Quantized scalar field in the stationary coordinate systems of flat spacetime

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A canonical quantization of the free scalar field is presented for the six classes of stationary coordinate systems in Minkowski space. The corresponding vacuum states are found to be restricted to two possibilities: Those in coordinate systems without event horizons are the Minkowski vacuum; those in coordinate systems with event horizons are the Fulling vacuum. The vacuum state is not in general the lowest energy state, but is stable due to the presence of additional symmetries, and a particle interpretation remains valid. These vacuum states are not compatible with vacuums defined by means of observer-based detectors.

I. INTRODUCTION

It is well known that the vacuum and particle states defined by canonical quantum field theory are not coordinate independent. This was first discussed by Fulling,[1] who showed that the vacuum defined in a coordinate system adapted to a uniformly accelerating observer differs from the usual vacuum of Minkowski coordinates. In fact, it has since been shown that the Minkowski vacuum can be expressed as an infinite sum of multiparticle states with a thermal distribution according to the canonical definition of the accelerating observer.[2] These results have led us to investigate the nature of the canonical quantum field theory in other coordinate systems in flat spacetime. In particular, "What other vacuum states are possible?"

We restrict ourselves to stationary systems, as only in such systems can the definition of particle states be time independent. The stationary coordinate systems in flat spacetime have been described in a previous paper.[3] There are six classes of systems corresponding to the six distinct types of timelike Killing vector fields in flat spacetime, which have been classified on the basis of the six types of stationary world lines in Minkowski space. Quantization in coordinate systems of classes A, B, and C has been performed previously. Minkowski coordinates belong to class A, which are adapted to inertial observers. Pseudocylindrical coordinates, adapted to uniformly accelerating observers, belong to class B. Quantization in this system, discussed above, yields the Fulling vacuum. Quantization in rotating coordinates, belonging to class C, has also been carried out.[4,5] The vacuum defined in this system is equivalent to the Minkowski vacuum. On the basis of these results we conjectured that vacuums differ when one system has an event horizon and the other does not. In this paper, quantization in the remaining stationary systems is performed with results that support this conjecture. In addition we find that there are only two stationary vacuums in flat spacetime: the Minkowski vacuum and the Fulling vacuum. The Minkowski vacuum is found in class A, C, and D systems, which have no event horizon. The Fulling vacuum is found in class B, E, and F systems, which have identical event horizons.

The remaining sections of this paper are arranged as follows. In Sec. II we review the formalism of canonical quantum field theory that is pertinent to the types of coordinate systems we treat.[6] In Secs. III through VIII we summarize the previous results and extend them to the remaining classes of stationary coordinate systems. In Sec. IX we discuss our results and indicate why they imply that a vacuum defined via canonical quantization is not compatible with a vacuum defined via an observer-based detector.

II. QUANTUM FIELD THEORY

The first step in quantizing the scalar field is to solve the Klein-Gordon equation, which in flat space is

$$\psi_{;\mu}{}^{\mu} + M^2\psi = 0\,, \tag{1}$$

for normal modes appropriate to the coordinate system in question. The coordinate systems we are interested in are based upon a Killing vector K which is timelike in at least some portion of Minkowski space. In general, when a Killing vector is present, normal modes may be chosen to satisfy

$$\mathfrak{L}_K\psi = -iE\psi\,, \tag{2}$$

where $\mathfrak{L}_K$ is the Lie derivative with respect to K.

Now, the presence of a Killing vector further implies that a coordinate system exists in which the metric is independent of one of the coordinates and in which the Killing vector consists only of a unit component in that coordinate direction. The coordinate systems of interest are just such systems adapted to all such Killing vectors K. If we call x^0 the coordinate associated with K, then in such a system Eq. (2) becomes

$$\frac{\partial}{\partial x^0}\psi = -iE\psi, \tag{3}$$

so that ψ can be separated in the form

$$\psi = e^{-iEx^0}U(\vec{x}). \tag{4}$$

If there are additional independent Killing vectors which commute with K and each other, the coordinate system may be further specialized and ψ further separated into exponentials in terms of coordinates $(x^1, \ldots)$ associated with each Killing vector. All of the coordinate systems we deal with are adapted to at least one such additional Killing vector.

Inner products are defined by

$$\langle \psi_1, \psi_2\rangle_\Sigma = i\int_\Sigma \psi_1^* \overleftrightarrow{f}^\mu \psi_2\, d\Sigma_\mu, \tag{5}$$

where

$$\overleftrightarrow{f}^\mu = g^{1/2}g^{\mu\nu}\frac{\overrightarrow{\partial}}{\partial x^\nu} - \frac{\overleftarrow{\partial}}{\partial x^\nu}g^{1/2}g^{\mu\nu} \tag{6}$$

is the Wronskian associated with the Klein-Gordon equation and Σ is generally some spacelike hypersurface. For simplicity we shall always choose for Σ one of the x^0=constant surfaces associated with K. In fact, we can easily arrange for the surfaces $x^0 = 0$ to coincide in five of the six systems treated.

