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Methods for deriving solutions in generalized theories of gravitation: The Einstein-Cartan theory

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In relativistic theories of gravitation that use, besides the metric, additional geometric objects in the description of the gravitational field, there exist two ways to obtain simplifying assumptions. One is to use symmetries of the geometric objects involved and the other is to use their *a priori* physical role. However, these two methods are related via the field equations; thus an arbitrary selection of simplifying assumptions may be inconsistent. We illustrate this point in the Einstein-Cartan theory and show that (a) under a certain assumption well-known solutions of this theory use incompatible simplifying assumptions and (b) a new solution which is compatible with the cosmological principle, and for which torsion cannot represent spin, exists.

I. INTRODUCTION

The Einstein-Cartan theory[1,2] and the numerous other theories of gravitation with torsion that have been proposed lately[3] make it necessary to study common features of the field equations of these theories.

In this paper we shall consider the class of generalized theories of gravitation which is defined by the following requirements: (i) The gravitational field is described by the metric g_{ab} (signature $+++-$) and some other geometric objects (which we denote with the generic letter S). (ii) The theory reduces to general relativity when the extra geometric objects involved in the description of the gravitational field vanish. Obviously, this class of theories contains all the aforementioned theories of gravitation.

It is reasonable to expect that the field equations of any member in this class shall be at least as complex as the corresponding equations of general relativity. This implies then that we shall have again to consider only special classes of solutions which shall be derived by means of simplifying assumptions.

One might suggest that these assumptions should be sought in the symmetries of the gravitational field. However, a new situation appears here which is foreign to general relativity. As a rule, the introduction of new geometric objects in the description of the gravitational field is due to the fact that we wish to geometrize properties of matter other than inertia which is already geometrized by the metric. Thus, the extra geometric objects introduced have an *a priori* physical role. For example, in the Einstein-Cartan theory torsion is taken to geometrize (i.e., to be coupled to) the intrinsic spin density of matter. Thus the spin part of the Poincaré group changes the geometry of space-time, just as the energy-momentum part does.[1]

Now, it is possible to use the physical role of these objects and to construct simplifying assumptions without referring to the symmetries of the gravitational field or the metric. Evidently these assumptions do not suffice for the simplification of the field equations because they do not involve the metric. Hence further assumptions concerning the metric have to be introduced which are, as a rule, symmetry assumptions as in general relativity.

Summarizing, we have the following procedure to simplify the field equations of these generalized theories: (a) To simplify the metric, use symmetry assumptions, expressed in terms of Killing vectors as in general relativity. (b) To simplify the extra geometric objects S, use their *a priori* assumed physical role and/or additional symmetries (not necessarily related to the ones already assumed for the metric).

The purpose of this paper is to discuss the general procedure outlined above. It will be shown that, under certain circumstances, it can lead to mathematical inconsistencies; further, it may exclude perfectly consistent solutions of the theory. The reason for this is that the metric and the other geometric objects S are related via the field equations; thus it is possible that the requirement that the metric admits a certain symmetry implies that some (or all) of the objects S *must* admit the same symmetry. Evidently, this induced symmetry introduces a new simplification for these objects (similar to the one of the metric) which may be incompatible with the simplification(s) introduced independently from their *a priori* physical role.

In the following we shall illustrate this point in the Einstein-Cartan theory. In Sec. II we describe briefly the Einstein-Cartan theory and show that if Eq. (16) holds, then many and well-known solutions of this theory are inconsistent because they use imcompatible simplifying assumptions. In Sec. III we use the symmetries of the gravitational field to simplify the field equations of the theory without making any specific *a priori* assumption about the physical role of torsion. As a result we obtain a new solution of the theory in which torsion does not represent spin. Finally, conclusions are noted in Sec. IV.

