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# Dirac electron in space-times with torsion: Spinor propagation, spin precession, and nongeodesic orbits

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The WKB limit of the noniterated Dirac equation in a Riemann-Cartan space-time is discussed. It is shown that within this framework the behavior of a Dirac particle is dominated by the new connection $\Gamma^{*}_{\mu\nu}{}^{\lambda} = \{{}^{\lambda}_{\mu\nu}\} - 3K_{[\mu\nu\epsilon]}g^{\epsilon\lambda}$ formed from the Christoffel connection and the contortion. The relevant effects are the following: (i) The normalized Dirac spinor is parallel propagated by $\Gamma^{*}_{\mu\nu}{}^{\lambda}$ along the particle's orbit. (ii) The same is true for the spin vector. By this the gyrogravitational ratio is specified as well. (iii) The particle orbit is nongeodesic. The respective "force" is of the usual form with the spin coupled to the curvature tensor $R^{*\alpha\beta\gamma\delta}(\Gamma^{*})$ of the connection $\Gamma^{*}_{\mu\nu}{}^{\lambda}$. The orbit is thereby defined by the streamlines of the conserved convection four-current obtained from the Dirac current by a Gordon decomposition. The cumulative effects (ii) and (iii) can in principle be used to detect torsion by measuring the spin precession of a massive spin-1/2 particle or by measuring its orbit in a Stern-Gerlach type of experiment.

## I. INTRODUCTION

Why does one study the Dirac electron in a Riemann-Cartan space-time $U_4$? Two facts link a $U_4$ theory of gravitation with quantum mechanics:

(i) One basic motivation for the introduction of the $U_4$ theory is that general relativity in Riemann space $V_4$ turns out to be nonrenormalizable. It has no correct quantum version within the usual field-theoretical framework. On the other hand, gauge theories have been successfully renormalized, and $U_4$ gravity can be obtained by a certain type of gauging.

(ii) The second fact is that Riemann-Cartan gravity shows its genuine influence on matter only in quantum-mechanical effects when interacting with massive elementary particles with spin. The usual gravitational field equations in Riemann-Cartan space-time $U_4$ (as well as in the special case of a "teleparallelism" Weitzenböck space-time $A_4$) agree on the macroscopic scale up to high orders with Einstein's theory.[1] Furthermore, independent of the gravitational field equations, gauge fields (like the electromagnetic field) on the one hand and matter with no net elementary-particle spin on the other are influenced in a $U_4$ by only the Riemann-Christoffel part of the connection and the curvature.[2]

Today, the appropriate structure of a quantum theory of gravitation is still one of the important open questions of theoretical physics. Although we do not know the correct quantum version of gravity as represented by space-time curvature, we can treat the influence of gravity on quantum-mechanical systems in an external-field approach. This approach works well, for example, for electromagnetic fields in flat space-time. Thus, we assume that a $U_4$ theory of gravity (or its special cases (is the correct macroscopic theory of gravity to use as a semiclassical approximation to quantum gravity. There are reasons to believe that embedding quantum-mechanical systems into the appropriate curved space-time will represent correctly the influence of gravity as an external field at least as long as (a) the effects of gravity are not of generic quantum-field-theoretical nature (containing, for example, radiative corrections), which would call for a quantized and renormalizable theory of gravity and (b) as long as the intended statements are macroscopic by nature. The latter presupposes that all geometrical manipulations which are necessary to give results an operational meaning refer to macroscopic clocks and macroscopic length scales.

The two types of experiment described below, a Stern-Gerlach-type experiment and an experiment to demonstrate spin polarization of spin-$\frac{1}{2}$ particles, fulfill the conditions (a) and (b).

In the following we discuss the Dirac field in a classical background $U_4$ geometry characterized by a given affine connection $\Gamma_{\alpha\beta}{}^{\gamma}$. Our results will therefore be independent of the field equations for the metric and the torsion. It has been shown by the author[3] that in the WKB limit in the limiting case of a Riemann-Cartan space with vanishing torsion (i.e., a Riemann space $V_4$), the spin vector $S_0^{\alpha}$ of a Dirac particle is parallel propagated along the particle trajectory with tangent vector $u^{\alpha}$:

$$S^{\mu}_{0;\epsilon}u^{\epsilon} = 0\,. \qquad (1.1)$$

The subscript zero indicates the lowest order in an expansion in $\hbar$ [cf. (6.10)]. The semicolon denotes the covariant derivative with regard to the Christoffel connection $\{{}^{\gamma}_{\alpha\beta}\}$. Hayashi and Shirafuji[4] on the other hand have discussed the other limiting case of a Riemann-Cartan space $U_4$ with identical zero curvature (i.e., the teleparallelism Weitzenböck space-time $A_4$). Their calculation is based on a second-order wave equation derived from the first-order Dirac equation and on a two-component spinor. They obtained

$$S^{\mu}_{0;\epsilon}u^{\epsilon}=3K^{[\epsilon\alpha\mu]}S_{0\alpha}u_{\epsilon}+O(\hbar), \tag{1.2}$$

where $K_{\alpha\beta\gamma}$ is the contortion tensor. Rumpf[5] was the first to discuss the spin motion of a Dirac electron in the full Riemann-Cartan space-time $U_4$. He obtained, with the algebraical method of Corben, the operator equation of motion

$$\dot{w}^{\mu}=3K^{[\epsilon\alpha\mu]}(x)w_{\alpha}\dot{x}_{\epsilon}+O(\hbar), \tag{1.3}$$

where $w^{\mu}$ is an operator constructed in analogy to the Pauli-Lubanski vector and the $x^{\epsilon}$ represents three space and one time operator. The dot denotes the derivative with respect to an additionally introduced $c$-number proper time on which a generalized Heisenberg picture is based. Note that (1.3) contains products of operators which may not be equated to products of average values. Accordingly (1.3) cannot simply be read as an Ehrenfest-type equation for quantum-mechanical mean values. Despite the fact that there seems to be no satisfactory procedure to link Corben's method with the usual quantum mechanics, the method appears to have a certain formal power. It leads to equations which, as we will see, successfully "mirror" the equations obtained totally within the usual quantum mechanics.

