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Center-of-mass correction in the MIT bag model

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(Received 30 June 1980)

The Peierls-Yoccoz method is used to treat the center-of-mass (c.m.) motion of hadrons in the MIT bag. The results are found to agree with those obtained in the intuitive procedure of Liu and Wong. The method of Donoghue and Johnson for making c.m. corrections is found to be incorrect. The pion decay constant F_π is calculated from the now known c.m. motion of the pion in the bag. It turns out to depend strongly on the three-momentum of the pion.

A possible prescription for correcting the effects of center-of-mass (c.m.) motion of a hadron in a bag in the Massachusetts Institute of Technology (MIT) bag model has been described elsewhere.[1] It is based on certain intuitive arguments which are difficult to justify in precise terms. It gives the correct nonrelativistic limit, and is most questionable in the most relativistic systems, for example, a massless hadron constructed from n massless quarks. In this case, it gives a c.m. momentum of $2.043\sqrt{n}/R$, where R is the bag radius. Unfortunately, the correct result is not known, so that the prescription cannot be tested.

However, there are standard methods in nuclear physics[2] for handling such c.m. problems. In this paper, one such method, the Peierls-Yoccoz projection[3] of the generator-coordinate method,[2,4] is applied to this problem. The use of an approximate quark wave function[5] in a bag leads to simple expressions for various quantities describing c.m. effects. In the special case of a massless hadron containing n massless quarks, this procedure yields a result of $2.10\sqrt{n}/R_{\text{eq}}$, where R_{eq} is an independently defined equivalent bag radius. This result shows that the prescription of Ref. 1 appears to be quite accurate.

In addition, I point out that the procedure of deriving the c.m. motion of a pion in a bag from a matrix element of the axial-vector current, which has been proposed recently by Donoghue and Johnson,[6] is incorrect. The pion decay constant F_π can still be estimated from this matrix element of the axial-vector current and the *known* c.m. motion of the pion in the bag. The result is found to be $\simeq 220$ MeV. The limitations of this result are also discussed.

In order to permit a comparison later, the notation of Ref. 6 is used where possible. As in Ref. 6, we are interested in the decomposition of a static hadron bag state $|H_B(\vec{x})\rangle$ with bag center at $\vec{x}$ into components $\phi(\vec{p})$ of plane-wave momentum eigenstates $|H(\vec{p})\rangle$:

$$|H_B(\vec{x})\rangle = \int d^3p\, e^{i\vec{p}\cdot\vec{x}}\left[\frac{\phi(\vec{p})}{W_H(p)}\right]|H(\vec{p})\rangle. \tag{1}$$

The inverse relation is

$$|H(\vec{p})\rangle = \frac{1}{(2\pi)^3}\left[\frac{W_H(p)}{\phi(\vec{p})}\right]\int d^3x\, e^{-i\vec{x}\cdot\vec{p}}|H_B(\vec{x})\rangle, \tag{2}$$

where $W_H(p)$ is the normalization of the plane wave:

$$\langle H(\vec{p})|H(\vec{p}\,')\rangle = (2\pi)^3\delta(\vec{p}-\vec{p}\,')W_H(p). \tag{3}$$

W_H may be chosen to be

$$W_H(p) = \begin{cases} 2\omega_p \text{ for a meson} \\ E(p)/m_H \text{ for a baryon.} \end{cases} \tag{4}$$

The physical results do not depend on the choice of W_H.

Given $|H_B(\vec{x})\rangle$, the momentum eigenstate $|H(\vec{p})\rangle$ can be projected from it, as shown in Eq. (2). Its substitution into Eq. (3) leads immediately to the final result

$$\phi(\vec{p}) = [W_H(p)\bar{I}(\vec{p})/(2\pi)^3]^{1/2}, \tag{5}$$

where

$$\bar{I}(\vec{p}) = \frac{1}{(2\pi)^3}\int d^3r\, e^{-i\vec{r}\cdot\vec{p}}\langle H_B(0)|H_B(\vec{r})\rangle \tag{6}$$

is the Fourier transform of the Hill-Wheeler overlap function.[4] This result can be used with Eq. (1) to calculate expectation values:

$$\langle H_B(0)\,|F(\vec{p})\,|H_B(0)\rangle = \int d^3p\,\tilde{I}(\vec{p})F(\vec{p})\,. \tag{7}$$

For the kinetic energy of c.m. motion, the function to be used is $(m_H{}^2+p^2)^{1/2}-m_H$, where m_H is the hadron mass.