II. THE CASE OF THE EINSTEIN-CARTAN THEORY

The Einstein-Cartan (or U_4) theory of gravitation[1] is an extension of general relativity to a space-time with an asymmetric (or Riemann-Cartan) connection defined by the requirement[4]

$$g_{ab|c}=0\,. \tag{1}$$

Equation (1) gives[1]

$$\Gamma^a{}_{bc}=\left\{{a \atop bc}\right\}-K_{bc}{}^a\,, \tag{2}$$

where

$$K_{bc}{}^a=-S_{bc}{}^a+S_c{}^a{}_b-S^a{}_{bc}=-K_b{}^a{}_c \tag{3}$$

is the so-called contortion tensor and $S_{ab}{}^c=\Gamma^c_{[ab]}$ is the torsion tensor. From Eq. (3) we also have

$$S_{bc}{}^a=-K_{[bc]}{}^a\,. \tag{4}$$

The curvature tensor $B_{abc}{}^d$ of $\Gamma^a{}_{bc}$ is defined by

$$B_{abc}{}^d=2\partial_{[a}\Gamma^d{}_{b]c}+2\Gamma^d{}_{[a|e|}\Gamma^e{}_{b]c}\,. \tag{5}$$

It can be shown that $B_{abc}{}^d$ has essentially only one contraction $B_{ij}\equiv B_{kij}{}^k$ (not symmetric in general); hence a unique curvature scalar $B\equiv B^i{}_i$.

The action in Einstein-Cartan theory is taken to be (we take $c=1$)

$$I=\int\left(\frac{1}{2k}\sqrt{-g}B+\mathcal{L}_M\right)d^4x\,, \tag{6}$$

where $\sqrt{-g}B$ is the Lagrangian density of the gravitational field and L_M is the minimally coupled Lagrangian density of the matter field(s) ψ (indices supressed; $k=8\pi G$ where G is Newton's gravitational constant).

In general, $\mathcal{L}_M=\mathcal{L}_M(\psi;\psi_{,a};g_{ab};g_{ab,c};S_{ab}{}^c)$ or, because of Eq. (4), $\mathcal{L}_M=\mathcal{L}_M(\psi;\psi_{,a};g_{ab};g_{ab,c};K_{ab}{}^c)$. Variation of the action I with respect to the metric tensor g^{ij} and the contortion tensor $K_{ab}{}^c$ yields the following set of field equations[1]:

$$\hat{G}^{ab}=k\tilde{\sigma}^{ab}\,, \tag{7}$$

$$T^{abc}=k\tau^{abc}\,, \tag{8}$$

where $\hat{G}^{ab}=R^{ab}-\frac{1}{2}g^{ab}R$ is the usual Einstein tensor of general relativity, and

$$T^{abc}\equiv S^{abc}+2g^{c[a}S^{b]d}{}_d\,. \tag{9}$$

In Eq. (7)

$$\tilde{\sigma}^{ab}\equiv\sigma^{ab}+k[-4\tau^{ac}{}_{[d}\tau^{bd}{}_{c]}-2\tau^{acd}\tau^b{}_{cd}+\tau^{cda}\tau_{cd}{}^b+\tfrac{1}{2}g^{ab}(4\tau_c{}^d{}_{[e}\tau^{ce}{}_{d]}+\tau^{cde}\tau_{cde})] \tag{10a}$$

$$\equiv\sigma^{ab}+k\Lambda^{ab} \tag{10b}$$

where

$$\sigma^{ab}\equiv\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g_{ab}}\,. \tag{11}$$

σ^{ab} is the symmetric stress-energy tensor of matter and reduces to the usual one of general relativity when torsion vanishes.

In Eq. (9) the quantity

$$\tau_a{}^{bc}\equiv\frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}_M}{\partial K_{cb}{}^a} \tag{12}$$

is interpreted as the spin angular momentum tensor and at each space-time point equals the sum of the intrinsic spin densities of all the physical fields at that point.

We note that the Einstein-Cartan theory reduces to general relativity when torsion vanishes, hence this theory belongs to the class of theories we considered in the Introduction. The extra geometric object involved is the torsion and its *a priori* physical role is the geometrization of spin.