The inner product (5) separates the mode functions satisfying the Klein-Gordon equation into modes which have positive or negative norm. If a mode ψ has a positive norm (positive inner product with itself), then ψ^* has a negative norm. This property is generally *not* coincident with that of positive and negative frequency, which is determined by the sign of the eigenvalue E in Eq. (3). It is the former property rather than the latter which is important in defining particle states when the field is quantized.

In each of our coordinate systems the normal to the x^0=constant surfaces is also a Killing vector in the form $\xi = c_1K + c_2V$, where V is one of the additional Killing vectors characterizing the coordinate system and c_1 and c_2 are constants. If v is the eigenvalue of a mode relative to V, i.e.,

$$\mathcal{L}_V\psi = iv\psi, \tag{7}$$

then the positive norm is coincident with the sign of the eigenvalue of ψ with respect to ξ,

$$\mathcal{L}_\xi\psi = -i(c_1E - c_2v)\psi, \tag{8}$$

i.e., $c_1E - c_2v > 0$ for all positive-norm modes, and vice versa. Modes are normalized on the above-mentioned choices for Σ.

The field may be quantized by defining a field operator Φ and its conjugate momentum Π, defined as the projection of the vector density

$$\Pi^\mu = g^{1/2}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\Phi \tag{9}$$

onto the normal to Σ, and invoking the commutation relations

$$[\Phi(x^0,\vec{x}), \Pi(x^0,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}'), \tag{10a}$$

$$[\Phi(x^0,\vec{x}), \Phi(x^0,\vec{x}')] = 0, \tag{10b}$$

$$[\Pi(x^0,\vec{x}), \Pi(x^0,\vec{x}')] = 0. \tag{10c}$$

The field operator is expressed in terms of positive-norm modes as

$$\Phi = \sum_i (a_i\psi_i + a_i^\dagger\psi_i^*). \tag{11}$$

Insertion of this expansion into Eqs. (9) and (10) yields the usual commutation relations for a and $a^\dagger$,

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{12a}$$

$$[a_i, a_j] = 0, \tag{12b}$$

$$[a_i^\dagger, a_j^\dagger] = 0. \tag{12c}$$

The operators a and $a^\dagger$ are therefore defined as annihilation and creation operators, respectively, and a vacuum state is defined by

$$a_i|0\rangle = 0 \tag{13}$$

for all a_i, one-particle states by $|i\rangle = a_i^\dagger|0\rangle$, etc. The annihilation and creation operators may be expressed in terms of the field by

$$a_j = i\int_\Sigma \psi_j^*\overleftrightarrow{f}^\mu\Phi\, d\Sigma_\mu, \tag{14a}$$

$$a_j^\dagger = -i\int_\Sigma \psi_j\overleftrightarrow{f}^\mu\Phi\, d\Sigma_\mu. \tag{14b}$$

For each Killing vector U present, the classical expression

$$P_U = \int_\Sigma U_\mu T^{\mu\nu}d\Sigma_\nu, \tag{15}$$

where $T^{\mu\nu}$ is the stress-energy tensor, is a conserved quantity. Thus, if we define a stress-energy tensor operator as

$$\mathfrak{T}_{\mu\nu}=\Phi_{,\mu}\Phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}(\Phi_{,\sigma}\Phi^{,\sigma}-M^2\Phi^2)\,, \tag{16}$$

then P_U defines an operator

$$\mathfrak{P}_U=\int_\Sigma U_\mu \mathcal{T}^{\mu\nu}d\Sigma_\nu\,. \tag{17}$$

The natural definition of energy is the quantity $\mathfrak{R}\equiv\mathfrak{P}_K$. Insertion of Eq. (16) into Eq. (17) yields

$$\mathfrak{R}=\tfrac{1}{2}\sum_j E_j(a_j a_j^\dagger+a_j^\dagger a_j)\,, \tag{18}$$

which is not generally bounded below, as E is not generally bounded below for positive-norm modes. However, let us consider the operator

$$\mathfrak{P}_\xi=\tfrac{1}{2}\sum_j (c_1E_j-c_2v_j)(a_ja_j^\dagger+a_j^\dagger a_j)\,. \tag{19}$$

Matrix elements of $\mathfrak{P}_\xi$ *are* positive definite. The vacuum state thus represents the lowest state of the quantity c_1E-c_2v. It also corresponds to $E=0=v$. As E and v are good quantum numbers for particle states, the vacuum state is not unstable, since by Eq. (19) it is the *only* state with $E=v=0$.

