PHYSICAL REVIEW D VOLUME 24, NUMBER 5 1 SEPTEMBER 1981

Logarithmic approximations, quark form factors, and quantum chromodynamics

S. D. Ellis, N. Fleishon, and W. J. Stirling
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
(Received 20 April 1981)

The structure of perturbative quantum chromodynamics in kinematic regimes characterized by two large but different mass scales is investigated. The double-leading-logarithm approximation is reviewed, and the roles of the various elements of this approximation are emphasized. In particular, the question of whether a suppression of cross sections in the limit of small angles and large ratios of kinematic variables is predicted by the theory is treated at various levels of approximation. Improved analysis suggests that no suppression is predicted. It is argued, however, that a definitive answer can only arise from an exact analysis.

I. INTRODUCTION

An area of considerable interest and development in the last two years is the perturbative study[1-4] of the behavior of quantum chromodynamics (QCD) in processes characterized by two large but very different momentum scales. There are two examples of such processes which are of particular experimental interest. The first is lepton-pair production in hadronic reactions where the total energy s, the pair mass Q^2, and the pair transverse momentum squared Q_T^2 are all large (compared to 1 GeV^2), and the ratio s/Q_T^2 is also large. In the simplest case this process can be viewed as proceeding through the annihilation of a quark-antiquark pair into a virtual photon and any number of gluons. The second example is electron-positron annihilation into hadrons where the total energy W is large and hadrons are simultaneously observed at relative angles approaching 180° such that the relative transverse momenta of the hadrons are sizable but much smaller than W.

These limiting configurations are most easily understood by first considering the simple kinematic regime where all momentum variables are large and all ratios are of order unity. For the case of lepton-pair production this corresponds to very large Q_T^2, comparable to s, and for lepton annihilation to the detection of hadrons at relative angles well away from 0° or 180°. In this regime straightforward, low-order perturbative analyses[5] of QCD are adequate to understand the structure of the theory and most of the features of existing data.[6]

As one moves into the kinematic regions where ratios become large and relative angles small or nearly 180°, the infrared structure of the theory ensures the emergence of logarithms of these ratios or angles. Order by order in perturbation theory the leading contributions in this limit have the form $\alpha_s^N \ln(1/\eta)]^{2N}$, where α_s is the strong coupling constant and the large ratio (small angle) is denoted by $1/\eta(\eta)$, $\eta \ll 1$. The approximation of keeping only these leading contributions, called the double-leading-logarithm approximation (DLLA), is by now well understood.[4] These contributions correspond to the emission of N soft, collinear gluons which, by virtue of their soft nature, are emitted *independently*. This independence ensures that the sum over N yields an exponential Sudakov-type[7] form factor:

$$F_{\text{DLLA}}(\eta) = \exp\left(-\frac{C_F\alpha_s}{2\pi}\ln^2\frac{1}{\eta}\right). \tag{1.1}$$

The factor C_F is the "Casimir coefficient" for the quarks, equal to $\frac{4}{3}$ for SU(3). The form factor has the interesting feature of being strongly *damped* in the limit $\eta \to 0$, while any individual, *fixed-order* term in the perturbative expansion is *divergent* in this limit. In this approximation the experimental quantities mentioned above will be a low-order divergent perturbative result multiplied by Eq. (1.1) [essentially a derivative of Eq. (1.1)]. Thus, as functions of η, they will exhibit a maximum at an η value intermediate between 0 and those values of η appropriate to low-order perturbation theory [i.e., when Eq. (1.1) is essentially unity] and a zero (dip) as η goes to 0.

The question which is central to this paper is whether such a peak is actually a *prediction* of the full theory. To answer this question the role of nonleading contributions must be considered. The leading contributions, order by order in α_s, have conspired to cancel at $\eta = 0$ leaving this region open to dominance by nonleading terms. In this context nonleading refers to contributions smaller both by powers of $[\ln(1/\eta)]^{-1}$ and powers of η as $\eta \to 0$.

It is also necessary to note that in practice, experiments are performed at finite energies where large ratios and small angles correspond to transverse momenta which, in fact, are not typically large on the scale of 1 $(\text{GeV})^2$. Since the connection between the incident and produced hadrons and the quark and gluon degrees of freedom of the perturbation theory, in either the initial or final states, involves a summation over relative transverse momenta corresponding to the nonzero size of hadrons, there exists a range of ratios and angles $[1/\eta \gtrsim s\times(1\ \mathrm{fm})^2]$ where the details of any perturbative analysis, no matter how elegant, will be lost in the smearing effects of this summation process. In the present work this question is largely ignored except to note that no phenomenological significance can be ascribed to the precise limit $\eta=0$ in the perturbation theory.