The aim of the calculations below is to give a *genuine quantum-mechanical derivation* of (1.2) for the *full* Riemann-Cartan space-time. This is done by *deducing the WKB limit* of the propagation equation for a Dirac particle. After a Gordon decomposition of the Dirac current, this equation then enables one to show how torsion, coupled to elementary-particle spin, forces the particle onto a nongeodesic[6] orbit. The spin precession and this nongeodesic orbit represent effects which, in principle, could form a basis for a measurement of the torsion.

To see how the linearly independent components of the spinorial part of the WKB limit of the Dirac wave function propagate and to demonstrate in detail how the kinematical properties of the congruence of streamlines interfere, we base the following discussion on the first-order Dirac equation itself, rather than on a second-order wave function.

## II. DIRAC THEORY IN A RIEMANN-CARTAN SPACE-TIME

A Riemann-Cartan space-time $U_4$ possesses the metric-compatible affine connection[7]

$$\Gamma_{\alpha\beta}{}^{\gamma}=\left\{{}^{\gamma}_{\alpha\beta}\right\}-K_{\alpha\beta}{}^{\gamma}, \tag{2.1a}$$

$$K_{\alpha\beta\gamma}=K_{\alpha[\beta\gamma]}, \tag{2.1b}$$

where $\left\{{}^{\gamma}_{\alpha\beta}\right\}$ denoted the usual Christoffel connection and $K_{\alpha\beta}{}^{\gamma}$ is the contortion tensor related to the torsion $S_{\alpha\beta}{}^{\gamma}$ by

$$S_{\alpha\beta}{}^{\gamma}=\Gamma_{[\alpha\beta]}{}^{\gamma}=-K_{[\alpha\beta]}{}^{\gamma}. \tag{2.2}$$

We introduce covariant derivatives with the full Cartan connection $\Gamma_{\alpha\beta}{}^{\gamma}$ and the Christoffel connection $\left\{{}^{\gamma}_{\alpha\beta}\right\}$ and use the following notations:

$$(\ )_{\parallel\alpha}=\nabla^{\Gamma}_{\alpha}(\ ), \tag{2.3a}$$

$$(\ )_{;\alpha}=\nabla^{\{\ \}}_{\alpha}(\ ), \tag{2.3b}$$

$$\nabla_{\alpha}=\nabla^{\Gamma}_{\alpha}\ \text{ or }\ \nabla^{\{\ \}}_{\alpha} \tag{2.3c}$$

and

$$A^{\alpha}_{\ \parallel\beta}=A^{\alpha}_{\ ,\beta}+\Gamma_{\beta\epsilon}{}^{\alpha}A^{\epsilon}. \tag{2.4}$$

Both connections are metric:

$$\nabla_{\epsilon}g_{\alpha\beta}=0. \tag{2.5}$$

Additionally, the completely antisymmetric Levi-Civita tensor[8] satisfies

$$\nabla_{\epsilon}\eta^{\alpha\beta\gamma\delta}=0. \tag{2.6}$$

To introduce spinors we define a tetrad field $h^{\alpha}_{a}(x)$ such that

$$h^{\alpha}_{a}h^{\beta}_{b}\eta^{ab}=g^{\alpha\beta}. \tag{2.7}$$

By means of a set of standard Dirac matrices (Ref. 9) $\gamma^{a}$, we introduce the generalized Dirac matrices

$$\gamma^{\alpha}=h^{\alpha}_{a}\gamma^{a}, \tag{2.8a}$$

$$\gamma^{5}=-\frac{i}{4!}\eta_{\alpha\beta\gamma\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}=i\gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)}, \tag{2.8b}$$

which obey

$$\gamma^{(a}\gamma^{b)}=\eta^{ab}, \tag{2.9a}$$

$$\gamma^{(\alpha}\gamma^{\beta)}=g^{\alpha\beta}, \tag{2.9b}$$

$$\gamma^{\alpha}\gamma^{5}=-\gamma^{5}\gamma^{\alpha}. \tag{2.9c}$$

Spinor derivatives are given by

$$\nabla_{\alpha}\Psi=\Psi_{,\alpha}+\Gamma_{\alpha}\Psi \tag{2.10a}$$

and

$$\nabla_{\alpha}\overline{\Psi}=\overline{\Psi}_{,\alpha}-\overline{\Psi}\Gamma_{\alpha}, \tag{2.10b}$$

where we have introduced the adjoint spinor $\overline{\Psi}=\Psi^{\dagger}\gamma^{(0)}$ and

$$\Gamma_{\alpha}=\tfrac{1}{4}(\nabla_{\alpha}h^{\epsilon}_{a})h^{b}_{\epsilon}\gamma_{b}\gamma^{a}. \tag{2.11}$$