If we have two sets of modes ψ_i and $\tilde{\psi}_i$ available for the field Φ, they may be related by an expression

$$\tilde{\psi}_i=\sum_j(\alpha_{ij}\psi_j+\beta_{ij}\psi_j^*)\,. \tag{20}$$

If the two sets of modes have been normalized relative to the same surface Σ, then the coefficients α and β satisfy the relations

$$\sum_k\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}_{ik}\begin{pmatrix}\alpha^\dagger & -\beta^-\\ -\beta^\dagger & \alpha^-\end{pmatrix}_{kj}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta_{ij}\,. \tag{21}$$

Inserting Eq. (20) into the expression for a_i in terms of the field, we obtain the following Bogoliubov transformation relating $\tilde{a}$ to a and $a^\dagger$:

$$\tilde{a}_i=\sum_j(\alpha_{ij}^*a_j-\beta_{ij}^*a_j^\dagger)\,. \tag{22}$$

When the coefficients β are zero, the vacuums defined by both sets of modes are equivalent. If, on the other hand, the β are nonzero, the two quantizations are inequivalent, and, in fact, the "old" vacuum will contain "new" particles, i.e.,

$$_{\text{old}}\left\langle 0\left|\sum_i\tilde{a}_i^\dagger\tilde{a}_i\right|0\right\rangle_{\text{old}}=\sum_{i,j}\beta_{ij}\beta_{ji}^\dagger\,. \tag{23}$$

This sum is not necessarily convergent.

The old vacuum may be expressed in terms of the new vacuum as

$$|0\rangle_{\text{old}}=\frac{1}{c}\exp\left(\frac{i}{2}\sum_{j,k}\tilde{a}_j^\dagger V_{jk}\tilde{a}_k^\dagger\right)|0\rangle_{\text{new}}\,, \tag{24}$$

where V_{jk} is the "pair-creation amplitude"

$$V_{jk}=i\sum_l\beta_{jl}^*\alpha_{lk}^{-1-}\,, \tag{25}$$

and c is the vacuum-to-vacuum amplitude $_{\text{new}}\langle 0|0\rangle_{\text{old}}$, which itself may be divergent. For a boson field the existence of α^{-1} is guaranteed.

Note that α and β may be expressed as

$$\alpha_{ij}=\langle\psi_j,\tilde{\psi}_i\rangle_\Sigma \tag{26a}$$

$$\beta_{ij}=-\langle\psi_j^*,\tilde{\psi}_i\rangle_\Sigma\,. \tag{26b}$$

III. CLASS-A COORDINATES

Class-A coordinate systems are based on a Killing vector field with components $(1,0,0,0)$ in an inertial frame. The most familiar class-A system is rectangular Minkowski coordinates (t,x,y,z). In these coordinates, the Klein-Gordon equation is

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+M^2\right)\psi=0\,. \tag{27}$$

Positive-norm modes in these coordinates are

$$\psi=\frac{1}{(2\pi)^{3/2}(2\omega)^{1/2}}e^{-i\omega t}e^{i\vec{k}\cdot\vec{x}}\,, \tag{28}$$

where $\omega^2=|\vec{k}|^2+M^2$ and $\omega>0$. This coordinate system is static, so K and ξ are equal, so that these modes are also of positive frequency. Thus, the vacuum state in this system is the lowest energy state. The field operator is expanded in terms of these modes as

$$\Phi=\int d^3k[a(\vec{k})\psi(\vec{k})+a^\dagger(\vec{k})\psi^*(\vec{k})]\,, \tag{29}$$

and the Minkowski vacuum is defined by

$$a(\vec{k})|0\rangle_M=0\,. \tag{30}$$

Another class-A coordinate system is obtained by using cylindrical coordinates on the $t=$constant hypersurfaces. In these coordinates (t,r,θ,z) the Klein-Gordon equation is

$$\left(\frac{\partial^2}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\partial^2}{\partial z^2}+M^2\right)\psi=0\,. \tag{31}$$

Positive-norm modes in these coordinates, chosen to be well behaved at $r=0$, are

$$\hat{\psi}=\frac{1}{2\pi(2\omega)^{1/2}}e^{-i\omega t}e^{im\theta}e^{ik_zz}J_m(qr)\,, \tag{32}$$

where $\omega^2=k_z^2+q^2+M^2$, $\omega>0$, and, of course, they are also of positive frequency. In terms of these modes, the field operator is

$$\Phi = \sum_m \int q\,dq\,dk_z[\hat{a}(q,m,k_z)\hat{\psi}(q,m,k_z) + \hat{a}^\dagger(q,m,k_z)\hat{\psi}^*(q,m,k_z)]. \quad (33)$$

Relating the modes ψ and $\hat{\psi}$ by a Bogoliubov transformation, we find that

$$\alpha(q,m,k_z';\vec{k}) = \frac{1}{(2\pi)^{1/2}}\left(\frac{k_x - ik_y}{q}\right)^m \frac{\delta(q-(k_x^2+k_y^2)^{1/2})}{q} \times \delta(k_z - k_z'), \quad (34a)$$

$$\beta(q,m,k_z';\vec{k}) = 0. \quad (34b)$$

The latter result verifies that the vacuum states in rectangular and cylindrical Minkowski coordinates are indeed the same.