Most of the solutions of Einstein-Cartan theory produced so far use this physical role to simplify torsion. Thus, because spin is represented by a spacelike vector s_a *(say)* in physically interesting situations, torsion shall be defined in terms of a vector. This has led to the consideration of convective torsion defined as follows:

$$S_{ab}{}^c\equiv\eta_{ab}{}^{de}s_d u_e u^c\,, \tag{13a}$$

$$s^d s_d=s^2>0\,. \tag{13b}$$

Equations (13a) and (13b) define torsion in the region of space-time filled up with a fluid whose particles have intrinsic spin s^a and four-velocity u^a. [According to the second field equation (12), torsion vanishes in empty space.]

This type of torsion is simple to deal with (because it has three nonvanishing components only) and it has been used extensively in deriving important and well-known solutions of Einstein-Cartan theory.[5,6] In all these solutions the simplification of the metric is via symmetry assumptions as in general relativity.

Using the field equations we investigate the extent to which the symmetries of the torsion are related to the symmetries of the metric. Let $\underline{\xi}$ be a Killing vector field, i.e.,

$$L_{\underline{\xi}} g_{ab} = 0 \,, \tag{14}$$

where L denotes Lie differentiation. It is well known that Eq. (14) implies

$$L_{\underline{\xi}} \hat{G}^{ab} = 0 \,. \tag{15}$$

Also, because σ^{ab} in Eq. (10a) represents the stress-energy-momentum tensor [cf. Eq. (11)], it is reasonable to require

$$L_{\underline{\xi}} \sigma^{ab} = 0 \,. \tag{16}$$

This requirement is strongly suggested by the fact that the Einstein-Cartan theory is considered a reasonable, slight modification of general relativity. However, Eq. (16) is a strong condition on σ_{ab} which, unlike the case of general relativity, does not follow from the field equations (7) and (14). Equations (15) and (16) and the field equations (10b) now give the constraint

$$L_{\underline{\xi}} \Lambda^{ab} = 0 \,. \tag{17}$$

What restrictions does Eq. (17) impose on the form of the torsion tensor? We have not been able to answer this question in full generality. However, we have answered it in two physically important cases.

First, there is the case of convective torsion. Second, it is the case of torsion with two irreducible parts only (cf. Appendix A), the vector part ${}^V S_{ab}{}^c$ and the antisymmetric part ${}^A S_{ab}{}^c$. The ${}^V S_{ab}{}^c$ is important because ${}^V S_{ab}{}^c$ is the type of torsion which results if the gauge invariance of the electromagnetic field[7] (or other gauge fields[8]) is reestablished in a Riemann-Cartan space-time. The ${}^A S_{ac}{}^c$ is the type of torsion for which the autoparallels of the Riemann-Cartan connection $\Gamma^a{}_{ab}$, defined by Eq. (2), coincide with the geodesics of the Riemannian connection $\{{}^{a}_{bc}\}$. In both these cases we have shown that the torsion and the metric *must* have the same symmetries. Thus symmetry assumptions concerning the simplification of the metric induce automatically symmetry assumptions simplifying the torsion. The proofs are rather involved and can be found in Appendices B and C, respectively.

The fact that for convective torsion the metric and torsion must have the same symmetries imposes severe restrictions on the form of the torsion.[9] In particular, if Eq. (16) is assumed, it proves that the solutions of the Einstein-Cartan theory discovered by Kopzyński and by Trautman, in which the singularity is avoided, must be reconsidered.[5] This reconsideration has also been suggested in Ref. 9 on the assumption that the cosmological principle still holds in a Riemann-Cartan space-time. Here we have shown that if Eq. (16) is assumed, this reconsideration follows directly from the field equations. Similar arguments apply to other solutions of Einstein-Cartan theory using convective torsion.[10]