For the Dirac matrices $\gamma^{\alpha}$ we then obtain

$$\gamma^{\alpha}{}_{\parallel\mu}=\gamma^{\alpha}{}_{,\mu}+\Gamma_{\mu\epsilon}{}^{\alpha}\gamma^{\epsilon}+\Gamma_{\mu}\gamma^{\alpha}-\gamma^{\alpha}\Gamma_{\mu}=0 \tag{2.12}$$

and the corresponding relation $\gamma^{\alpha}{}_{;\mu}=0$ when $\Gamma_{\alpha\beta}{}^{\gamma}$ is replaced by $\{{}^{\gamma}_{\alpha\beta}\}$:

$$\nabla_{\mu}\gamma^{\alpha}=0, \tag{2.13a}$$

$$\nabla_{\mu}\gamma^{5}=0. \tag{2.13b}$$

The Dirac Lagrangian minimally coupled with regard to the $U_4$ connection implies the Riemann-Cartan *Dirac equation* (with $c=1$)

$$i\gamma^{\mu}\Psi_{\|\mu}+\frac{i}{2}\gamma^{\mu}K_{\epsilon\mu}{}^{\epsilon}\Psi-\frac{m}{\hbar}\Psi=0 \tag{2.14a}$$

which can be written as

$$i\gamma^{\mu}\Psi_{;\mu}+\frac{i}{4}K_{[\alpha\beta\gamma]}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\Psi-\frac{m}{\hbar}\Psi=0. \tag{2.14b}$$

The adjoint equation is

$$i\bar{\Psi}_{\|\mu}\gamma^{\mu}+\frac{i}{2}\bar{\Psi}\gamma^{\mu}K_{\epsilon\mu}{}^{\epsilon}+\frac{m}{\hbar}\Psi=0. \tag{2.15}$$

The *Dirac current*

$$j^{\alpha}=\bar{\Psi}\gamma^{\alpha}\Psi \tag{2.16}$$

is conserved:

$$j^{\alpha}{}_{;\alpha}=0. \tag{2.17}$$

## III. WKB APPROXIMATION

To obtain the behavior of the Dirac wave function in the semiclassical limit, we introduce the *WKB expansion*

$$\Psi(x)=\exp[iS(x)/\hbar]\sum_{n=0}^{\infty}(-i\hbar)^{n}a_{n}(x) \tag{3.1}$$

and restrict ourselves to situations in which the applicability conditions of a WKB approximation are fulfilled. We may assume $S(x)$ to be real. The $a_n(x)$ are spinors. Inserting (3.1) into the Dirac equation (2.14) and equating the coefficients of the different orders of $\hbar$ to zero, we obtain for the first orders

$$(\gamma^{\alpha}S_{;\alpha}+m)a_{0}(x)=0 \tag{3.2}$$

and

$$\begin{aligned}(\gamma^{\alpha}S_{;\alpha}+m)a_{1}(x)&=-\gamma^{\alpha}a_{0\|\alpha}-\tfrac{1}{2}\gamma^{\alpha}K_{\epsilon\alpha}{}^{\epsilon}a_{0}\\&=-\gamma^{\alpha}a_{0;\alpha}-\tfrac{1}{4}K_{[\alpha\beta\gamma]}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}a_{0}.\end{aligned} \tag{3.3}$$

Introducing

$$p_{\alpha}=-S_{,\alpha}, \tag{3.4}$$

we obtain as a consequence of (3.2) the *Hamilton-Jacobi equation*

$$p_{\alpha}p^{\alpha}=m^{2}. \tag{3.5}$$

The timelike congruence orthogonal to the hypersurfaces of constant "phase" $S(x)$ is described by the tangent vector field $u^{\alpha}$,

$$u_{\alpha}(x)=\frac{1}{m}p_{\alpha}=-\frac{1}{m}S_{,\alpha}, \tag{3.6a}$$

$$u^{\alpha}u_{\alpha}=1. \tag{3.6b}$$

One consequence of (3.6a) used below is that $u_{\alpha;\beta}=u_{\beta;\alpha}$. We will see later in Sec. VII that the $u^{\alpha}$ congruence describes the motion of the Dirac matter in the completely classical limit in consequence of

$$(j^{\epsilon}j_{\epsilon})^{-1/2}j^{\alpha}=u^{\alpha}+O(\hbar). \tag{3.7}$$

Because of (3.6a) and (3.6b) we find

$$u_{\alpha;\epsilon}u^{\epsilon}=0. \tag{3.8}$$

*So to order zero in $\hbar$ (i.e., in the completely classical limit), the trajectories of the Dirac current in a Riemann-Cartan space-time are geodesics.*

Equation (3.6a) specifies as well the remaining kinematical properties of the $u^{\alpha}$ congruence:

$$u_{\alpha;\beta}=\hat{\sigma}_{\alpha\beta}+\tfrac{1}{3}\hat{\Theta}P_{\alpha\beta}, \tag{3.9}$$

where $P_{\alpha\beta}$ is the tensor projecting onto the space orthogonal to $u^{\alpha}$,

$$P_{\alpha\beta}=g_{\alpha\beta}-u_{\alpha}u_{\beta}, \tag{3.10}$$

and the expansion $\hat{\Theta}$ and the shear $\hat{\sigma}_{\alpha\beta}$ of the congruence are given by

$$\hat{\Theta}=u^{\alpha}{}_{;\alpha}, \tag{3.11}$$

$$\hat{\sigma}_{\alpha\beta}=u_{(\kappa;\lambda)}P^{\kappa}_{\alpha}P^{\lambda}_{\beta}-\tfrac{1}{3}\hat{\Theta}P_{\alpha\beta}, \tag{3.12}$$

which imply

$$\hat{\sigma}_{[\alpha\beta]}=0, \tag{3.13a}$$

$$\hat{\sigma}^{\epsilon}{}_{\epsilon}=0, \tag{3.13b}$$

$$\hat{\sigma}_{\alpha\beta}u^{\beta}=0. \tag{3.13c}$$

## IV. ALGEBRAICAL CONSEQUENCES

The WKB equation (3.2) fixes $a_0(x)$ only algebraically. Accordingly the general solution $a_0(x)$ has the form

$$a_{0}(x)=\beta_{1}(x)b_{01}(x)+\beta_{2}(x)b_{02}(x), \tag{4.1}$$

where $b_{01}(x)$ and $b_{02}(x)$ are the two well-known linearly independent momentum-space solutions of (3.2):