IV. CLASS-*C* COORDINATES

Class-*C* coordinate systems are based on a Killing vector field with components $(1+\kappa x, \kappa t - \tau y, \tau x, 0)$, $\tau > \kappa$. The most familiar class-*C* coordinate system is rotating coordinates $(t, r, \bar{\theta}, z)$, where $\bar{\theta} = \theta - \Omega t$. In these coordinates, the Klein-Gordon equation is

$$\left[\left(\frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial\bar{\theta}}\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\bar{\theta}^2} - \frac{\partial^2}{\partial z^2} + M^2\right]\psi = 0, \quad (35)$$

where Ω is a constant. Positive-norm modes in these coordinates, again chosen to be well behaved at $r=0$, are

$$\bar{\psi} = \frac{1}{2\pi[2(\bar{\omega}+\bar{m}\Omega)]^{1/2}} e^{-i\bar{\omega}t} e^{i\bar{m}\bar{\theta}} e^{ik_z z} J_{\bar{m}}(qr), \quad (36)$$

with $(\bar{\omega}+\bar{m}\Omega)^2 = k_z^2 + q^2 + M^2$ and $\bar{\omega}+\bar{m}\Omega > 0$. It is evident that these modes are not generally of positive frequency, and the vacuum will thus not be the lowest energy state. The field operator is expanded in terms of these modes as

$$\Phi = \sum_{\bar{m}} \int q\,dq\,dk_z[\bar{a}(q,\bar{m},k_z)\bar{\psi}(q,\bar{m},k_z) + \bar{a}^\dagger(q,\bar{m},k_z)\bar{\psi}^*(q,\bar{m},k_z)]. \quad (37)$$

The rotating modes $\bar{\psi}$ are identical to the cylindrical modes $\hat{\psi}$ when both modes are expressed in the same coordinate system, if ω is replaced by $\bar{\omega}+\bar{m}\Omega$ in the latter, so the Bogoliubov transformation between the two sets of modes is

$$\alpha(q',\bar{m},k_z';q,m,k_z) = \frac{\delta(q-q')}{q}\delta_{m\bar{m}}\delta(k_z - k_z'), \quad (38a)$$

$$\beta(q',\bar{m},k_z';q,m,k_z) = 0. \quad (38b)$$

Thus, the vacuum defined in rotating coordinates is just the Minkowski vacuum. In fact, the field theories in rotating and in cylindrical coordinates differ only by a redefinition of the energy of a mode.

V. CLASS-*D* COORDINATES

Class-*D* coordinate systems are based on a Killing vector field with components $(1+\kappa x, \kappa t - \kappa y, \kappa x, 0)$. We have selected what we regard as a "natural" coordinate system for this class, null parabolic coordinates, defined by $t = \frac{1}{6}\kappa^2\tilde{t}^3 + (\kappa\tilde{x}+\frac{1}{2})\tilde{t} + \tilde{y}$, $x = \frac{1}{2}\kappa\tilde{t}^2 + \tilde{x} - 1/2\kappa$, and $y = \frac{1}{6}\kappa^2\tilde{t}^3 + (\kappa\tilde{x}-\frac{1}{2})\tilde{t} + \tilde{y}$. In these coordinates the Klein-Gordon equation is

$$\left(2\frac{\partial^2}{\partial\tilde{t}\partial\tilde{y}} - \frac{\partial^2}{\partial\tilde{x}^2} - 2\kappa\tilde{x}\frac{\partial^2}{\partial\tilde{y}^2} - \frac{\partial^2}{\partial z^2} + M^2\right)\psi = 0. \quad (39)$$

Positive-norm modes in these coordinates, chosen to be well behaved as $\tilde{x} \to \infty$, are

$$\tilde{\psi} = \frac{1}{4\pi}\left(\frac{4}{\kappa l^2}\right)^{1/6} e^{-i\tilde{\omega}\tilde{t}} e^{-il\tilde{y}} e^{ik_z z}\,\mathrm{Ai}(\lambda + \alpha\tilde{x}), \quad (40)$$

for $l > 0$. In this expression, $\alpha = (2\kappa l^2)^{1/3}$, $\lambda = \alpha^{-2} \times (k_z^2 - l\tilde{\omega} + M^2)$, and Ai is an Airy function.[7] Unlike the other coordinate systems treated, the hypersurface $\tilde{t} = 0$ is actually a *null* surface, corresponding to the hyperplane $t = y$ in rectangular Minkowski coordinates. Such a surface is not a Cauchy surface for Minkowski space, strictly speaking. However, it is a satisfactory initial data surface for a massive field, and requires further consideration only for a massless field. This is the "front" form of quantization discussed by Dirac.[8]

Here, as in rotating coordinates, positive-norm modes need not be positive-frequency modes as well. Moreover, the energy $\tilde{\omega}$ plays no role whatsoever in defining positive-norm modes, that being characterized solely by l.