It is in the context of the above-mentioned phenomenological and theoretical interests that the present paper attempts to study the structure of the *nonleading perturbative* contributions. In Sec. II the relevant details of the DLLA are reviewed and Eq. (1.1) derived. The shape of cross sections in the DLLA is discussed. The specific approximations which are inherent in the DLLA are emphasized in Sec. III. Of particular interest is the role of approximate transverse-momentum conservation. The use of the Bessel-Fourier transform to exactly conserve transverse momentum, which has already been studied by several authors[3,8] is discussed in detail in Sec. IV. The limitations of the results of Sec. IV are covered in Sec. V with special attention to the role of energy conservation. Finally Sec. VI includes a review and a discussion of results.

These results include the observation that, while nonleading logarithmic contributions associated with an exact treatment of transverse-momentum conservation serve to fill in the zero at $\eta=0$, there are further nonleading contributions associated with energy conservation which may be of comparable influence but more difficult yet to treat precisely. Furthermore the usual "soft" approximation to the full gluon-emission matrix element, while adequate for the DLLA, is not sufficient for the degree of precision desired here. To treat properly the question of whether a small-η peak is actually *predicted* by the perturbative analysis of QCD will require a much more complete analysis than presently appears practical. Stated more succinctly, the question of the absence or the presence of a peak, which is surely very subtle when studied in momentum space, remains subtle also when considered in Bessel-transform space.

II. THE DOUBLE-LEADING-LOGARITHM APPROXIMATION IN TRANSVERSE-MOMENTUM SPACE

The calculation of the double-leading-logarithm approximation (DLLA)[2,3] to the process of hadronic lepton-pair production (Drell-Yan process) at small transverse momentum and to the measurement of energy-energy correlations in electron-positron annihilation near the back-to-back configuration have been discussed in detail in Ref. 4. Only the important features of the analysis will be reviewed here.

From the standpoint of perturbative QCD it is both convenient and sufficient to study simply the process of quark-antiquark of total energy s annihilating into a virtual photon with large mass squared Q^2 and transverse momentum Q_T relative to the quark/antiquark direction plus any number of gluons as illustrated in Fig. 1. This eliminates the, for the present purposes, irrelevant complications of treating the distributions of quarks in hadrons for the complete lepton-pair production process. The electron-positron process is related by crossing. The specific cross section to be calculated is

$$\frac{d\sigma}{dQ_T{}^2}\equiv\int_0^{Q_{\max}{}^2} dQ^2\left(\frac{Q^2}{s}\right)\frac{d^2\sigma}{dQ^2\,dQ_T{}^2}, \tag{2.1}$$

a close analog of the energy-weighted correlation function for the lepton annihilation process. This cross section is insensitive[4] to those perturbative corrections which serve to renormalize the quark distributions within the incident hadrons. For the process $q\bar{q}\to\gamma^*+Ng$ this cross section has the form (all parton masses zero)

$$\frac{1}{\sigma_0}\frac{d\sigma^{(N)}}{dQ_T{}^2}=\prod_{i=1}^{N}\left[\int d\theta_i d\mu_i\frac{d^2k_{Ti}}{\pi s}\delta\left(\theta_i\mu_i-\frac{k_{Ti}{}^2}{s}\right)\right]M^{(N)}(\theta,\mu,\vec{k}_T;s)\pi\delta\left(\sum_j^N\vec{k}_{T_j}+\vec{Q}_T\right)$$
$$\times\theta\left(1-\sum_k^N(\theta_k+\mu_k)+\left(\sum_l^N\theta_l\right)\left(\sum_m^N\mu_m\right)-\frac{Q_T{}^2}{s}\right), \tag{2.2}$$

where, with the quark (antiquark) four-momentum labeled p_1 (p_2),

$$s=(p_1+p_2)^2,\quad \sigma_0=\frac{4\pi\alpha^2}{9s},\quad k_i^\mu\equiv\theta_i p_2^\mu+\mu_i p_1^\mu+\vec{k}_{Ti}, \tag{2.3}$$

and α is the fine-structure constant. The δ functions inside the product ensure massless gluons. The final θ function arises from the constraint that Q^2 be positive and is the remnant of overall energy and longitudinal energy conservation. The trans-

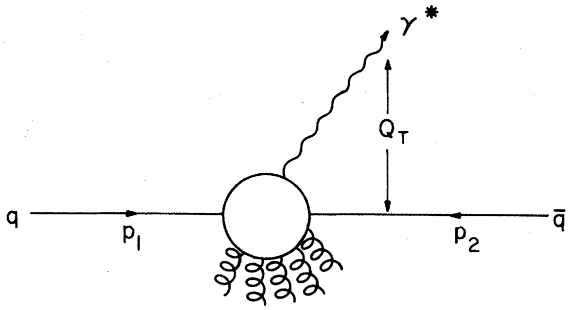


FIG. 1. The annihilation subprocess $q\bar{q}\to\gamma^* + Ng$. The quark (antiquark) has momentum p_1 (p_2) and the large mass virtual photon γ^* has momentum Q with component Q_T transverse to the quark-antiquark axis.

verse-momentum-conserving δ function was used to write the argument of the θ function (Q^2/s) in the form shown. The only other relevant restriction not explicitly indicated is that $\mu_i, \theta_i \geq 0$ corresponding to positive energy (outgoing) gluons.