$$b_{01}=\left(\frac{p^{(0)}+m}{2m}\right)^{1/2}\begin{pmatrix}1\\0\\ \dfrac{p^{(3)}}{p^{(0)}+m}\\ \dfrac{p^{(1)}+ip^{(2)}}{p^{(0)}+m}\end{pmatrix}, \tag{4.2a}$$

$$b_{02}=\left[\frac{p^{(0)}+m}{2m}\right]^{1/2}\begin{bmatrix}0\\1\\ \frac{p^{(1)}-ip^{(2)}}{p^{(0)}+m}\\ -\frac{p^{(3)}}{p^{(0)}+m}\end{bmatrix} \tag{4.2b}$$

with

$$p^a=p^\alpha h^a_\alpha. \tag{4.3}$$

The complex functions $\beta_1(x)$ and $\beta_2(x)$ in (4.1) are still to be determined.

In a Riemann-Cartan space-time, local Lorentz rotations of the tetrad are coupled with local spin transformations. Equations containing spinors remain invariant if both transformations are performed together. Accordingly these equations can be verified without loss of generality by choosing a particular tetrad field. The following choice proves to be convenient: We restrict to an arbitrary but fixed world line of the $u^\alpha$ congruence and let $\overset{*}{=}$ denote equality along that world line. (The asterisk can be omitted if the respective equation is invariant against tetrad spin transformations.) We choose the timelike tetrad vector $h^\alpha{}_{(0)}$ parallel to $u^\alpha$,

$$h^\alpha{}_{(0)}\overset{*}{=}u^\alpha. \tag{4.4}$$

Using parallel propagation with the Christoffel connection, we then propagate the tetrad parallel along the chosen $u^\alpha$ line [which is consistent because of equation (3.8)] and as well parallel into the neighborhood of this world line. This construction leads to a tetrad field in a tube, which apart from (4.4) fulfills on the world line

$$h^\alpha_{a;\epsilon}\overset{*}{=}0. \tag{4.5}$$

Note that in the neighborhood of the world line the tetrad vector $h^\alpha_{(0)}$ will in general not be parallel to the tangent vector $u^\alpha$ of the congruence.

An immediate consequence is

$$\Psi_{;\alpha}\overset{*}{=}\Psi_{,\alpha}. \tag{4.6}$$

Furthermore, referring to the choice (4.5) and usind the kinematical properties (3.9) and (3.13) of the $u^\alpha$ congruence, we obtain

$$p_{(0),\epsilon}h^\epsilon_b\overset{*}{=}0, \tag{4.7a}$$

$$p_{\hat{a},\epsilon}h^\epsilon_{(0)}\overset{*}{=}0, \tag{4.7b}$$

$$p_{\hat{a},\epsilon}h^\epsilon_{\hat{b}}\overset{*}{=}m\,(\hat{\sigma}_{\alpha\beta}+\tfrac{1}{3}\,\hat{\Theta}P_{\alpha\beta})h^\alpha_{\hat{a}}h^\beta_{\hat{b}}. \tag{4.7c}$$

With (4.4) the solutions $b_{01}$ and $b_{02}$ of (4.2) reduce to

$$b_{01}\overset{*}{=}\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad b_{02}\overset{*}{=}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \tag{4.8}$$

which implies

$$\bar{b}_{01}\gamma^\mu b_{01}\overset{*}{=}u^\mu, \tag{4.9a}$$

$$\bar{b}_{02}\gamma^\mu b_{02}\overset{*}{=}u^\mu, \tag{4.9b}$$

$$\bar{b}_{01}\gamma^\mu b_{02}\overset{*}{=}0, \tag{4.9c}$$

$$\bar{b}_{02}\gamma^\mu b_{01}\overset{*}{=}0. \tag{4.9d}$$

We find from (4.2) and (4.7)

$$\bar{b}_{01}\gamma^\epsilon b_{01,\epsilon}\overset{*}{=}\frac{\hat{\Theta}}{2}, \tag{4.10a}$$

$$\bar{b}_{02}\gamma^\epsilon b_{02,\epsilon}\overset{*}{=}\frac{\hat{\Theta}}{2}, \tag{4.10b}$$

$$\bar{b}_{01}\gamma^\epsilon b_{02,\epsilon}\overset{*}{=}0, \tag{4.10c}$$

$$\bar{b}_{02}\gamma^\epsilon b_{01,\epsilon}\overset{*}{=}0, \tag{4.10d}$$

and with (4.5) also (for $i=1,2$)

$$b_{0i\Vert\epsilon}u^\epsilon\overset{*}{=}\tfrac{1}{4}u^\epsilon K_{\epsilon\kappa\lambda}\gamma^\kappa\gamma^\lambda b_{0i}. \tag{4.11}$$

For later use we note the following algebraical relations which can be proven from (4.8) and the particular form of our Dirac matrices:

$$\gamma^\delta\gamma^5 b_{0j}=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}\sigma_{\alpha\beta}b_{0j}u_\gamma, \tag{4.12}$$

where

$$\sigma^{\alpha\beta}=i\,\gamma^{[\alpha}\gamma^{\beta]}. \tag{4.13}$$

Similarly we find

$$K_{[\alpha\beta\gamma]}\gamma^\alpha\gamma^\beta\gamma^\gamma b_{0j}=-3i\,K_{[\alpha\beta\gamma]}\sigma^{\alpha\beta}b_{0j}u^\gamma. \tag{4.14}$$

## V. SOLVABILITY CONDITIONS AND SPINOR PROPAGATION

Because in general the $u^\alpha$ congruence will expand or contract, the density of the Dirac field will vary and the absolute value $\bar{a}_0a_0$ of the spinor $a_0$ will not remain constant. To separate out this density effect, we introduce a normalized spinor $b_0$ proportional to $a_0$ by

$$a_0(x)=f(x)b_0(x), \tag{5.1a}$$

$$\bar{b}_0b_0=1, \tag{5.1b}$$

where $f(x)$ is a real function which is given by

$$f^2(x)=\bar{a}_0a_0=\beta_1^*(x)\beta_1(x)+\beta_2^*(x)\beta_2(x). \tag{5.2}$$

In the following we will show how $f(x)$, $b_0(x)$, and $a_0$ propagate along the $u^\alpha$ world lines.