Finally, an especially interesting solution is the one produced by Rosenbaum *et al.*[11] In this solution the metric is taken to be everywhere η_{ab} (i.e., space-time is Riemann flat) and the torsion is assumed to be of the convective form Eq. (13). Thus, this solution corresponds to what one may call "special relativity with torsion." However, according to our results, if Eq. (16) is assumed, then this solution is not possible, because in a maximally symmetric four-dimensional space-time convective torsion can never be form invariant.[9]

III. A NEW SOLUTION OF THE EINSTEIN-CARTAN THEORY

We consider again the Einstein-Cartan theory and make no *a priori* assumption(s) concerning the physical role of torsion. Having then no alternative we construct simplifying assumptions using the symmetries of the gravitational field. Because the gravitational field is defined by both metric and torsion, we require that a vector $\underline{\xi}$ shall generate a symmetry of the gravitational field if and only if

$$L_{\underline{\xi}} g_{ab} = 0 \,, \tag{18}$$

$$L_{\underline{\xi}} S_{ab}{}^c = 0 \,. \tag{19}$$

Since both g_{ab} and $S_{ab}{}^c$ are considered to be cosmic tensor fields, Eqs. (18) and (19) are compatible with the cosmological principle as stated by Weinberg[12] and allow us to construct analogs of the Friedmann-Robertson-Walker (FRW) models in Einstein-Cartan theory. As has been shown,[9] these models are excluded if the convective form of torsion is used.

The cosmological principle requires the existence of six Killing vectors generating the three-dimensional maximally symmetric hypersurfaces of homogeneity. Equation (18) indicates, as usual, that the metric has the Robertson-Walker form[12]

$$d\sigma^2 = g_{ab}dx^a dx^b = -dt^2 + R^2(t)d\sigma^2 , \quad (20)$$

where $R(t)$ is the cosmic scale factor and

$$ds^2 = \frac{dr^2}{1-\alpha r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 ,$$

where $\alpha = 0, \pm 1$ is the constant curvature of the cosmic time hypersurfaces. Equation (19) and application of the theorem of Ref. 9 give the following nonvanishing components for torsion:

$$S_{01}{}^1 = S_{02}{}^2 = S_{03}{}^3 \equiv F , \quad (21)$$

$$S_{123} = S_{[123]} \equiv f , \quad (22)$$

where F, f are arbitrary functions of cosmic time t.

Indices 1, 2, 3, enumerate coordinates in the maximally symmetric hypersurfaces of constant time. In the comoving frame the four-velocity u^a has components (1, 0, 0, 0). Using this frame and Eqs. (21) and (22), we show easily that the nonvanishing irreducible parts of torsion are

$$^V S_{ab}{}^c = \tfrac{1}{3}(\delta^c_b u_a - \delta^c_a u_b)F , \quad (23)$$

$$^A S_{ab}{}^c = \eta_{ab}{}^{cd} u_d f . \quad (24)$$

Hence

$$S_{ab}{}^c = \tfrac{1}{3}(\delta^c_b u_a - \delta^c_a u_b)F + \eta_{ab}{}^{cd} u_d f . \quad (25)$$

From Eqs. (23) and (24) it follows that torsion can be associated with the four-velocity u^a only and not with a spacelike vector.[13] We conclude then that this type of torsion does not represent spin.

From Eq. (25) we find

$$K_{ab}{}^c = 2S_b{}^c{}_a - 3\,{}^A S_{ab}{}^c = 2\,{}^V S_b{}^c{}_a - {}^A S_b{}^c{}_a$$
$$= \tfrac{2}{3}(g^c{}_a u_b - g_{ab}u^c)F - \eta_{ab}{}^{cd} u_d f . \quad (26)$$

One important property of this solution is that it does not violate the so-called "Weyl's postulate," i.e., that "free fall" takes place along the geodesics of the metric. Indeed, using Eq. (26) we find[14]

$$\dot{u}^a \equiv u^a{}_{|b} u^b = u^a{}_{;b} u^b = 0 .$$

We turn now to the field equation (7). We find, after a lengthy calculation,

$$\Lambda^{ab} - \tfrac{1}{2} g^{ab}\Lambda^c{}_c = \tfrac{8}{3} S^a S^b + {}^A S^{acd}\, {}^A S^b{}_{cd}$$
$$= (\tfrac{8}{3}F^2 + 2f^2)u^a u^b + 2g^{ab} f^2 , \quad (27)$$

where $S^a \equiv S^a{}_b{}^b$.