The squared matrix element $M^{(N)}$ describing the emission of N gluons is, in general, a complicated function depending on all the θ_i, μ_i, and $\vec{k}_{T_i}$ in a coupled fashion. However, it is largest when the θ_i and μ_i are small ($\ll 1$), i.e., when the gluons are *soft*. In the limit of small θ_i and μ_i, the most singular part of $M^{(N)}$ can be expressed as

$$M_s^{(N)} = \prod_{i=1}^{N}\left[\frac{\alpha_s C_F}{\pi}\frac{s^2}{(2n\cdot k_i)D_i}\right], \tag{2.4}$$

where, in this limit

$$D_i = s\sum_{j=1}^{i}\mu_j . \tag{2.5}$$

This approximation to $M^{(N)}$ ignores terms of the form $(\mu_i\cdot\theta_j)s$ and $\vec{k}_{T_i}\cdot\vec{k}_{T_j}$ in the full expression for D_i. The vector $n^\mu = p_1^\mu + p_2^\mu$ defines the noncovariant planar gauge chosen to simplify the structure of the calculation. The final result is, of course, gauge independent. In the chosen gauge the result in Eq. (2.4) arises solely from the "ladder" diagrams.

To the extent that the μ_i integrals are treated symmetrically, as they formally are in Eqs. (2.2) and (2.4), the replacement

$$\prod_{i=1}^{N}\frac{1}{D_i}\to\frac{1}{N!}\frac{1}{s^N}\prod_{i=1}^{N}\frac{1}{\mu_i} \tag{2.6}$$

is correct. Thus the most singular part of the matrix element, Eq. (2.4), can be written in a totally *factorized* form.

The restriction to small θ_i and μ_i further guarantees that the θ function can be simply replaced by *independently, completely factorized* limits on the θ_i and μ_i integrals. Thus in this approximation of small μ_i and θ_i, the μ_i and θ_i integrals and integrands are completely *factorized*. In order to actually evaluate the integrals an upper limit must be chosen since it is no longer supplied by the θ function. While the specific choice is irrelevant to the result for the leading-logarithmic contribution, the choice will affect nonleading terms. This question will be discussed in detail later. For the present the simplest choice is

$$0\leq\theta_i, \mu_i\leq 1 , \tag{2.7}$$

corresponding to the maximum individual value allowed by the θ function ($Q_T^2=0$). This is an *overestimate* of the actually allowed θ, μ phase space. Evaluating the μ_i and θ_i integrals yields the "soft gluon" cross section as

$$\left.\frac{1}{\sigma_0}\frac{d\sigma^{(N)}}{dQ_T^2}\right|_{\rm soft} \equiv \frac{1}{N!}\left(\frac{\alpha_s C_F}{\pi}\right)^N\prod_{i=1}^{N}\left[\int\frac{d^2k_{Ti}}{\pi k_{Ti}^2}\ln\frac{s}{k_{Ti}^2}\theta\left(1-\frac{k_{Ti}^2}{s}\right)\right]\pi\delta\left(\sum_i\vec{k}_{Ti}+\vec{Q}_T\right), \tag{2.8}$$

where α_s is treated as fixed for now.

The last step necessary to obtain the DLLA is to observe that the maximum number of large logarithms arises from the regions of integration where the k_{T_i} are *strong ordered*, i.e.,

$$k_{T_a}^2\ll k_{T_b}^2\ll\ldots,\quad k_{T_c}^2\simeq Q_T^2\ll s , \tag{2.9}$$

where $(a,b,\ldots,c)$ is *any* permutation of $(1,2,\ldots,N)$. The sum over permutations yields a factor $N!$ times N *nested* integrals with the δ function affecting only the last (largest k_{T_i}) integral. Alternatively the sum over permutations can be used to write the N k_{T_i} integrals as independent integrals with upper limit Q_T and to replace $\delta(\sum\vec{k}_{T_i}+\vec{Q}_T)$ by $\sum_i\delta(\vec{k}_{T_i}+\vec{Q}_T)$. In either approach the result is

$$\begin{aligned}\left.\frac{1}{\sigma_0}\frac{d\sigma^{(N)}}{dQ_T^2}\right|_{\rm DLLA} &= \frac{\alpha_s C_F}{\pi}\frac{\ln s/Q_T^2}{Q_T^2}\left(\frac{\alpha_s C_F}{\pi}\right)^{N-1}\int^{Q_T^2}\frac{d^2k_{T(N-1)}}{k_{T(N-1)}}\ln s/k_{T(N-1)}^2\cdots\int^{k_{T2}^2}\frac{dk_{T1}^2}{k_{T1}^2}\ln s/k_{T1}^2\,\theta(1-Q_T^2/s)\\ &= \frac{\alpha_s C_F}{\pi}\frac{\ln s/Q_T^2}{Q_T^2}\frac{1}{(N-1)!}\left(\frac{\alpha_s C_F}{\pi}\int^{Q_T^2}\frac{dk_T^2}{k_T^2}\ln s/k_T^2\right)^{N-1}\theta(1-Q_T^2/s) .\end{aligned} \tag{2.10}$$