The field operator is expanded in terms of the modes $\tilde{\psi}$ as

$$\Phi = \int dl\,dk_z d\lambda[\tilde{a}(l,k_z,\lambda)\tilde{\psi}(l,k_z,\lambda) + \tilde{a}^\dagger(l,k_z,\lambda)\tilde{\psi}^*(l,k_z,\lambda)]. \quad (41)$$

A Bogoliubov transformation between the modes $\tilde{\psi}$ and the rectangular Minkowski modes ψ yields

$$\alpha(l,k_z',\lambda;\vec{k}) = \frac{1}{2}\left(\frac{\omega}{\pi\kappa l}\right)^{1/2}\exp\left[i\,\frac{k_x}{2\kappa l^2}(k_x{}^2+l^2+\lambda\alpha^2)\right] \times\delta(k_z-k_z')\delta[l-(\omega-k_y)]\,, \quad (42a)$$

$$\beta(l,k_z',\lambda;\vec{k})=0\,. \quad (42b)$$

Thus, the vacuum defined in null parabolic coordinates is again the Minkowski vacuum.

VI. CLASS-*B* COORDINATES

Class-*B* coordinate systems are based on a Killing vector field with components $(1+\kappa x,\kappa t,0,0)$. A familiar class-*B* system is pseudocylindrical coordinates (τ,ξ,y,z), where $\xi=(x^2-t^2)^{1/2}$ and $\tau=\tanh^{-1}(t/x)$, in which the Klein-Gordon equation is

$$\left(\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2}-\frac{1}{\xi}\frac{\partial}{\partial\xi}\,\xi\,\frac{\partial}{\partial\xi}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+M^2\right)\psi=0\,. \quad (43)$$

Positive-norm modes in these coordinates may be chosen as

$$\phi_R=\frac{(\sinh\pi E)^{1/2}}{2\pi^2}\,e^{-iE\tau}e^{ik_yy}e^{ik_zz}K_{iE}(Q\xi)\,, \quad (44)$$

where $Q^2=k_y{}^2+k_z{}^2+M^2$ and $E>0$. K_{iE} is a Macdonald function,[9] a Bessel function of pure imaginary order and argument. As $\xi=(x^2-t^2)^{1/2}$ is defined only for $x>|t|$, these functions are only defined in the corresponding region of spacetime, the right Rindler wedge; hence the notation ϕ_R. By defining negative values of ξ by $\xi=-(x^2-t^2)^{1/2}$ for $-x>|t|$, we can define mode functions which cover the left Rindler wedge:

$$\phi_L=\frac{(\sinh\pi E)^{1/2}}{2\pi^2}\,e^{iE\tau}e^{ik_yy}e^{ik_zz}K_{iE}(-Q\xi)\,. \quad (45)$$

The factor $e^{iE\tau}$ has a different sign here than in ϕ_R because $\partial/\partial\tau$ is past directed in the left Rindler wedge, and thus the sign in the exponential must be altered if we still desire positive-norm modes to be given by $E>0$. Then the mode functions ϕ_R and ϕ_L are also positive-frequency modes.

The field operator is expanded in terms of the modes ϕ_R and ϕ_L as

$$\Phi=\int dE\,dk_y\,dk_z[b_R(E,k_y,k_z)\phi_R(E,k_y,k_z) +b_R^\dagger(E,k_y,k_z)\phi_R^*(E,k_y,k_z) +b_L(E,k_y,k_z)\phi_L(E,k_y,k_z) +b_L^\dagger(E,k_y,k_z)\phi_L^*(E,k_y,k_z)]\,. \quad (46)$$

The modes ϕ_R and ϕ_L are related to the rectangular Minkowski modes ψ by the Bogoliubov coefficients

$$\alpha_R(E,k_y',k_z';\vec{k})=[2\pi\omega(1-e^{-2\pi E})]^{-1/2}\left(\frac{\omega-k_x}{Q}\right)^{iE} \times\delta(k_y-k_y')\delta(k_z-k_z')\,, \quad (47a)$$

$$\alpha_L(E,k_y',k_z';\vec{k})=\alpha_R^*(E,k_y',k_z';\vec{k}) =[2\pi\omega(1-e^{-2\pi E})]^{-1/2}\left(\frac{\omega+k_x}{Q}\right)^{iE} \times\delta(k_y-k_y')\delta(k_z-k_z')\,, \quad (47b)$$

$$\beta_R(E,k_y',k_z';\vec{k})=-e^{-\pi E}\alpha_R(E,k_y',k_z';\vec{k})\,, \quad (47c)$$

$$\beta_L(E,k_y',k_z';\vec{k})=-e^{-\pi E}\alpha_L(E,k_y',k_z';\vec{k})\,. \quad (47d)$$