While the WKB equation (3.2) determines $a_0(x)$ algebraically, the differential behavior of $a_0(x)$ is restricted by the solvability condition of the WKB equation (3.3) of next higher order in $\hbar$. For a given $a_0(x)$, Eq. (3.3) is an inhomogeneous linear algebraical equation for $a_1$. Consequently the condition for the existence of a nontrivial solution $a_1$ of (3.3) is that all solutions of the corresponding transposed homogeneous equation are orthogonal to the inhomogeneity. Comparison of the homogeneous part of (3.3) with (3.2) shows that the solutions in question are the $\bar{b}_{01}(x)$ and $\bar{b}_{02}(x)$ of (4.1). Therefore, as *solvability conditions* of (3.3) we obtain differential conditions ($i=1, 2$)

$$b_{0i}(\gamma^\alpha a_{0;\alpha} + \tfrac{1}{4} K_{[\alpha\beta\gamma]}\gamma^\alpha\gamma^\beta\gamma^\gamma a_0) = 0. \tag{5.3}$$

Inserting into (5.3) the relations (4.8), (4.9), and (4.10), which represent our knowledge of $a_0$ as far as that was obtained before, we find

$$\beta_{1,\epsilon}u^\epsilon = -\tfrac{1}{2}\hat{\Theta}\beta_1 - \tfrac{1}{4}K_{[\alpha\beta\gamma]}\bar{b}_{01}\gamma^\alpha\gamma^\beta\gamma^\gamma b_{01}\beta_1 - \tfrac{1}{4}K_{[\alpha\beta\gamma]}\bar{b}_{01}\gamma^\alpha\gamma^\beta\gamma^\gamma b_{02}\beta_2, \tag{5.4a}$$

$$\beta_{2,\epsilon}u^\epsilon = -\tfrac{1}{2}\hat{\Theta}\beta_2 - \tfrac{1}{4}K_{[\alpha\beta\gamma]}\bar{b}_{02}\gamma^\alpha\gamma^\beta\gamma^\gamma b_{01}\beta_1 - \tfrac{1}{4}K_{[\alpha\beta\gamma]}\bar{b}_{02}\gamma^\alpha\gamma^\beta\gamma^\gamma b_{02}\beta_2. \tag{5.4b}$$

A consequence of (5.2) and (5.4) then is

$$f_{,\epsilon}u^\epsilon = -\tfrac{1}{2}\hat{\Theta}f. \tag{5.5}$$

The propagation of $a_0$ is finally obtained by differentiating (4.1) and using (4.11) and (5.4). The result is

$$a_{0\|\epsilon}u^\epsilon = -\frac{\hat{\Theta}}{2}a_0 + \frac{i}{2}K_{[\alpha\beta]\epsilon}\sigma^{\alpha\beta}a_0u^\epsilon, \tag{5.6}$$

where we made use of the fact that according to (4.8) $b_{0i}(\cdots)b_{0j}$ represents one component of the matrix $(\cdots)$. Additionally we used the generally valid relation

$$\tfrac{3}{2}K_{[\alpha\beta\gamma]} = K_{[\alpha\beta]\gamma} + \tfrac{1}{2}K_{\gamma\alpha\beta}. \tag{5.7}$$

Direct consequences of (5.5) and (5.6) are *the propagation equations for the normalized spinors $b_0(x)$ and $\bar{b}_0(x)$*,

$$b_{0\|\epsilon}u^\epsilon = \frac{i}{2}K_{[\alpha\beta]\epsilon}\sigma^{\alpha\beta}b_0u^\epsilon, \tag{5.8a}$$

$$\bar{b}_{0\|\epsilon}u^\epsilon = -\frac{i}{2}K_{[\alpha\beta]\epsilon}\bar{b}_0\sigma^{\alpha\beta}u^\epsilon. \tag{5.8b}$$

## VI. THE MICROPHYSICALLY RELEVANT CONNECTION AND SPIN PRECESSION

The way a Dirac spinor is propagated in the WKB limit in a Riemann-Cartan space-time can be given a simple geometrical meaning. To do so we introduce in addition to $\Gamma_{\mu\nu}{}^\lambda$ and $\{{}^{\lambda}_{\mu\nu}\}$ the *new connection*

$$\Gamma^*_{\mu\nu}{}^\lambda = \Gamma_{\mu\nu}{}^\lambda + 2S_\nu{}^\lambda{}_\mu. \tag{6.1}$$

It can be rewritten as

$$\Gamma^*_{\mu\nu}{}^\lambda = \{{}^{\lambda}_{\mu\nu}\} + 3S_{[\mu\nu\epsilon]}g^{\epsilon\lambda} \tag{6.2a}$$

$$= \{{}^{\lambda}_{\mu\nu}\} - 3K_{[\mu\nu\epsilon]}g^{\epsilon\lambda} \tag{6.2b}$$

$$= \Gamma_{\mu\nu}{}^\lambda - 2K_{[\nu\epsilon]\mu}g^{\epsilon\lambda}. \tag{6.2c}$$

The covariant derivative based on $\Gamma^*_{\mu\nu}{}^\lambda$ as connection is denoted by $\nabla^*_\alpha$.