The manifest divergences at the lower limits of the k_{Ti} integrals are canceled by the corresponding virtual contributions. The details of this cancellation are discussed, for example, in Ref. 4. The precise details are not essential to the present discussion although the implicit inclusion of the virtual contributions is, of course, essential. The appearance of the "Sudakov form factor" below can, in fact, be viewed as the result of *incomplete* cancellation between real and virtual contributions. Canceling the explicit divergences and summing over N yields the usual DLLA result:

$$\frac{1}{\sigma_0}\frac{d\sigma}{dQ_T^2}\bigg|_{\mathrm{DLLA}}=\frac{\alpha_s C_F}{\pi}\frac{\ln s/Q_T^2}{Q_T^2}\exp\left(-\frac{\alpha_s C_F}{2\pi}\ln^2 s/Q_T^2\right)\theta(1-Q_T^2/s)\,, \tag{2.11}$$

which is just the Sudakov-type form factor, evaluated at s/Q_T^2, times the one-gluon cross section. It is convenient to return to the general notation of the Introduction and define $\eta=Q_T^2/s$ and $\lambda=\alpha_s C_F/\pi$. Thus Eq. (2.11) can be written as

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\bigg|_{\mathrm{DLLA}}=\frac{\lambda}{\eta}\ln\frac{1}{\eta}\exp\left(-\frac{\lambda}{2}\ln^2\eta\right)\theta(1-\eta)$$
$$=\frac{d}{d\eta}F_{\mathrm{DLLA}}(\eta)\theta(1-\eta) \tag{2.12}$$

with $F_{\mathrm{DLLA}}(\eta)$ identified from Eq. (1.1).

The characteristic feature of Eq. (2.12) is the existence of a peak at small η. This is illustrated in Fig. 2 where the solid line is the DLLA result of Eq. (2.12), plotted for the case $\lambda=0.08$ [$\alpha_s=0.2$ for SU(3)], and the dashed line is the single-gluon result [Eq. (2.12) without the exponential factor]. As expected the two cross sections are essentially identical for η near 1. The effects of larger numbers of gluons only become important for $\lambda\ln^2 1/\eta \gtrsim O(1)$, i.e., $\eta \lesssim e^{-1/\sqrt{\lambda}} \ll 1$ for $\lambda<1$, as indicated in Fig. 2. For even smaller η the DLLA cross section exhibits a maximum for $\eta \sim e^{-1/\lambda}$ and then vanishes at $\eta=0$. The definition of the cross section in Eq. (2.12) is such as to ensure area 1 under the curve. It is important to note that a negligible fraction, $\sim e^{-1/2\lambda}$, of this area corresponds to the region between $\eta=0$ and the peak.

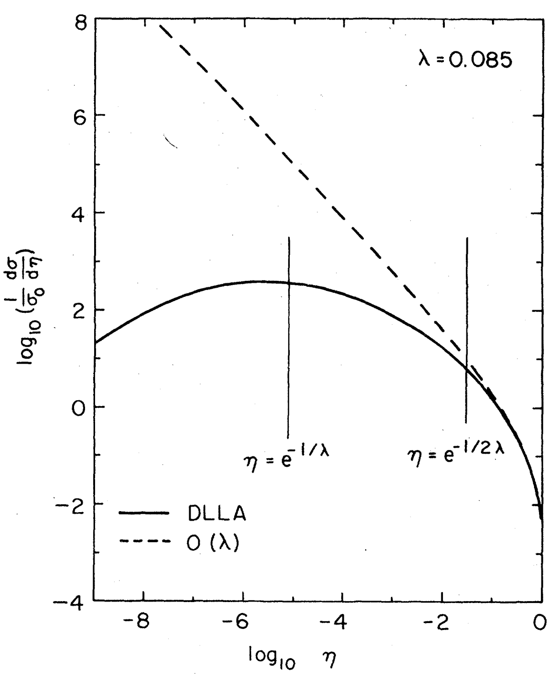


FIG. 2. Theoretical approximations to the cross section defined in the text. The solid line is the exponentially damped DLLA, Eq. (2.12). The dashed line is the corresponding one-gluon contribution which diverges at $\eta=0$.

The question to be explored here is whether this characteristic peaked behavior is a real effect or purely an artifact of the DLLA. As mentioned earlier, the present analysis will not include the smearing effects of relating quarks to hadrons (which will surely fill in the zero at $\eta=0$ to some extent) but will deal instead with the effects of relaxing the above approximations.

III. BEYOND THE DLLA

In order to study the effects of improving upon the DLLA of the previous section it is helpful to first list the approximations made in progressing from Eq. (2.2) to Eq. (2.12). First the soft limit, small θ_i and μ_i, was taken so that the following steps were appropriate.