One immediately sees that the vacuum in pseudocylindrical coordinates, the Fulling vacuum, defined by

$$b_{R(L)}(E,k_y,k_z)\,|0\rangle_F=0\,, \quad (48)$$

is not equivalent to the Minkowski vacuum. In fact, since

$$V_{ij}^{RL}=V_{ij}^{LR}=i\sum_k\beta_{Rik}^*\alpha_{Lkj}^{-1\sim} \quad (49)$$

or

$$V^{RL}(E',k_y',k_z';E,k_y,k_z)=-ie^{-\pi E}\delta(E-E') \times\delta(k_y-k_y')\delta(k_z-k_z')\,, \quad (50)$$

$$V_{ij}^{RR}=V_{ij}^{LL}=0\,, \quad (51)$$

the Minkowski vacuum $|0\rangle_M$ is related to the Fulling vacuum $|0\rangle_F$ by

$$|0\rangle_M=\frac{1}{c}\exp\left[\int dE\,dk_y\,dk_z e^{-\pi E}b_R^\dagger(E,k_y,k_z)b_L^\dagger(E,k_y,k_z)\right]|0\rangle_F\,. \quad (52)$$

Thus, the Minkowski vacuum consists of a mixture of multiparticle Fulling states composed of particle pairs, one each in the right and left Rindler wedges, distributed in a thermal fashion.

The number of Fulling particles N_F present in the Minkowski vacuum diverges. Even the number of monochromatic Fulling particles $N_F(E)$ diverges.

It is convenient to define modes in another class-B coordinate system, obtained by transforming the coordinates y,z into polar coordinates r,θ. Positive-norm modes in this coordinate system are

$$\hat{\phi}_R=\left(\frac{\sinh\pi E}{2\pi^3}\right)^{1/2}e^{-iE\tau}e^{im\theta}J_m(qr)K_{iE}(Q\xi)\,, \quad (53)$$

with $Q^2=q^2+M^2$ and $E>0$, on the right Rindler wedge and $\hat{\phi}_L$ defined on the left Rindler wedge in the same manner as ϕ_L. These modes are related to the modes ϕ_R and ϕ_L in precisely the same manner as the modes of rectangular and cylindrical Minkowski coordinates:

$$\alpha_{R(L)}(E',q,m;E,k_y,k_z)=\frac{1}{(2\pi)^{1/2}}\left(\frac{k_z-ik_y}{q}\right)^m\frac{\delta(q-(k_y^2+k_z^2)^{1/2})}{q}\delta(E-E')\,, \quad (54a)$$

$$\beta_{R(L)}(E',q,m;E,k_y,k_z)=0\,. \quad (54b)$$

VII. CLASS-E COORDINATES

Class-E coordinate systems are based on a Killing vector field with components $(1+\kappa x,\kappa t-\tau y,\tau x,0)$, $\kappa>\tau$. We have chosen as a natural coordinate system for this class pseudorotating coordinates. In these coordinates $(\tau,\xi,\bar{y},z)$, where $\bar{y}=y-\Omega\tau$, the Klein-Gordon equation is

$$\left[\frac{1}{\xi^2}\left(\frac{\partial}{\partial\tau}-\bar{\Omega}\frac{\partial}{\partial\bar{y}}\right)^2-\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}-\frac{\partial^2}{\partial\bar{y}^2}-\frac{\partial^2}{\partial z^2}+M^2\right]\psi=0\,, \quad (55)$$

where $\bar{\Omega}$ is a constant. Positive-norm modes in these coordinates may be chosen as

$$\bar{\phi}_R=\frac{[\sinh\pi(\bar{E}+\bar{k}_y\bar{\Omega})]^{1/2}}{2\pi^2}\times e^{-i\bar{E}\tau}e^{i\bar{k}_y\bar{y}}e^{ik_zz}K_{i(\bar{E}+\bar{k}_y\bar{\Omega})}(Q\xi)\,, \quad (56)$$

where $Q^2=\bar{k}_y^2+k_z^2+M^2$ and $\bar{E}+\bar{k}_y\bar{\Omega}>0$, again defined only on the right Rindler wedge. Mode functions $\bar{\phi}_L$ are defined on the left Rindler wedge precisely as in Sec. VI. These mode functions are not, in general, of positive frequency.