This new connection is metric:

$$\nabla^*_\epsilon g_{\alpha\beta} = 0, \tag{6.3}$$

and

$$\nabla^*_\epsilon \eta^{\alpha\beta\gamma\delta} = 0, \tag{6.4}$$

and

$$\nabla^*_\epsilon \gamma^\alpha = 0, \tag{6.5a}$$

$$\nabla_\epsilon \gamma^5 = 0. \tag{6.5b}$$

Furthermore the following relations to the Christoffel derivative $\nabla^{\{\}}_\epsilon$ will be useful:

$$(\nabla^*_\epsilon A^\alpha)A^\epsilon = A^\alpha{}_{;\epsilon}A^\epsilon, \tag{6.6a}$$

$$\nabla^*_\epsilon A^\epsilon = A^\epsilon{}_{;\epsilon}. \tag{6.6b}$$

The consequences of the preceding paragraph can be formulated. In the quasiclassical limit of the Dirac equation in a Riemann-Cartan space-time, *the spinor part $a_0$ of lowest order in $\hbar$ in a WKB expansion is propagated according to*

$$(\nabla^*_\epsilon a_0)u^\epsilon = -\frac{\hat{\Theta}}{2}a_0. \tag{6.7}$$

*Normalizing $a_0$ in (5.1) leads to $b_0$. The main result is that the normalized spinor $b_0$ is parallel propagated with respect to the new connection $\Gamma^*_{\alpha\beta}{}^\gamma$ along the $u^\alpha$ congruence orthogonal to the surfaces of constant phase S:*

$$(\nabla^*_\epsilon b_0)u^\epsilon = 0, \tag{6.8a}$$

$$(\nabla^*_\epsilon \bar{b}_0)u^\epsilon = 0. \tag{6.8b}$$

It is this equation (6.8) which determines the behavior of the localized physical quantities.

The WKB approximation to a Dirac solution $\Psi(x)$ describes a stream of "free" particles, i.e., particles which are influenced by metric and torsion only. The spin density is related to $\bar{\Psi}\sigma^{\alpha\beta}\Psi$ and the particle number density to $\bar{\Psi}\Psi$. Both are observer-independent constructions. See Ref. 3 for further interpretation. Accordingly the *spin vector*

$$S^{\alpha}=\tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}u_{\beta}\,\frac{\bar{\Psi}\sigma_{\gamma\delta}\Psi}{\bar{\Psi}\Psi} \tag{6.9}$$

represents the components of the spin of the particle. The tangent vector to the "orbit" of the particle as defined by the convection current is denoted as $v^{\alpha}$. It is related to $u^{\alpha}$ by $v^{\alpha}=u^{\alpha}+O(\hbar)$ [compare (7.6)] and will be specified below in Eq. (7.5).

Introducing the WKB expansion we find

$$S^{\alpha}=\tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}u_{\beta}\,\bar{b}_0\sigma_{\gamma\delta}\,b_0+O(\hbar)=S_0^{\ \alpha}+O(\hbar). \tag{6.10}$$

Because of (4.12) it can also be written in the lowest order of $\hbar$ as

$$S_0^{\ \alpha}=\bar{b}_0\gamma^5\gamma^{\alpha}b_0, \tag{6.11}$$

which may be the more familiar expression in the framework of a Riemann-Cartan theory.

An almost immediate consequence of the propagation equation (6.8) is then

$$(\nabla^{*}_{\epsilon}S_0^{\alpha})u^{\epsilon}=0. \tag{6.12}$$

Equation (6.12) follows from (6.10) using (6.4), (6.5a), (6.6a), and (3.8) or from (6.11) with (6.5) and (6.8). *In a Riemann-Cartan space-time therefore the localized Dirac spin vector is to the lowest order in $\hbar$ of a WKB expansion parallel transported with respect to the new connection $\Gamma^{*}_{\lambda\mu}{}^{\nu}$ of (6.1) along the particle orbit.*

Because the behavior of classical matter without net intrinsic spin is governed by the Christoffel connection $\{^{\lambda}_{\alpha\beta}\}$ alone, we decompose (6.12) using (6.2b) to obtain a comparison:

$$S^{\mu}_{0;\epsilon}u^{\epsilon}=3K^{[\mu\kappa\lambda]}S_{0\lambda}u_{\kappa}. \tag{6.13}$$

In terms of the *axial-vector part* $a^{\mu}$ of the torsion or the contortion tensor

$$a^{\mu}=\tfrac{1}{6}\,\eta^{\mu\alpha\beta\gamma}K_{[\alpha\beta\gamma]}, \tag{6.14}$$

the spin propagation equation can be written

$$S^{\mu}_{0;\epsilon}u^{\epsilon}=-3\eta^{\mu\alpha\beta\gamma}a_{\alpha}S_{0\beta}u_{\gamma}. \tag{6.15}$$

How can this spin precession be measured? In a Riemann-Cartan space-time the usual macroscopic gyroscope is Fermi propagated with regard to the Christoffel connection $\{^{\gamma}_{\alpha\beta}\}$. The most direct test of Eq. (6.15) is therefore to compare the motion of the spin vector with the motion of the axes of rotation of three orthogonally oriented gyroscopes. For the components with respect to these three axes, we have the precession

$$\dot{\vec{S}}_0=3\vec{S}_0\times\vec{a}, \tag{6.16}$$

which may be used to detect torsion. Note the factor 3 which is typical for the gyrogravitational ratio if torsion is involved.