(1) Approximate the full matrix element by the most singular piece (in this limit), as in Eq. (2.4), and thus, using the symmetrization substitution of Eq. (2.6), write the matrix element in fully *factorized* form.

(2) Ignore the constraints of the (energy-conserving) θ function so that the θ_i and μ_i integrals can be treated *independently*.

(3) Set the upper limits of all the μ_i and θ_i integrals (arbitrarily) to 1, independent of s, Q_T^2 (and the other θ_i and μ_i), to yield simple results.

These three approximations (1), (2), (3) led to the soft form of Eq. (2.8) which clearly treats correctly only the small μ_i, θ_i phase space. Note, in particular, that unbounded energy in the gluons is allowed as $N\to\infty$.

One more step was required to obtain the final result.

(4) Keep only the *strong-ordered* regions of k_{T_i} phase space [Eq. (2.9)] so that, instead of being coupled, the k_{T_i} integrals are either simply nested or written as N equal contributions each of which involves *independent* k_{T_i} integrals [Eq. (2.10)]. The transverse-momentum-conserving δ function is then a trivial constraint.

Note that this last step excludes regions of k_{T_i} phase space where several k_{T_i} are of order Q_T or where any are larger than Q_T. The k_{T_i} phase space is *underestimated*.

Thus it is the sum of all four steps (1), (2), (3), (4) which lead to the final, simple, fully *factorized* expression in Eq. (2.10). This factorization is essential to the exponential DLLA result of Eqs. (2.11) and (2.12).

The major purpose of what follows is to study how the result for $(1/\sigma_0)(d\sigma/d\eta)$ changes as the approximations (1), (2), (3), (4) are altered, i.e., are improved upon. In order to perform this task it is useful to introduce another definition. In general the cross section can be written in the form

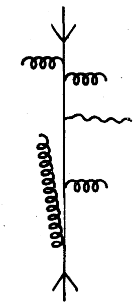
$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta} \equiv \frac{1}{\eta}\sum_{N=1}^{\infty}\lambda^N\sum_{m=0}^{2N-1}\mathfrak{A}_m^{(N)}(\eta)\left(\ln\frac{1}{\eta}\right)^{2N-1-m}. \tag{3.1}$$

The $\mathfrak{A}_m^{(N)}(\eta)$ are of the form of a constant, $\mathfrak{A}_m^{(N)}(0)$, plus terms which vanish as $\eta\to 0$. The terms which vanish as $\eta\to 0$ are the contributions which are nonleading due to powers of η as mentioned in the Introduction. It is therefore natural to define the logarithm approximation (LA) as an approximation to Eq. (3.1) where $\mathfrak{A}_m^{(N)}(\eta)$ is replaced by $\mathfrak{A}_m^{(N)}(0)$:

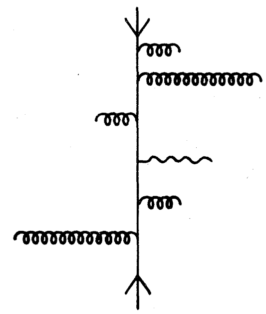
$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{\text{LA}} \equiv \frac{1}{\eta}\sum_{N=1}^{\infty}\lambda^N\sum_{m=0}^{2N-1}\mathfrak{A}_m^{(N)}(0)\left(\ln\frac{1}{\eta}\right)^{2N-1-m}. \tag{3.2}$$

The DLLA [Eq. (2.12)] corresponds to setting all $\mathfrak{A}_m^{(N)}(0)$ to zero except $\mathfrak{A}_0^{(N)}(0)=(-2)^{1-N}/(N-1)!$

Returning now to the central question of corrections to the DLLA, consider the contribution from configurations with a *single* energetic ($Q_T/\sqrt{s}\ll\mu_i$ or $\theta_i\leqslant 1$) but essentially collinear gluon ($k_{T_i}\approx 0$) as illustrated in Fig. 3(a). This correction to the DLLA was studied in some detail in Ref. 4 and is reviewed briefly here for completeness. Since the single energetic gluon effectively factors out from the other soft ($\mu_j,\theta_j\approx 0$) gluons, approximations (1), (2), (3), (4) can be maintained for all but the energetic gluon. Furthermore, since the energetic gluon is collinear, approximation (4) is unaffected and only (1), (2), (3) must be treated differently for this *single* gluon. In particular the correct energetic gluon matrix element is required. This correction to the DLLA was evaluated in Ref. 4 to a precision corresponding to next-to-double-leading-logarithm accuracy (NDLLA: $\alpha_s^{\,N}[\ln(1/\eta)]^{2N-2}$). The final result of Ref. 4, including also the effects of the running coupling to the same accuracy, was qualitatively similar to the DLLA. The fixed-coupling-constant analog of Eq. (2.12) is



(a)



(b)

FIG. 3. Contributions to the annihilation subprocess $q\bar{q}\to\gamma^*+Ng$ which are not included in the double-leading-logarithm approximation (DLLA). (a) Contributions containing a hard collinear gluon and (N-1) soft gluons. These contributions are summed in the next-to-double-leading-logarithm approximation (NDLLA) of Ref. 4. (b) Contributions containing a number of hard gluons each emitted with large transverse momentum.

