Exact formula for ($\mu \to e\gamma$)-type processes in the standard model

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We compute exactly all the one-loop contributions to the amplitude $A \to B\gamma$, where A and B are fermions of the same charge, but different flavor, in the standard electroweak gauge model. The result is of importance if all mass effects are to be correctly taken into account.

In the standard electroweak gauge model[1] with one Higgs doublet, the decay $A \to \beta\gamma$, where A and B are fermions of the same charge but different flavor, is of considerable importance, since it is sensitive to the masses and mixing angles of all fermions, known or unknown, to which A and B are coupled via the charged-current mixing matrix. Early calculations of such a process, e.g., $\mu \to e\gamma$,[2] have always been done with the following two approximations. (1) All external masses are much smaller than the W-boson mass. (2) All internal masses are much smaller than the W-boson mass. More recently, a calculation where only the first approximation is taken was done by Inami and Lim.[3] Now we have obtained the exact formula for such a decay, discarding even the first approximation. Our result is of importance, especially if there are heavy fermions, such as heavy neutral leptons, whose decay into other channels are somehow suppressed.

As in the calculation of the effective $Zd\bar{s}$ coupling,[4] there are two classes of diagrams to be considered, as shown in Figs. 1 and 2. In the limit of vanishing external momenta, the total contribution of these diagrams to $A \to B\gamma$ is exactly zero, a result easily obtained[4] by looking at only those terms of the effective $Zd\bar{s}$ coupling which are proportional to $\sin^2\theta_W$. In fact, gauge invariance of the $A \to B\gamma$ amplitude requires it to be of the form

$$M = q^\mu \epsilon^\nu \bar{B}(p_2)\sigma_{\mu\nu}(c_1 R + c_2 L)A(p_1) \,, \tag{1}$$

where $q = p_1 - p_2$ is the momentum of the photon, and c_1, c_2 are quantities to be computed for a given model. In the standard model,[1] the basic charged-current interaction is left-handed, so that c_1 will be proportional to m_A and c_2 to m_B. In the limit $m_A, m_B \ll M_W$, c_1/m_A will also be equal to c_2/m_B.

Let A, B be fermions of charge Q and weak iso-

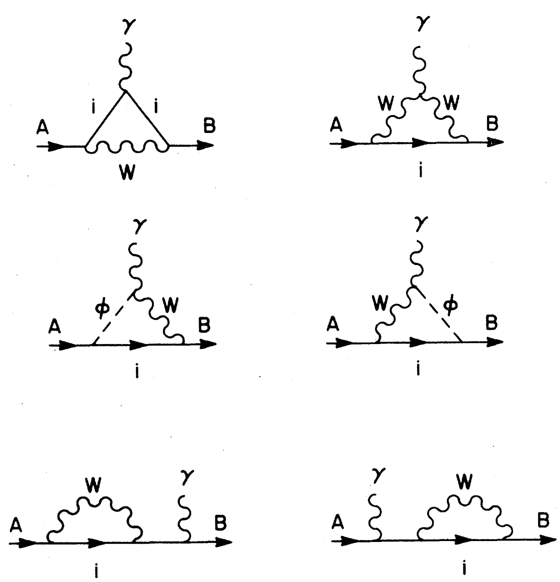

FIG. 1. One-loop diagrams contributing to $A \to B\gamma$ involving W and ϕ.

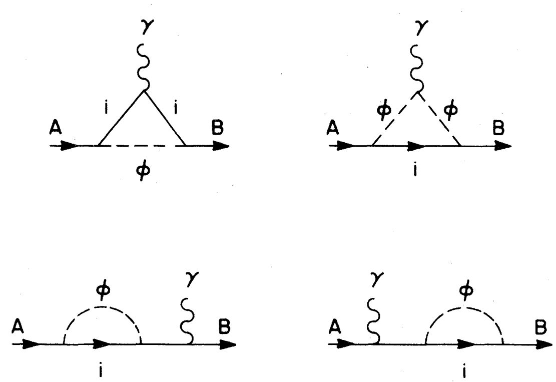

FIG. 2. One-loop diagrams contributing to $A \to B\gamma$ involving only ϕ.