## VII. GORDON DECOMPOSITION AND NONGEODESIC ORBIT

Because of Dirac Eqs. (2.14) and (2.15), the Dirac current $j^{\mu}$ of (2.16) can be decomposed in the sense of Gordon decomposition into

$$j^{\mu}=j^{\mu}_c+j^{\mu}_M, \tag{7.1}$$

with $j^{\mu}_M$ defined by

$$j^{\mu}_M=\frac{\hbar}{2m}\,(\bar{\Psi}\sigma^{\mu\epsilon}\Psi)_{;\epsilon} \tag{7.2}$$

and $j^{\mu}_c$ defined by

$$j_{\mu c}=\frac{\hbar}{2mi}\,(\bar{\Psi}_{\parallel\mu}\Psi-\bar{\Psi}\Psi_{\parallel\mu})+\frac{\hbar}{2m}\,\bar{\Psi}\sigma^{\alpha\beta}\Psi K_{\alpha\beta\mu} \tag{7.3a}$$

or

$$j_{\mu c}=\frac{\hbar}{2mi}\,[(\nabla^{*}_{\mu}\bar{\Psi})\Psi-\bar{\Psi}\nabla^{*}_{\mu}\Psi], \tag{7.3b}$$

which mirrors exactly its Riemann-space analog. These currents are conserved separately, $j^{\mu}_M$ because of properties of $\nabla^{\{\}}_{[\alpha}\nabla^{\{\}}_{\beta]}$ and then $j^{\mu}_c$ by (2.17):

$$j^{\mu}_{c;\mu}=0, \tag{7.4a}$$

$$j^{\mu}_{M;\mu}=0. \tag{7.4b}$$

Using the same arguments as used in Ref. 3 for a Riemann space without torsion, one obtains from the structure of (7.2) that $j^{\mu}_M$ is the curl of the spin density. (Since the spin of the electron is coupled to a magnetic dipole moment, this curl is equivalent to an electric current in Maxwell theory.) Because of its origin, $j^{\alpha}_M$ has the meaning of a *magnetization current*. The remaining part $j^{\alpha}_c$ of the total electric current $j^{\alpha}$ has accordingly the meaning of a *convection current*. In correspondence with their interpretations, both currents are separately conserved.

We relate our concept of particle orbits to electromagnetic measurements and therefore base the definition of motion on the convection current. The current $j^{\mu}_c$ of (7.3) defines a congruence of timelike curves with tangent vector $v^{\alpha}$,

$$v^{\alpha}=(j^{\epsilon}_c j_{c\epsilon})^{-1/2}j^{\alpha}_c, \tag{7.5a}$$

$$v^{\alpha}v_{\alpha}=1. \tag{7.5b}$$

Introducing our WKB expansion, we find that $j^{\alpha}_c$ agrees to lowest order in $\hbar$ with the $u^{\alpha}$ congruence, which is the completely classical limit of the orbit

$$j_{c\alpha}=f^2u_\alpha+\frac{\hbar}{i}u_\alpha(\bar{a}_0a_1-\bar{a}_1a_0)+\frac{\hbar}{2mi}f^2[(\nabla^*_\alpha\bar{b}_0)b_0-b_0\nabla^*_\alpha b_0]+O(\hbar^2). \quad (7.6)$$

This, with (6.8), implies

$$v_\alpha=u_\alpha+\frac{\hbar}{2mi}[(\nabla^*_\alpha\bar{b}_0)b_0-\bar{b}_0\nabla^*_\alpha b_0]+O(\hbar^2). \quad (7.7)$$

Based on this expression the nongeodesic behavior of the $v^\alpha$ orbit can be given the form

$$v_{\alpha;\epsilon}v^\epsilon=(\nabla^*_\epsilon v_\alpha)v^\epsilon=2(\nabla^*_{[\epsilon}v_{\alpha]})v^\epsilon =\frac{\hbar}{2mi}\{(\nabla^*_{[\epsilon}\nabla^*_{\alpha]}\bar{b}_0)b_0-\bar{b}_0(\nabla^*_{[\epsilon}\nabla^*_{\alpha]}b_0)\}u^\epsilon+O(\hbar^2). \quad (7.8)$$

To obtain the "force" on the right-hand side, we used (6.6a) and (3.6b).

We introduce the curvature tensor of the $\Gamma^*$ connection

$$R^*_{\alpha\beta\gamma}{}^\delta=2\Gamma^*{}_{[\beta|\gamma}{}^\delta{}_{,|\alpha]}+2\Gamma^*_{[\alpha|\epsilon}{}^\delta\Gamma^*_{|\beta]\gamma}{}^\epsilon. \quad (7.9)$$

The generalized Ricci identity for spinors then reads

$$\nabla^*_{[\alpha}\nabla^*_{\beta]}\Psi=\tfrac{1}{8}R^*_{\alpha\beta\kappa\lambda}\gamma^{[\lambda}\gamma^{\kappa]}\Psi-\Gamma^*_{[\alpha\beta]}{}^\kappa\nabla^*_\kappa\Psi, \quad (7.10a)$$

$$\nabla^*_{[\alpha}\nabla^*_{\beta]}\bar{\Psi}=-\tfrac{1}{8}R^*_{\alpha\beta\kappa\lambda}\bar{\Psi}\gamma^{[\lambda}\gamma^{\kappa]}-\Gamma^*_{[\alpha\beta]}{}^\kappa\nabla^*_\kappa\bar{\Psi}. \quad (7.10b)$$

Because of these identities, the force equation (7.8) can be given the form [10]

$$m\,v_{\alpha;\epsilon}v^\epsilon=\frac{1}{2}\left(\frac{\hbar}{2}\right)R^*_{\alpha\beta\kappa\lambda}\bar{b}_0\sigma^{\kappa\lambda}b_0u^\beta+O(\hbar^2). \quad (7.11)$$

*The particle orbit is nongeodesic. The force is of the usual form but with spin coupling to the curvature tensor* $R^*_{\alpha\beta\gamma\delta}$ *of the new connection* $\Gamma^*_{\alpha\beta}{}^\gamma$ *of (6.1).*

Macroscopic test bodies and light rays move on geodesics constructed from the Christoffel connection $\{{}^\gamma_{\alpha\beta}\}$. Macroscopic frames of reference will be based on this. Accordingly, relative to them, a Dirac particle will not move freely but will experience the force given by the right-hand side of (7.11). Its influence on the orbit can be measures in a Stern-Gerlach type of experiment.