This method is called a Peierls-Yoccoz (PY) projection[3] in nuclear physics. It projects out exact eigenstates of the center-of-mass momentum $\vec{p}$ by taking appropriate linear integrals of bag states at rest at different locations. However, the internal state of the hadron in $|H(\vec{p})\rangle$ may vary with $\vec{p}$. One result is that the total mass of the hadron may not come out to be the expected Galilean invariant.[7] In a similar way, certain expectation values calculated this way may contain spurious $\vec{p}$ dependences when none is expected. If this should occur, it is a reflection of the imperfections contained in the given static state $|H_B(\vec{x})\rangle$, which causes the internal state of the hadron in $|H(\vec{p})\rangle$ to vary with $\vec{p}$. Such spurious $\vec{p}$ dependences can only be removed by improving the given $|H_B(\vec{x})\rangle$ itself.

One procedure for doing this in nuclear physics is called the Peierls-Thouless[7] method, which requires the construction of suitably boosted states $|H_B(\vec{v},\vec{x})\rangle$, where $\vec{v}$ is the velocity. This is not easy to do, especially in the present relativistic context where the Lorentz invariance of the internal structure is required. (For example, the boosted state must also be time dependent.) In the present paper, I shall simply follow Ref. 6 by restricting myself to the original static MIT bag state. This means that a PY projection is the best that can be done.

It is obvious that the PY projection is very different from the Donoghue-Johnson[6] (DJ) treatment of the c.m. motion of the pion in the bag which starts with the pion decay matrix element

$$\langle 0\,|\bar{u}(\vec{x})\gamma^0\gamma_5 d(\vec{x})\,|\pi(\vec{p})\rangle = i\sqrt{2}F_\pi p^0 e^{i\vec{p}\cdot\vec{x}}\,. \tag{8}$$

Transforming to the bag state, one finds

$$\langle 0\,|\bar{u}(\vec{x})\gamma^0\gamma_5 d(\vec{x})\,|\pi_B(0)\rangle = i\sqrt{2}F_\pi\int d^3p\,e^{i\vec{p}\cdot\vec{x}}p^0\left[\frac{\phi_{DJ}(\vec{p})}{W_H(p)}\right]. \tag{9}$$

That is,

$$\phi_{DJ}(\vec{p}) = \frac{W_H(p)}{p^0}\,\frac{\sqrt{3}}{F_\pi}\tilde{f}_A(\vec{p})\,, \tag{10}$$

where

$$\tilde{f}_A(\vec{p}) = \frac{1}{(2\pi)^3}\int d^3x\,e^{-i\vec{x}\cdot\vec{p}}\,\frac{1}{i\sqrt{6}}\langle 0|\,\bar{u}(\vec{x})\gamma^0\gamma_5 d(\vec{x})\,|\pi_B(0)\rangle\,. \tag{11}$$

The energy p^0 will be specified later.

In this approach, the pion decay constant F_π turns out to be a normalization constant determined by the condition that $\langle\pi_B(0)\,|\pi_B(0)\rangle=1$. Equation (10) can now be used in Eq. (1) to give the expectation value

$$\langle\pi_B(0)|\,F(\vec{p})\,|\pi_B(0)\rangle = (2\pi)^3\int\frac{d^3p}{W_H(p)}\left[\frac{\sqrt{3}W_H(p)}{F_\pi p^0}\tilde{f}_A(\vec{p})\right]^2 F(\vec{p})\,, \tag{12}$$

when the plane-wave normalization (3) is assumed to be valid. This is the DJ result.

Unfortunately the plane-wave normalization is not correctly given by Eq. (3). Given $\phi_{DJ}(\vec{p})$, it can be calculated directly from Eq. (2), and is found to be

$$\langle\pi(\vec{p})\,|\pi(\vec{p}\,')\rangle_{DJ} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')W_H(p)\times\left\{\frac{W_H(p)}{(2\pi)^3}\left[\frac{F_\pi p^0}{\sqrt{3}W_H(p)\tilde{f}_A(\vec{p})}\right]^2\tilde{I}(\vec{p})\right\}. \tag{13}$$

The additional factor in Eq. (13) when inserted into Eq. (12) gives just the Peierls-Yoccoz result shown in Eq. (7). This development shows quite clearly that the c.m. motion of the pion in the bag has nothing to do with the axial-vector current. In particular, if F_π is a constant (i.e., a Lorentz scalar), this additional factor cannot be unity for all values of $\vec{p}$. Thus the plane-wave normalization, Eq. (3), used in Ref. 6 is *inconsistent* with the assumption made there that F_π is a constant. We shall come back to the question of the pion decay later.