Field equation (7) can be written

$$R^{ab} = k(\sigma^{ab} - \tfrac{1}{2} g^{ab}\sigma^c{}_c) + k^2(\Lambda^{ab} - \tfrac{1}{2} g^{ab}\Lambda^c{}_c) . \quad (28)$$

From Eq. (16) it follows that σ^{ab} has necessarily the form of a perfect fluid,[12] i.e.,

$$\sigma^{ab} = (\rho + p)u^a u^b + p g^{ab} , \quad (29)$$

where ρ is the energy density of matter and p is the isotropic pressure. From Eqs. (27), (28), and (29) it follows that

$$R^{ab} = k(\tilde{\rho} + \tilde{p})u^a u^b + \tfrac{1}{2}k(\tilde{\rho} - \tilde{p})g^{ab} , \quad (30)$$

where

$$\tilde{\rho} \equiv \rho + \tfrac{4}{3}kF^2 + 3kf^2 , \quad (31)$$

$$\tilde{p} \equiv p + \tfrac{4}{3}kF^2 - kf^2 \quad (32)$$

are the "effective" density and the "effective" pressure, respectively. Equation (30) is identical with the corresponding equation of general relativity except that $\tilde{\rho}, \tilde{p}$ replace the usual ρ and p. In components Eq. (30) reads

$$3\ddot{R} = -\tfrac{1}{2}k(\tilde{\rho} + 3\tilde{p})R , \quad (33)$$

$$R\ddot{R} + \dot{R}^2 + 2\alpha = \tfrac{1}{2}k(\tilde{\rho} - \tilde{p})R^2 . \quad (34)$$

The conservation law $\hat{G}^{ab}{}_{;b} = 0$ gives

$$\frac{d}{dR}(\tilde{\rho}R^3) = -3R^2\tilde{p} . \quad (35)$$

Although the field equations (30) have an identical form with the corresponding equations of general relativity, the quantities they involve are different. Thus, in general, the physics they contain shall be different from that of general relativity.

The solution we found is not free of singularities. To show this we use Eqs. (31) and (32) to replace $\tilde{\rho}$ and $\tilde{p}$ in Eq. (33) and obtain

$$3\ddot{R} = -\tfrac{1}{2}k(\rho + 3p + \tfrac{8}{3}kF^2)R . \quad (36)$$

Thus as long as $\rho + 3p \geq 0$, we have $\ddot{R}/R < 0$ and singularity is reached at some finite time in the past. A detailed study of this solution is currently under investigation.

IV. CONCLUSIONS

The recent efforts to generalize general relativity have their roots in the desire to construct a theory which will encounter in a unified manner strong, electromagnetic, weak, and the very weak gravitational interactions. All the suggested theories, so far, are relativistic and belong to the class of theories we considered in the Introduction.

In the present paper we emphasized the great care which should be taken in the construction of solutions and the physical interpretation of the extra geometric objects introduced by these theories.

The results of Secs. II and III illustrate this point in the well-known Einstein-Cartan theory. Thus, in Sec. II we have shown that if Eq. (16)

is assumed, then the well-known and important solutions of this theory use inconsistent simplifying assumptions. Further, in Sec. III we presented a new solution of the theory which is compatible with the cosmological principle and in which torsion does not represent spin. This solution is excluded if the assumption that torsion is coupled to intrinsic spin of matter is used. Further, it is directly related to the standard cosmological model of general relativity and can be used for a direct observational verification of the Einstein-Cartan theory.

