that it cannot be treated in isolation, so that, for example, when the cavity is subjected to adiabatic change, we can obtain incorrect results by ignoring the free entrance into the cavity of zero-point radiation. For example the equilibrium condition,

$$\Delta S = 0 , \quad \Delta U = 0 , \tag{4.2}$$

on which equation (3.5) is based, cannot be valid for this system.

This means that Planck's own proof of his radiation formula was indeed inconsistent, as his critics claimed, but that the foundation for a correct proof, based on Planck's original, pre-1900 program, was in Einstein's mirror analysis of 1909.

Finally, from the point of view of modern information theory, the correct entropy for a natural radiation field, if we assume that "natural" means "Gaussian," is simply

$$S = k \int \frac{V\omega^2}{\pi^2 c^3} \ln \left[\rho(\omega)\right] d\omega . \tag{4.3}$$

Of course, if (4.2) were the appropriate equilibrium condition, the Rayleigh spectrum would follow immediately. However, until a more complete analysis can be made of the zero-point radiation entering the cavity during an adiabatic change, it is not possible to make a thermodynamical analysis based on the entropy function. It follows that, for the time being, the H-theorem analysis developed here is the most general analysis possible.

Note added in proof. I have recently received from Dr. A. Rueda of the University of Bogotá, Columbia, a paper [Uniandes Report No. AS 1349 (unpublished)] on the behavior of classical particles immersed in the classical electromagnetic zero-point field. This paper takes substantially the same view of the effect of wall collisions as I have expressed in the discussion above following my equation (3.13).

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[25] This hypothesis, which means that $\vec{E}$ (and $\vec{B}$) is a *Gaussian process*, has a curious history. Planck's hypothesis of *natural radiation* did not explicitly assume more than that $\vec{\theta}_{k\lambda}$ were independent and uniformly distributed. Einstein and Hopf were the first to consider higher moments of $\vec{E}$, and also the first to assume independence for $\vec{E}_{k\lambda}$. The assumption was challenged by Laue (Ref. 26), and while Einstein claimed to prove (Ref. 27), in his reply to Laue, that it was not an assumption, but a necessary consequence of the natural radiation hypothesis, all he actually did prove was that $\vec{E}_{k\lambda}$ are uncorrelated. In stochastic electrodynamics it has become customary to use the Gaussian hypothesis, on the grounds that natural radiation should be the most random possible. For the connection between the independence of $\vec{E}_{k\lambda}$ and the normality of $\vec{E}$ see Bartlett (Ref. 28).

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[31] See Refs. 10 and 11. See also the account given by T. S. Kuhn in *Black-Body Theory and the Quantum Discontinuity, 1894–1912* (Oxford University Press, New York, 1979).

[32] This analysis is closely modeled on that of Boyer. He included the effect of the wall by putting an extra impulse item into (2.1). I believe his derivation of (3.17) is flawed by his attempt to neglect the square of this large, but occasional, impulse. The crux of Boyer's argument, however, is in the paragraph between Eqs. (26) and (27) of Ref. 12, and there he is discussing energy, rather than momentum, exchange at the wall. My derivation of (3.17) should be regarded as an attempt at correcting this small blemish in Boyer's otherwise very impressive article.

[33] M. Planck, quoted in A. Hermann, *The Genesis of Quantum Theory, 1899–1913* (MIT Press, Cambridge, Mass., 1971), p. 23.