The field operator is expanded in terms of the modes $\bar{\phi}_R$ and $\bar{\phi}_L$ as

$$\Phi=\int d\bar{E}\,d\bar{k}_y dk_z[\bar{b}_R(\bar{E},\bar{k}_y,k_z)\bar{\phi}_R(\bar{E},\bar{k}_y,k_z)+\bar{b}_R^\dagger(\bar{E},\bar{k}_y,k_z)\bar{\phi}_R^*(\bar{E},\bar{k}_y,k_z)+\bar{b}_L(\bar{E},\bar{k}_y,k_z)\bar{\phi}_L(\bar{E},\bar{k}_y,k_z)+\bar{b}_L^\dagger(\bar{E},\bar{k}_y,k_z)\bar{\phi}_L^*(\bar{E},\bar{k}_y,k_z)]\,. \quad (57)$$

As in the relationship between modes in cylindrical Minkowski and rotating coordinates, the mode functions $\phi_{R(L)}$ and $\bar{\phi}_{R(L)}$ are functionally equivalent if E is replaced by $\bar{E}+\bar{k}_y\bar{\Omega}$ in the former. Thus, the Bogoliubov transformation between the two sets of modes is given by

$$\alpha_{R(L)}(\bar{E},\bar{k}_y,k_z';E,k_y,k_z)=\delta[E-(\bar{E}+\bar{k}_y\bar{\Omega})]\times\delta(k_y-\bar{k}_y)\delta(k_z-k_z')\,, \quad (58a)$$

$$\beta_{R(L)}(\bar{E},\bar{k}_y,k_z';E,k_y,k_z)=0\,. \quad (58b)$$

Thus, the vacuum state in pseudorotating coordinates is the Fulling vacuum, and the two field theories are identical up to a redefinition of energy.

VIII. CLASS-F COORDINATES

Class-F coordinate systems are based on the most general timelike Killing vector field, which has components $(1+\kappa x,\kappa t-\tau y,\tau x-\upsilon z,\upsilon y)$ in some inertial frame. We choose as the most natural class-F coordinate system rotating pseudocylindrical coordinates $(\tau,\xi,r,\tilde{\theta})$, where $\tilde{\theta}=\theta-\tilde{\Omega}\tau$, in which the Klein-Gordon equation is

$$\left[\frac{1}{\xi^2}\left(\frac{\partial}{\partial t}-\tilde{\Omega}\frac{\partial}{\partial\tilde{\theta}}\right)^2-\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\tilde{\theta}^2}+M^2\right]\psi=0\,, \quad (59)$$

where $\tilde{\Omega}$ is a constant. Positive-norm mode functions in these coordinates may be chosen as

$$\tilde{\phi}_R=\left[\frac{\sinh\pi(\tilde{E}+\tilde{m}\tilde{\Omega})}{2\pi^3}\right]^{1/2}e^{-i\tilde{E}\tau}e^{i\tilde{m}\tilde{\theta}}J_{\tilde{m}}(qr)\times K_{i(\tilde{E}+\tilde{m}\tilde{\Omega})}(Q\xi)\,, \quad (60)$$

where $Q^2=q^2+M^2$ and $\tilde{E}+\tilde{m}\tilde{\Omega}>0$, on the right Rindler wedge, with $\tilde{\phi}_L$ defined on the left Rindler wedge as before. These modes are not generally of positive frequency.

The field operator is expanded in terms of these modes as

$$\Phi=\sum_{\tilde{m}}\int q\,dq\,d\tilde{E}[\tilde{b}_R(\tilde{E},q,\tilde{m})\tilde{\phi}_R(\tilde{E},q,\tilde{m})+\tilde{b}_R^\dagger(\tilde{E},q,\tilde{m})\tilde{\phi}_R^*(\tilde{E},q,\tilde{m})+\tilde{b}_L(\tilde{E},q,\tilde{m})\tilde{\phi}_L(\tilde{E},q,\tilde{m})+\tilde{b}_L^\dagger(\tilde{E},q,\tilde{m})\tilde{\phi}_L(\tilde{E},q,\tilde{m})]\,. \quad (61)$$

Once again, the mode functions $\tilde{\phi}_{R(L)}$ are the same as the modes $\hat{\phi}_{R(L)}$ of Sec. VI if E is replaced by $\tilde{E}+\tilde{m}\tilde{\Omega}$ in the latter. The Bogoliubov transformation between these two sets of modes therefore yields

$$\alpha_{R(L)}(\tilde{E},q',\tilde{m};E,q,m) = \delta[E-(\tilde{E}+\tilde{m}\Omega)]\delta_{m\tilde{m}}\frac{\delta(q-q')}{q}, \tag{62a}$$

$$\beta_{R(L)}(\tilde{E},q',\tilde{m};E,q,m)=0, \tag{62b}$$

so the vacuum state in rotating pseudocylindrical coordinates is once again the Fulling vacuum.

IX. DISCUSSION

We have developed a canonical quantization of the free scalar field in all of the classes of stationary coordinate systems in flat space. There are only two possibilities for the vacuum state. In those coordinate systems without an event horizon, it is the Minkowski vacuum. In those systems with an event horizon, it is the Fulling vacuum. These two vacuums are not equivalent. The presence of an event horizon thus appears to play a key role in the definition of a vacuum state. The vacuum state defined in each case, while not in general the lowest energy state, is nevertheless stable, owing to the presence of a conserved quantity in each case, associated with the definition of positive-norm mode functions, for which the vacuum *is* a minimum. The use of these symmetries in this situation merely represents a particular observer's explanation of the stability of the Minkowski or Fulling vacuum.