## VIII. CONCLUSION

The question "How does one—at least in principle—measure torsion?" is not a new one.[11] From the very beginning the recipe has been: use the elementary-particle spin. But even for nonvanishing spin density a macroscopic body will not be affected by torsion, if the elementary-particle spin integrates to zero. Such a test body is sensitive to the Christoffel connection $\{{}^\gamma_{\alpha\beta}\}$ only. [See Yasskin and Stoeger (Ref. 2) for this as well as for the previous literature.] For spin-polarized macroscopic bodies, the Papapetrou-type equations for motion and precession are of the usual structure with $\{{}^\gamma_{\alpha\beta}\}$ being replaced by the Cartan connection $\Gamma^\gamma_{\alpha\beta}$. But these equations contain ambiguities and nondeterministic elements as Yasskin and Stoeger (Ref. 2) have pointed out. Therefore a genuine *quantum-mechanical treatment* of a massive spin-$\frac{1}{2}$ elementary particle itself seems to be the only promising approach which remains.

Quantum-mechanical measurements will refer to macroscopic instruments which themselves react only on the Christoffel part of the connection. This has the advantage, that in most cases, the measurements will directly reflect the influence of the torsion.

Relevant effects which demonstrate the influence of torsion on a quantum-mechanical system can mainly be expected for quantum systems of macroscopic extension. Apart from superfluids and supercurrents, these systems are most adequately described in the framework of a *WKB approximation*. It is important to note that the contribution to the torsion from the spin of the "measuring" Dirac particle itself is of order $\hbar$ and thus falls into the terms neglected in the approximation. This is true for any Riemann-Cartan theory obtained from variational principle by varying the torsion. Furthermore, the WKB limit has the advantage of being free of the difficulties of interpretation which are usually involved in a full Hilbert-space treatment of quantum mechanics in curved space-time.

Our discussion above has led to the result that the WKB limit of the Dirac equation in a Riemann-Cartan space-time is dominated by the new connection $\Gamma^*_{\mu\nu}{}^\lambda=\{{}^\lambda_{\mu\nu}\}-3K_{[\mu\nu\epsilon]}g^{\epsilon\lambda}$: (i) The normalized Dirac spinor is parallel propagated by $\Gamma^*_{\mu\nu}{}^\lambda$ along the particle's orbit. (The orbit is thereby defined by the streamlines of the conserved convection four-current obtained from the Dirac current by a Gordon decomposition). (ii) The same is true for the spin vector. (iii) The particle orbit is nongeodesic. The respective "force" is of the usual form but with the spin coupled to the curvature tensor $R^{*\alpha\beta\gamma\delta}(\Gamma^*)$ of the connection $\Gamma^*_{\mu\nu}{}^\lambda$.

Two experiments are immediately related to the results (ii) and (iii) above: (a) measurement of the spin precession and (b) measurement of the orbit with a Stern-Gerlach type of experiment. Both are *cumulative effects*. They require solely that one waits long enough or lets the particle travel far enough.

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[6]Throughout this paper a *geodesic* means the extremal (shortest or longest) path between two points as measured by the metric $g_{\alpha\beta}$. Its differential equation contains the Christoffel connection $\{{}^{\gamma}_{\alpha\beta}\}$ only.

[7]We use the following conventions: Signature of the metric tensor $g_{\alpha\beta}$: (+ ---). A comma subscript as in $,\alpha$ denotes the partial derivative. $\alpha, \beta, \ldots = 0, \ldots, 3$ are tensor indices raised and lowered with $g_{\alpha\beta}$. $a, b, \ldots = 0, \ldots, 3$ and $\hat{a}, \hat{b}, \ldots = 1, 2, 3$ are tetrad indices raised and lowered with $\eta_{ab} = \mathrm{diag}\,(+1, -1, -1, -1)$. The corresponding object is a Riemann scalar with regard to $a, b, \ldots$. Particular values of $a, b, \ldots$ are denoted by brackets: $A^{(1)} = A^{a=1}$. Symmetrization: $A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$. Antisymmetrization: $A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$ and

$$A_{[\alpha\beta\gamma]} = \tfrac{1}{3}(A_{\alpha[\beta\gamma]} + A_{\beta[\gamma\alpha]} + A_{\gamma[\alpha\beta]})$$

with vertical bars denoting exclusion from this process,

$$A_{[\alpha|\beta|\gamma]} = \tfrac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}).$$

[8]$\eta_{\alpha\beta\gamma\delta}$ may be introduced by means of $\eta^{\alpha\beta\gamma\delta} = h^{\alpha}_{a} h^{\beta}_{b} h^{\gamma}_{c} h^{\delta}_{d} \epsilon^{abcd}$, where the totally antisymmetric symbol $\epsilon^{abcd}$ is normalized according to $\epsilon^{0123} = +1$.

[9]Our conventions are as follows:

$$\gamma^{\hat{a}} = \begin{pmatrix} 0 & \sigma^{\hat{a}} \\ -\sigma^{\hat{a}} & 0 \end{pmatrix}, \quad \gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{5} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\sigma^{\hat{a}}$ are the standard Pauli spin matrices.

[10]Within a sypersymmetric formulation for classical dynamics and using an appropriate Lagrangian and Grassmann variables to represent the spin, Rumpf has derived a classical equation which mirrors equation (7.11) exactly [H. Rumpf, University of Vienna report, 1981 (unpublished)].

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