The solution is defined up to two arbitrary functions f, F which are not constrained by the first set of field equations (7) of the theory. Bounds on these functions can be found from the observed values of the Hubble constant and the deceleration parameter.

The method we employed to derive this solution can be applied to any other member of the class of theories considered in this paper and produce, if it exists, a solution compatible with the cosmological principle. This solution may be excluded, (e.g., as in Einstein-Cartan theory) by the *a priori* physical role assumed for the extra geometric objects of the theory. Clearly, the same remarks apply to solutions with other types of symmetry.

Finally, another useful aspect of this method is that if the theory uses a Lagrangian approach—which is most probable—one may use the symmetries of the gravitational field to obtain constants of motion and/or conserved quantities. One case where this has been done can be found in Ref. 15.

APPENDIX A

It can be shown that torsion can be decomposed with respect to the orthogonal group in the following manner:

$$S_{ab}{}^{c} = {}^{V}S_{ab}{}^{c} + {}^{T}S_{ab}{}^{c} + {}^{A}S_{ab}{}^{c}, \tag{A1}$$

where

$${}^{V}S_{ab}{}^{c} \equiv \tfrac{1}{3}(\delta^{c}_{b}S_{a} - S^{c}_{a}S_{b}), \tag{A2}$$

$${}^{T}S_{ab}{}^{c} \equiv \tfrac{2}{3}(S_{ab}{}^{c} - S^{c}{}_{[ab]} - S_{[a}\delta^{c}_{b]}), \tag{A3}$$

$${}^{A}S_{ab}{}^{c} \equiv S_{[abc]}, \tag{A4}$$

$$(S_{a} \equiv S_{ab}{}^{b}).$$

${}^{V}S_{ab}{}^{c}$, ${}^{T}S_{ab}{}^{c}$, and ${}^{A}S_{ab}{}^{c}$ are called the "vector part," the "traceless part," and the "antisymmetric part" of torsion, respectively.

APPENDIX B

We prove that if the torsion tensor has the form of convective torsion defined in Eq. (13), the field equations of the Einstein-Cartan theory (7) imply that, if $\underline{\xi}$ is a Killing vector of the metric, i.e., $L_{\underline{\xi}}g_{ab}=0$, then $\underline{\xi}$ leaves also invariant the form of torsion, i.e., $L_{\underline{\xi}}S_{ab}{}^{c}=0$. From Eqs. (10) and (13) we find

$$\frac{1}{k^2}(\Lambda^{a}{}_{b} - \tfrac{1}{2}\delta^{a}{}_{b}\Lambda^{c}{}_{c}) = 2S^{a}S_{b} + (S^{c}S_{c})u^{a}u_{b}$$
$$= 2(\delta^{a}{}_{b} + u^{a}u_{b})S^{2} - 2S^{a}S_{b} + S^{2}u^{a}u_{b},$$

where $S_{a} \equiv S_{ab}{}^{b}$ and $S^{2} = S_{a}S^{a}$. Let $\underline{\xi}$ be a Killing vector. Then Eq. (17) gives

$$2L_{\underline{\xi}}[(\delta^{a}{}_{b} + u^{a}u_{b})S^{2}] - 2L_{\underline{\xi}}(S^{a}S_{b}) + L_{\underline{\xi}}(S^{2}u^{a}u_{b}) = 0. \tag{B1}$$

Contract a, b (use $L_{\underline{\xi}}g_{ab}=0$) to find

$$L_{\underline{\xi}}S^{2} = 0. \tag{B2}$$

Equation (B1) becomes, using Eq. (B2),

$$3L_{\underline{\xi}}(u^{a}u_{b})S^{2} - 2L_{\underline{\xi}}(S^{a}S_{b}) = 0.$$

Then contract with S_{a} and use that $S^{2} \neq 0$ to obtain

$$3(L_{\underline{\xi}}u^{a})S_{a}u_{b} - 2(L_{\underline{\xi}}S_{b}) = 0. \tag{B3}$$