Given Eq. (7), it is a simple matter to evaluate various c.m. corrections, once the Hill-Wheeler function $\tilde{I}(\vec{p})$ is calculated, perhaps by numerical means. At the moment, a qualitative understanding of the general features of these corrections as applied to hadrons is perhaps more important than precise numerical results. To this end, we approximate the 1s quark wave function in the bag by the simple Gaussian expression

$$\psi(\vec{r}_1) = [R_0{}^3\pi^{3/2}(1+\tfrac{3}{2}\beta^2)]^{-1/2}e^{-r_1{}^2/2R_0{}^2}\begin{pmatrix}1\\ i\beta\dfrac{\vec{\sigma}\cdot\vec{r}_1}{R_0}\end{pmatrix} \tag{14}$$

used by Duck.[5] The size parameter R_0 can be related to the bag radius R by requiring that the wave function in Eq. (14) reproduces the correct mean-square radius

$$\langle r^2\rangle = \tfrac{3}{2}R_0{}^2\left(\frac{1+\frac{5}{2}\beta^2}{1+\frac{3}{2}\beta^2}\right) = \frac{R^2}{f^2(m,x)}\,. \tag{15}$$

Here $f(m,x)$ is a function of the quark mass m and bag eigenvalue x which relates $\langle r^2\rangle$ and R^2 in the bag model.[8] For $m=0$, $x=2.043$, $f(0,2.043)=1.372$ is obtained.

The parameter β determines the probability $p_l=\frac{3}{2}\beta^2/(1+\frac{3}{2}\beta^2)$ that the quark is in the *lower*, or relativistic, or small component of the wave function. For the case $m=0$, it has been estimated[9] to be 0.36, in another problem of the bag model.

For the wave function of Eq. (14), the Hill-Wheeler overlap for n 1s quarks is just

$$I_n(\vec{r})=\langle H_B(0)\,|\,H_B(\vec{r})\rangle=\left[e^{-r^2/4R_0^2}\left(1-c\frac{r^2}{R_0^2}\right)\right]^n, \tag{16}$$

where

$$c=\beta^2/(4+6\beta^2)\,.$$

Hence

$$\tilde{I}_2(\vec{p})=\frac{R_0^{\,3}}{(2\pi)^{3/2}}e^{-p^2R_0^2/2}\times[1-6c+15c^2+(1-5c)2cp^2R_0^{\,2}+c^2p^4R_0^{\,4}]\,, \tag{17}$$

while $\tilde{I}_3(p)$ is proportional to $\exp(-\frac{1}{3}p^2R_0^{\,2})$.

These overlap functions permit ready estimates of c.m. effects via Eq. (7). As illustrations, we give below the results for $\langle p_n\rangle$ and $\langle p^2\rangle_n$ for a hadron of n quarks after the parameter R_0 has been eliminated in favor of the bag radius R (now called an equivalent bag radius $R_{\rm eq}$) with the help of Eq. (15):

$$\langle p\rangle_n=\frac{\sqrt{n}}{R_{\rm eq}}\left[\left(\frac{6}{\pi}\,\frac{1+\frac{5}{2}\beta^2}{1+\frac{3}{2}\beta^2}\right)^{1/2}f(m,x)(1+2c)\right](1+a_n)\,, \tag{18}$$

$$\langle p^2\rangle_n=\frac{n}{R_{\rm eq}^{\,2}}\left[\frac{9}{4}\left(\frac{1+\frac{5}{2}\beta^2}{1+\frac{3}{2}\beta^2}\right)f^2(m,x)(1+4c)\right](1+b_n)\,, \tag{19}$$

where

$$a_2=-\frac{c^2}{1+2c}\,,\quad a_3=-\frac{4}{3}\frac{c^2}{1+2c}(1-\tfrac{34}{9}c)\,,\quad b_2=0\,,\quad b_3=\frac{320}{27}\frac{c^3}{1+4c}\,. \tag{20}$$

For massless quarks, $\beta=0.36$, $c=0.027$, so that all a_n, b_n terms may be ignored. Hence

$$\langle p\rangle_n=\frac{2.10}{R_{\rm eq}}\sqrt{n}\,,\quad \langle p^2\rangle_n=\frac{5.19}{R_{\rm eq}^{\,2}}n\,. \tag{21}$$

These results are independent of the hadron mass. In comparison, the method of Ref. 1 gives for massless hadrons the result $\langle p^k\rangle_n=(2.043\sqrt{n}/R)^k$, where the $n^{k/2}$ dependence comes from the intuitive argument that $\langle R_{\rm c.m.}^{\,2}\rangle=R^2/n$. In the present method, the same dependence on n appears simply because the Hill-Wheeler overlap in Eq. (16) is just the nth power of the single-quark overlap. This shows that the intuitive argument holds also for a quantal system, even in the extreme relativistic limit. Even the numerical proportionality constants are in good agreement, although there is no mathematical reason why this should be true. I conclude from this that the method of Ref. 1 appears to be physically reasonable and quantitatively useful.