spin $-\frac{1}{2}$, so that they are coupled via the W boson to fermions i of charge $Q+1$ and weak isospin $+\frac{1}{2}$. Owing to the unitarity of the charged-current mixing matrix, each of the divergent diagrams in Fig. 1 becomes finite after summing over all internal fermions, whereas the divergences of the diagrams in Fig. 2 cancel among themselves for each internal fermion. Let $a=m_A^2/M_W^2$, $b=m_B^2/M_W^2$, and $x_i=m_i^2/M_W^2$, then the total relative contribution of each internal fermion to $A\to B\gamma$ is given by[5]

$$F_1^{(i)}=\frac{1}{4(a-b)}\int_0^1 dy\left(Q+\frac{1}{y}\right)\left(-3+2y-\frac{x_i y^2}{1-y}\right)\ln\frac{f_i(a)}{f_i(b)} \tag{2}$$

and

$$F_2^{(i)}=\frac{1}{8}\int_0^1 dy\left(Q+\frac{1}{y}\right)\left\{-y\ln\frac{f_i(a)}{f_i(b)}+\frac{(4+2x_i-a-b)}{(a-b)}\left[2y+\frac{f_i(a)+f_i(b)}{(1-y)(a-b)}\ln\frac{f_i(a)}{f_i(b)}\right]\right\}, \tag{3}$$

where

$$f_i(a)=1-(1-x_i)y-ay(1-y)\,. \tag{4}$$

Let λ_{ij} denote the charged-current mixing matrix which connects the fermions of weak isospin $-\frac{1}{2}$ to those of weak isospin $+\frac{1}{2}$, then

$$c_1=\frac{G_F}{\sqrt{2}}\left(\frac{e}{2\pi^2}\right)m_A\sum_i\lambda_{iA}\lambda_{iB}^*\left(F_1^{(i)}+F_2^{(i)}\right) \tag{5}$$

and

$$c_2=\frac{G_F}{\sqrt{2}}\left(\frac{e}{2\pi^2}\right)m_B\sum_i\lambda_{iA}\lambda_{iB}^*\left(F_1^{(i)}-F_2^{(i)}\right)\,. \tag{6}$$

Whereas $F_1^{(i)}$ is symmetric with respect to the exchange of A and B, $F_2^{(i)}$ is antisymmetric. Hence in the limit $a,b\ll 1$, the former can be proportional to unity, but the latter can only be proportional to $a-b$ and therefore negligible. In that case, $c_1/m_A=c_2/m_B$ as was noted earlier. Furthermore, if $a=b$ but otherwise not restricted in magnitude, then $F_2^{(i)}$ is identically zero, and $c_1=c_2$ exactly. This means that there is no one-loop weak-interaction contribution to the CP-nonconserving electric dipole moment of any fermion, e.g., the neutron, in the standard model, regardless of whether or not there is a CP-nonconserving phase in the charged-current mixing matrix. For $A\to B\gamma$, since $a\neq b$, there are in general both a transition magnetic dipole moment which is proportional to F_1 and a transition electric dipole moment which is proportional to F_2.

To recover the result of Ref. 3, we take the approximation a, $b\ll 1$, and for convenience, drop the index i. Then

$$F_1=Q\left[\frac{1}{12}+\frac{1-5x-2x^2}{8(1-x)^3}-\frac{3x^2\ln x}{4(1-x)^4}\right]+\frac{1}{8}+\frac{3-9x}{8(1-x)^2}-\frac{3x^2\ln x}{4(1-x)^3} \tag{7}$$

and

$$F_2=\frac{1}{8}(a-b)\int_0^1 dy\frac{(Qy+1)y(1-y)}{[1-(1-x)y]}\times\left\{1-\frac{(2+x)y(1-y)}{3[1-(1-x)y]}\right\}, \tag{8}$$

which is of course negligible compared to F_1. If we also take the approximation $x\ll 1$, then we obtain the well-known result

$$F_1=-\frac{2Q+3}{8}x\,, \tag{9}$$

where we have eliminated the constant terms since they will add up to zero after summing over all internal fermions. If A,B have weak isospin $+\frac{1}{2}$ instead, then all the above expressions are changed by changing Q into $-Q$ and then changing the overall sign. Thus for $\mu\to e\gamma$, $F_1=-x/8$, whereas for $\nu_1\to\nu_2\gamma$, $F_1=3x/8$.

Consider as an example the case $1-a\ll 1$, $b\ll 1$, $x\ll 1$. According to the exact formulas,

$$F_1=\frac{3Q}{8}+\frac{\pi^2}{8}-\frac{1}{2},\quad F_2=\frac{5}{4}-\frac{\pi^2}{8}, \tag{10}$$

whereas according to the approximate formulas,

$$F_1=\frac{5Q}{24}+\frac{1}{2},\quad F_2=0. \tag{11}$$

The numerical difference is obviously significant.

Finally, for completeness, we have also computed the exact $A\to B\gamma$ decay rate:

$$\Gamma=\frac{\alpha G_F^2 m_A^5}{32\pi^4}\left(1-\frac{b}{a}\right)^3\left(\left|\sum_i\lambda_{iA}\lambda_{iB}^*[F_1^{(i)}+F_2^{(i)}]\right|^2+\frac{b}{a}\left|\sum_i\lambda_{iA}\lambda_{iB}^*[F_1^{(i)}-F_2^{(i)}]\right|^2\right). \tag{12}$$

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[5]We drop the constant term $\frac{1}{8}Q+\frac{1}{4}$ in F_1, in anticipation that it does not contribute after summing over all internal fermions.