Contract again with u^{b} and use $u_{a}S^{a}=0$ to find

$$(L_{\underline{\xi}}u^{a})S_{a} = 0. \tag{B4}$$

Equations (B3) and (B4) give

$$L_{\underline{\xi}}S_{b} = 0. \tag{B5}$$

From Eqs. (B1), (B5), and $u^{a}u_{a} = -1$ it follows that

$$L_{\underline{\xi}}u^{a} = 0. \tag{B6}$$

Finally, Eqs. (B5) and (B6) imply

$$L_{\underline{\xi}}S_{ab}{}^{c} = 0. \tag{B7}$$

APPENDIX C

Here we assume torsion to be of the form

$$S_{ab}{}^{c} = {}^{V}S_{ab}{}^{c} + {}^{A}S_{ab}{}^{c},$$

where ${}^{V}S_{ab}{}^{c}$ and ${}^{A}S_{ab}{}^{c}$ are defined in Appendix A. We prove as in Appendix B that if $\underline{\xi}$ is a Killing vector, then from the field equations it follows that $L_{\xi}S_{ab}{}^{c}=0$.

We define a vector r^{a} by

$${}^{A}S_{abc} = \eta_{abcd}r^{d}.$$

If $S_{ab}{}^{b} \equiv S_{a}$ we have

$${}^{V}S_{ab}{}^{c} = \tfrac{1}{3}(\delta^{c}{}_{b}S_{a} - \delta^{c}{}_{a}S_{b}).$$

Let $r^{2} = r^{a}r_{a}$ and $S^{2} = S^{a}S_{a}$. Equation (17) gives

$$L_{\underline{\xi}}(\tfrac{8}{3}S^{a}S_{b} - 2\delta^{a}{}_{b}r^{2} + 2r^{a}r_{b}) = 0 \tag{C1}$$

or

$$\tfrac{4}{3}(L_{\underline{\xi}}S^{a})S_{b} + \tfrac{4}{3}S^{a}(L_{\underline{\xi}}S_{b}) - \delta^{a}{}_{b}L_{\xi}r^{2}$$
$$+ (L_{\xi}r^{a})r_{b} + r^{a}(L_{\xi}r_{b}) = 0. \tag{C2}$$

Contract a, b in (C1) to find

$$L_\xi S^2 = \tfrac{9}{4} L_\xi r^2 . \tag{C3}$$

Contract Eq. (C2) with S_b to obtain

$$(L_\xi r^2) S_b + \tfrac{8}{3} S^2 (L_\xi S_b) + 2(r^c S_c)(L_\xi r_b) + 2 r_b [(L_\xi r^c) S_c] = 0 . \tag{C4}$$

Contract Eq. (C4) with S^b and r^b to obtain, respectively,

$$(r^c S_c)[(L_\xi r_a) S^a] = -S^2 L_\xi r^2 , \tag{C5}$$

$$\tfrac{4}{3} S^2 [(L_\xi S_a) r^a] + r^2 [(L_\xi r^a) S_a] + (r^c S_c) L_\xi r^2 = 0 . \tag{C6}$$

Contract Eq. (C4) with r^a and r^b to obtain, respectively,

$$\tfrac{8}{3} [(L_\xi S^a) r_a] S_b + \tfrac{8}{3} (S^a r_a) L_\xi S_b - r_b L_\xi r^2 + 2 r^2 L_\xi r_b = 0 , \tag{C7}$$

$$(S^c r_c)[(L_\xi S^a) r_a] = 0 . \tag{C8}$$

Case 1: $r^a S_a \neq 0$. Equation (C8) gives

$$(L_\xi S^a) r_a = 0 . \tag{C9}$$

Equations (C9), (C5), and (C6) give

$$[r^2 S^2 - (r^a S_a)^2] L_\xi r^2 = 0 . \tag{C10}$$

Case 1(*a*): $L_\xi r^2 = 0$. (C11)

Equations (C11) and (C3) give

$$L_\xi S^2 = 0 . \tag{C12}$$

Equations (C9), and (C12) give

$$(L_\xi S_a)(S^a + r^a) = 0$$

for all r^a. Because $S^a + r^a$ is an arbitrary vector, this gives

$$L_\xi S_a = 0 .$$

Case 1(b): $r^2 S^2 - (r^a S_a)^2 = 0$.