$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{\text{NDLLA}} = \frac{\partial}{\partial\eta}\exp\left[-\frac{\lambda}{2}\left(\ln^2\eta-3\ln\frac{1}{\eta}\right)\right], \tag{3.3a}$$

while with a running coupling, $\lambda(k_T^{\,2})\equiv 2\gamma/\ln(k_T^{\,2}/\Lambda^2)$, the result is

$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{\text{NDLAA, run}} = \frac{\partial}{\partial\eta}\exp\left\{+2\gamma\left[(\ln s/\Lambda^2-\tfrac{3}{2})\ln\left(1-\frac{\ln 1/\eta}{\ln s/\Lambda^2}\right)+\ln\frac{1}{\eta}\right]\right\}. \tag{3.3b}$$

The primary effect of these corrections is to shift the magnitude and position of the maximum.

The shift is, in fact, to larger η since the exponential in Eq. (3.3b) vanishes at $\eta=\Lambda^2/s>0$. For simplicity in the following analysis the coupling will be kept fixed and only the most singular piece of the matrix element kept.

For the question of interest here a more important correction to the DLLA arises from the configuration illustrated in Fig. 3(b). It corresponds to the emission of two gluons of large ($>Q_T$), almost equal and opposite $\vec{k}_T$ which balance the virtual photon (plus any number of soft gluons). Such a configuration is *not* included in the strong-ordered approximation (4). Nor, if the two gluons are sufficiently energetic, is it correctly described if energy conservation is ignored or if the matrix element is approximated as in approximations (1), (2), (3).

Consider first the impact of approximation (4). To investigate this question return to Eq. (2.8) prior to approximating the transverse-momentum-conserving δ function. In the present notation this becomes

$$\frac{1}{\sigma_0}\frac{d\sigma}{dQ_T^2}\bigg|_{(1),(2),(3)}=\sum_{N=1}^{\infty}\frac{\lambda^N}{N!}\prod_{i=1}^{N}\left\{\frac{1}{\pi}\int d^2k_{Ti}\left[\frac{\ln s/k_{Ti}^2}{k_{Ti}^2}\right]_+\theta(1-k_{Ti}^2/s)\right\}\pi\delta\left(\sum\vec{k}_{Ti}+\vec{Q}_T\right), \tag{3.4}$$

where the subscript (1), (2), (3) denotes that approximations (1), (2), (3) were made in deriving Eq. (3.4). The infrared-regularizing virtual contributions have been implicitly included in Eq. (3.4) by introducing the familiar $[\;]_+$ prescription[9] ($f_+(x)g(x)\equiv f(x)[g(x)-g(0)]$). To further simplify the analysis and focus on the relevant features, the LA, introduced in Eq. (3.2), can be taken in Eq. (3.4). For $N=1,2$ a detailed k_T-space calculation has been performed in this approximation to yield

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\bigg|_{(1),(2),(3),\mathrm{LA}}\simeq\frac{1}{\eta}\ln\frac{1}{\eta}+\frac{\lambda^2}{\eta}\left[-\tfrac{1}{2}\ln^3\frac{1}{\eta}+2\zeta(3)\right]. \tag{3.5}$$

This result serves to define the $\mathfrak{A}_m^{(1)}(0)$ and $\mathfrak{A}_m^{(2)}(0)$ of Eq. (3.2) in the approximation denoted by the subscripts. The symbol $\zeta(3)$ stands for the Riemann zeta function[10] of argument (3), $\zeta(3)=1.202\ldots.$ The leading terms are recognized as the DLLA result, Eq. (2.12). The expression in Eq. (2.12) indicates that in order λ^2 the inclusion of exact transverse-momentum conservation has no effect on the leading three orders of logarithms. Unfortunately it is technically very difficult to extend the evaluation of Eq. (3.4) beyond second order in λ directly in k_T space. Note in particular that the coupling of the integrals by the δ functions precludes any *simple* exponentiating result in k_T space in this improved approximation. A more complete analysis of Eq. (3.4) can be obtained by employing the Fourier-Bessel transform as discussed next.

IV. THE FOURIER-BESSEL-TRANSFORM METHOD

The expression for the cross section in Eq. (3.4) is a convolution of k_T integrals, and thus the integrands can be made to factorize by going to Fourier-transform or impact-parameter space. In the present context this approach was first suggested by Parisi and Petronzio[3] and more recently studied by others.[3,8] The transformed cross section is defined by

$$\hat{\sigma}(\vec{b})=\frac{1}{\pi}\int d^2Q_T\,e^{i\vec{b}\cdot\vec{Q}_T}\frac{1}{\sigma_0}\frac{d\sigma}{dQ_T^2}, \tag{4.1}$$

and the inverse integral is

$$\frac{1}{\sigma_0}\frac{d\sigma}{dQ_T^2}=\frac{1}{4\pi}\int d^2b\,e^{-i\vec{b}\cdot\vec{Q}_T}\hat{\sigma}(\vec{b}). \tag{4.2}$$