The clear distinction between positive frequency and positive norm has been an important element in our procedure. The latter property is the critical factor in defining creation and annihilation operators, and thus the vacuum and particle states. The former property is associated with energy, and in more conventional settings it is believed that expectation values of the energy, if not positive definite, must at least be bounded below for a stable vacuum to exist. In our circumstances the presence of additional symmetries has resolved this difficulty. Whether it can be so readily resolved in a more general geometry is at present unknown.

When does the distinction between positive frequency and positive norm arise? In the particular situations we have examined, there is no such distinction when one can find an initial-value hypersurface on which the energy-defining Killing vector is everywhere timelike. When no such surface exists, then we find that the connection between positive frequency and positive norm disappears. The existence or nonexistence of such a surface may therefore be the determining factor. As a case in point, we could artificially restrict the Killing vector defining class C systems to be everywhere timelike by inserting a cylindrical conductor of radius $R<\Omega^{-1}$. In this case the identification of positive frequency and positive norm is restored.

What alternatives are there to the above quantization procedure for defining a vacuum? Unruh[10] suggests using a detector to define a vacuum state for any observer as the state in which the detector has no response. Aside from philosophical grounds, a prime motivation for such a definition is that this response, the Fourier transform of the Wightman or autocorrelation function, which is the spectrum of vacuum fluctuations, is in accord with the results of canonical quantization in both class-A and class-B coordinate systems.

Let us consider this spectrum in more detail. It is essentially given by

$$S(E)=\int_{-\infty}^{\infty} e^{iEs}\langle\Phi(x(s))\,\Phi(x(0))\rangle\, ds\,, \tag{63}$$

where s is the proper time along the detector's world line $x(s)$. The Wightman function $\langle\Phi\Phi\rangle$ may be evaluated relative to any vacuum of interest. Evaluated relative to the Minkowski vacuum, the familiar result is

$$\langle\Phi(x')\Phi(x)\rangle_M=\frac{1}{4\pi^2[(\mathbf{x}'-\mathbf{x})^2-(t'-t-i\epsilon)^2]}\,. \tag{64}$$

For inertial motion, this becomes $-1/4\pi^2 \times(s-i\epsilon)^2$ and $S(E)=0$ for $E>0$, i.e., the detector "sees nothing." For motion with uniform acceleration ξ^{-1}, this expression is $-1/8\pi^2\xi^2 \times\{\cosh[(s-i\epsilon)/\xi]-1\}$ and the resulting spectrum for $S(E)$ is a Planck spectrum. More general stationary motions yield spectra which are distinctly non-Planckian.[11]

We have evaluated the Wightman function relative to the Fulling vacuum and the result, expressed in pseudocylindrical coordinates (τ,ξ,y,z), is

$$\langle\Phi(x')\Phi(x)\rangle_F=\frac{1}{4\pi^2\xi\xi'}\,\frac{1-\alpha^2}{\alpha}\times\frac{\tanh^{-1}\alpha}{(2\tanh^{-1}\alpha)^2-(\tau'-\tau-i\epsilon)^2}\,, \tag{65}$$

where

$$\alpha=\left[\frac{(\xi'-\xi)^2+(y'-y)^2+(z'-z)^2}{(\xi'+\xi)^2+(y'-y)^2+(z'-z)^2}\right]^{1/2} \tag{66}$$

For uniformly accelerated motion (ξ, y, and z constant) this reduces to $-1/4\pi^2(s-i\epsilon)^2$, and thus the detector will indeed see nothing in the Fulling vacuum. More general stationary motions give Wightman functions which clearly have nonvanishing Fourier transforms for $E>0$. The correlation between vacuum states defined via canonical quantum field theory and via a detector is thus broken for more general stationary motions, and we must conclude that the two definitions are inequivalent.

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[1]S. A. Fulling, Phys. Rev. D 7, 2850 (1973).

[2]W. G. Unruh, Phys. Rev. D 14, 870 (1976).

[3]J. R. Letaw and J. D. Pfautsch, J. Math. Phys. (to be published).

[4]G. Denardo and R. Percacci, Nuovo Cimento 48B, 81 (1978).

[5]J. R. Letaw and J. D. Pfautsch, Phys. Rev. D 22, 1345 (1980).

[6]The development in Sec. II is largely in the spirit of B. S. DeWitt, Phys. Rep. 19C, 295 (1975). Further details on some of this material may be found therein.

[7]Our convention and notation follow that of M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[8]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[9]E. C. Titchmarsh, *Eigenfunction Expansions* (Oxford University Press, London, 1958), Vol. I.

[10]W. Unruh (private communication).

[11]J. R. Letaw, Phys. Rev. D (to be published).