This gives, using Eq. (C3),

$$(S^2 + \tfrac{9}{4} r^2) L_\xi r^2 - 2 (r^a S_a) L_\xi (r^a S_a) = 0 . \tag{C13}$$

Equation (C9) gives

$$(r^a S_a) L_\xi (r^b S_b) = (r^a S_a)[(L_\xi r^b) S_b]$$

and Eq. (C5) implies further

$$(r^a S_a) L_\xi (r^a S_a) = -S^2 L_\xi r^2 .$$

Replacing this in (C13), we find $L_\xi r^2 = 0$ and the proof continues as in Case 1(a).

Case II: $r^a S_a = 0$. (C14)

Equations (C5) and (C3) give

$$L_\xi r^2 = 0 , \tag{C15a}$$

$$L_\xi S^2 = 0 . \tag{C15b}$$

Equations (C4) and (C6) become

$$\tfrac{4}{3} S^2 L_\xi S_a + r_a [(L_\xi r^b) S_b] = 0 , \tag{C16}$$

$$\tfrac{4}{3} S^2 [(L_\xi S_a) r^a] + r^2 [(L_\xi r^a) S_a] = 0 . \tag{C17}$$

From Eqs. (C14) and (C17), it follows that

$$(\tfrac{4}{3} S^2 - r^2)[(L_\xi S_a) r^a] = 0 . \tag{C18}$$

We distinguish three cases.

Case II(*a*): $\tfrac{4}{3} S^2 - r^2 \neq 0$.

Then

$$(L_\xi S_a) r^a = 0 . \tag{C19}$$

From Eqs. (C15b) and (C19) it follows that

$$(L_\xi S_a)(r^a + S^a) = 0 .$$

Thus

$$L_\xi S_a = 0 .$$

Equation (C3) implies $L_\xi r^2 = 0$ and Eqs. (C14), and (C19) give $(L_\xi r_a) S^a = 0$. It follows that $L_\xi r_a = 0$; thus $L_\xi S_{ab}{}^c = 0$.

Case II(*b*): $\tfrac{4}{3} S^2 - r^2 = 0$, $S^2 \neq 0$. Equations (C6) and (C14) give

$$(L_\xi S_a) r^a = (L_\xi r_a) S^a = 0 .$$

From this and Eq. (C15) it follows easily that

$$L_\xi S_{ab}{}^c = 0 .$$

Case II(*c*): $S^2 = r^2 = 0$. Equation (C4) gives

$$r_a [(L_\xi S_b) r^b] = 0 ,$$

from which follows

$$(L_\xi S_a) r^a = 0$$

and the case is reduced to Case II(b).

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[4]The notation in this paper is as follows. Covariant derivatives with respect to the asymmetric connection Γ^a_{bc} are denoted by a vertical line, and with respect to the Christoffel symbol $\{^a_{bc}\}$ by a semicolon: $A^a|_b = A^a{}_{,b} + \Gamma^a{}_{bc}A^c$, $A^a{}_{;b} = A^a{}_{,b} + \{^a_{bc}\}A^c$. Parentheses denote symmetrization and square brackets denote antisymmetrization. Finally, the Lagrangian derivative is defined as $\delta\mathfrak{L}(Q,\partial Q)/\delta Q \equiv \partial\mathfrak{L}/\partial Q - (\partial\mathfrak{L}/\partial_k Q)_{,k}$, where $\partial_k Q = \partial Q/\partial x^k$.
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