Substituting Eq. (3.4) into Eq. (4.1) gives

$$\hat{\sigma}(\vec{b})\bigg|_{(1),(2),(3)}=\sum_{N=0}^{\infty}\frac{1}{N!}\lambda^N\prod_{i=1}^{N}\left\{\frac{1}{\pi}\int d^2k_{Ti}\exp(-i\vec{b}\cdot\vec{k}_{Ti})\theta\left(1-\frac{k_{Ti}^2}{s}\right)\left[\frac{\ln s/k_{Ti}^2}{k_{Ti}^2}\right]_+\right\}. \tag{4.3}$$

Having transformed to impact-parameter space, one finds, as expected, a completely factorized set of k_{Ti} integrals, and $\hat{\sigma}(\vec{b})$ exponentiates, in the noted approximation,

$$\hat{\sigma}(\vec{b})\bigg|_{(1),(2),(3)}\equiv\exp[\nu(\vec{b})], \tag{4.4}$$

where

$$\nu(\vec{b})=\frac{\lambda}{\pi}\int d^2k_T\,\theta\left(1-\frac{k_T^2}{s}\right)\left[\frac{\ln s/k_T^2}{k_T^2}\right]_+e^{-i\vec{b}\cdot\vec{k}_T}$$
$$=\lambda\int_0^s dk_T^2\frac{1}{k_T^2}\ln\frac{s}{k_T^2}[J_0(k_Tb)-1]. \tag{4.5}$$

It is convenient to define dimensionless variables $x=k_T^2/s$, $z=b^2s$ in terms of which

$$\nu(z)=\lambda\int_0^1 dx\frac{1}{x}\ln\frac{1}{x}[J_0(\sqrt{xz})-1]\,, \qquad (4.6a)$$

$$\hat\sigma(z)=\exp[\nu(z)]\,. \qquad (4.6b)$$

Note that $\nu(0)=0$, $\hat\sigma(0)=1$, since the "total cross section" has been normalized to 1. The function $\nu(z)$ cannot be evaluated precisely analytically but various analytic approximations can be useful. For large z, the function $[J_0(\sqrt{xz})-1]$ is well approximated by the θ function, $-\theta(x-1/z)$, giving

$$\nu(z)\underset{z\to\infty}{\approx}-\frac{\lambda}{2}\ln^2 z\,. \qquad (4.7)$$

A more precise evaluation is obtained by performing a parts integration in Eq. (4.6) to find

$$\nu(z)=-2\lambda\int_0^{\sqrt{z}} dy\,J_1(y)\ln^2(y/\sqrt{z})$$
$$\equiv-\frac{\lambda}{2}b_0(\sqrt{z})\ln^2 z+2\lambda b_1(\sqrt{z})\ln z-2\lambda b_2(\sqrt{z})\,. \qquad (4.8)$$

The "logarithm coefficients" $b_r(\sqrt{z})$ defined in Eq. (4.8) are given by

$$b_r(\sqrt{z})=\int_0^{\sqrt{z}} dy\,J_1(y)\ln^r y\,. \qquad (4.9)$$

Similarly to the $\mathfrak{A}_m^{(N)}(\eta)$ the $b_r(\sqrt{z})$ are of the form of a constant, $b_r(\infty)$, plus terms which vanish as $z\to\infty$. The asymptotic limits $b_r(\infty)$ are given by

$$b_r(\infty)=\int_0^\infty dy\,J_1(y)\ln^r(y)\,, \qquad (4.10)$$

$$\Delta b_r\equiv b_r(\infty)-b_r(\sqrt{z})=\int_{\sqrt{z}}^\infty dy\,J_1(y)\ln^r(y)$$
$$\underset{z\gg r}{\longrightarrow}\ln^r\sqrt{z}\,J_0(\sqrt{z})\sim\left(\frac{2}{\pi\sqrt{z}}\right)^{1/2}\ln^r\sqrt{z}\cos\left(\sqrt{z}-\frac{\pi}{4}\right), \qquad (4.11)$$

where the next term in the asymptotic expansion is of the form $(r/\sqrt{z})\ln^{r-1}(\sqrt{z})J_1(\sqrt{z})$.