Let me now return briefly to the interesting question of F_π. Since $\phi(\vec{p})$ is now completely determined by Eq. (5), it can be used in Eq. (7) to calculate F_π in the MIT bag model. The result is

$$F_\pi^{\rm bag}(p^2)=[3W_H(p)(2\pi)^3/\tilde{I}_2(\vec{p})]^{1/2}\tilde{f}_A(\vec{p})/p^0\,. \tag{22}$$

For the approximate quark wave function shown in Eq. (14), and the matrix element in Eq. (11), but with a momentum-dependent F_π, I obtain

$$F_\pi^{\rm bag}(p^2)=F_\pi^{\rm bag}(0)\left[\frac{m_\pi W_H(p)}{2(p^0)^2}\right]^{1/2}g(p^2)\,, \tag{23}$$

where

$$F_\pi^{\rm bag}(0)=\left[\frac{6}{m_\pi R_0^{\,3}(2\pi)^{3/2}(1-6c+15c^2)}\right]^{1/2}\left(\frac{2-3\beta^2}{2+3\beta^2}\right) \tag{24}$$

and

$$g(p^2)=\frac{1+cp^2R_0^{\,2}/(1-\frac{3}{2}\beta^2)}{[1+p^2R_0^{\,2}(2c-10c^2+c^2p^2R_0^{\,2})/(1-6c+15c^2)]^{1/2}}\simeq 1.0\,. \tag{25}$$

Thus $F_\pi^{\rm bag}(0)$ in this theory is infinite when either the pion mass or its bag radius vanishes. This differs from the DJ result which is finite even for a massless pion.

It is interesting to obtain $F_\pi^{\rm bag}(0)$ for the observed pion mass and a finite bag radius deduced from the experimental pion charge radius[10] of $\langle r^2\rangle_{\rm ch}^{1/2}=0.56$ fm. The result is

$$F_\pi^{\rm bag}(0)\simeq 220\ \text{MeV} \tag{26}$$

for the equivalent bag radius of $R_{\rm eq}=3.9\sqrt{2}\ \text{GeV}^{-1}$, where the factor $\sqrt{2}$ allows for the pion motion in the bag. (This is equivalent to $R_0=3.1\ \text{GeV}^{-1}$.) This decay constant of a pion at rest is about twice the experimental value[11] of 94 MeV, but it should be considered only a rough estimate because of the crudeness of the wave function used.

At finite p^2, it is necessary to specify the energy p^0 in order to extract $F_\pi^{\rm bag}(p^2)$. If the projected pion in $|\pi(\vec{p})\rangle$ of Eq. (2) is on the energy shell, $p^0=\omega_p$ must hold. It then follows from Eq. (23) that $F_\pi^{\rm bag}(p^2)$ depends on p^2. This shows that given the DJ ansatz of Eq. (1), it is mathematically inconsistent to assume next, as is done in Ref. 6, that F_π in Eq. (23) is a constant when the relation $p^0=\omega_p$ is also used.

From the present very limited perspective, it would be more natural to allow F_π to depend on the squared three-momentum p^2. Equation (23) then shows that this momentum dependence comes primarily from a "kinematical" factor, which for an on-shell pion is just

$$\left[\frac{m_\pi W_H(p)}{2(p^0)^2}\right]^{1/2}=\left(\frac{m_\pi}{\omega_p}\right)^{1/2} . \tag{27}$$

Since the pion mass is small, this factor decreases rapidly with p^2. For example, $F_\pi^{\text{bag}}(\langle p^2\rangle)$ calculated at the average value $\langle p^2\rangle$ in the bag turns out to be only $\simeq$100 MeV, in rough agreement with the experimental value. The significance of this agreement is not clear, however, since $F_\pi^{\text{bag}}(p^2)$ is valid only for the projected state of good momentum $\vec{p}$, while $\langle\vec{p}\rangle=0$ holds for the unprojected bag state.

It is likely that the usual pion decay matrix element containing a Lorentz-invariant decay constant F_π cannot be recovered completely unless the theoretical projected pion states also have a Lorentz-invariant internal structure. This is a very interesting problem, but the problem solved here is much more modest. It is concerned merely with center-of-mass corrections involving only the quarks in the usual, i.e., static, MIT bag state $|H_B(\vec{x})\rangle$.

Note added in proof. I have found recently that I. Duck has done the same calculation for $F_\pi(0)$ in Phys. Lett. 64B, 163 (1976) without approximating the quark wave function in the bag by the modified Gaussian of Eq. (14). My result agrees with his when allowance is made of the different bag radii used.

I want to thank Dr. K. F. Liu for stimulating discussions. This work is supported in part by the National Science Foundation Contract No. PHY78-15811.

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