Consider now the LA to Eq. (4.8), $b_r(\sqrt{z})\to b_r(\infty)$. The $b_r(\infty)$ have a generating function of a simple analytic form[11] ($t<\frac{1}{2}$),

$$\sum_{r=0}^\infty\frac{1}{r!}t^r b_r(\infty)=\int_0^\infty dy\,y^t J_1(y)$$
$$=2^t\frac{\Gamma(1+t/2)}{\Gamma(1-t/2)}$$
$$=\exp\left[t(\ln 2-\gamma)-2\sum_{m=1}^\infty\frac{\zeta(2m+1)}{(2m+1)}\left(\frac{t}{2}\right)^{2m+1}\right]. \qquad (4.12)$$

In the last expression γ is Euler's constant, $\gamma=0.57721\ldots$. Thus the $b_r(\infty)$ of interest here are

$$b_0(\infty)=1\,,$$
$$b_1(\infty)=\ln 2-\gamma\,, \qquad (4.13)$$

and

$$b_2(\infty)=(\ln 2-\gamma)^2\,.$$

It is clearly useful to define

$$z_0\equiv\exp[2b_1(\infty)]=4e^{-2\gamma}=1.2609\ldots\,. \qquad (4.14)$$

Thus, in the LA appropriate for $z\gg1$, $b_r(\sqrt{z})\to b_r(\infty)$ and Eq. (4.8) can be expressed as

$$\nu(z)\Big|_{\mathrm{LA}}=-\frac{\lambda}{2}\ln^2 z/z_0\,, \qquad (4.15)$$

where the reader is reminded that approximations (1), (2), (3) are implicitly included in the result of Eq. (4.15). Note that the next correction to Eq. (4.15), for large z, is of order $\lambda J_2(\sqrt{z})/z$. In order to obtain an approximate expression useful for all z, one may multiply Eq. (4.15) by $\theta(z-z_0)$, approximately accounting for the fact that the true $b_r(\sqrt{z})$ vanish as $z\to0$, to yield

$$\tilde\nu(z)=-\frac{\lambda}{2}\ln^2 z/z_0\,\theta(z-z_0)\,, \qquad (4.16)$$

which is numerically a good approximation to the full result.

It is convenient to define "rescaled" coefficients

$$\bar b_r(\infty)\equiv\int_0^\infty dy\,J_1(y)\ln^r(y/\sqrt{z_0})\,, \qquad (4.17)$$

such that

$$\bar b_0(\infty)=1\,,$$
$$\bar b_1(\infty)=\bar b_2(\infty)=0\,,$$
$$\bar b_3(\infty)=-\tfrac{1}{2}\zeta(3)\,,$$

and

$$\sum_{r=0}^\infty\frac{1}{r!}t^r\bar b_r(\infty)=\exp\left[-2\sum_{m=1}^\infty\frac{\zeta(2m+1)}{2m+1}\left(\frac{t}{2}\right)^{2m+1}\right]. \qquad (4.18)$$

Returning to the transformed cross section $\hat\sigma(z)$ defined in Eq. (4.6b), note that in all the above discussed approximation schemes $\hat\sigma(z)$ decreases to zero faster than any inverse power of z as $z\to\infty$. While performing the inversion of $\hat\sigma(z)$ to obtain the η-space cross section, it is informative to first investigate the correspondence between logarithms in z space and η space. In terms of dimensionless, scalar variables, Eq. (4.2) becomes

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}=\tfrac{1}{4}\int_0^\infty dz\,J_0(\sqrt{z\eta})\hat\sigma(z)\,. \qquad (4.19)$$

Integrating this expression by parts and rescaling ($y=\sqrt{z\eta}$) yields

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\Big|_{(1),(2),(3)}=\frac{1}{\eta^2}\int_0^\infty dy\,y^2J_1(y)[-\nu'(y^2/\eta)]e^{\nu(y^2/\eta)}\,. \qquad (4.20)$$

[In order to eliminate the surface term which appears in the parts integration of Eq. (4.20), the fact that $\hat{\sigma}(z)$ vanishes faster than any inverse power as $z \to \infty$ was used. Thus, strictly speaking, one should be concerned about the discussion below when $e^{\nu(z)}$ is expanded in powers of λ where each term is individually divergent in the limit $z \to \infty$. It is possible, if somewhat laborious, to demonstrate that the manipulations performed below are valid. Alternatively one may proceed directly from Eq. (4.19) to derive the results below, again in a less economical fashion.]

Now, focusing on the LA in z space, Eq. (4.15), the corresponding η-space cross section is

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\bigg|_{(1),(2),(3),\,\mathrm{LA}\text{-}z} = \frac{\lambda}{\eta}\int_0^\infty dy\, J_1(y)\ln(y^2/\eta z_0)\, e^{-(\lambda/2)\ln^2(y^2/z_0\eta)}, \quad (4.21)$$

where again the subscripts of the left-hand side are to keep explicit account of the various approximations. Expanding the exponential yields integrals which have already been studied in Eqs. (4.10) and (4.12). [This feature is, in fact, the reason for performing the parts integration in Eq. (4.20).] Using the definition of $\bar{b}_r(\infty)$ in Eq. (4.17) and expanding the quantity $[\ln(y^2/z_0\eta)]^{2N-1}$ as a binomial yields

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\bigg|_{(1),(2),(3),\,\mathrm{LA}\text{-}z} &= \frac{-2}{\eta}\int_0^\infty dy\, J_1(y)\sum_{N=1}^\infty\left(\frac{-\lambda}{2}\right)^N\frac{1}{(N-1)!}\sum_{m=0}^{2N-1}\frac{2^m\ln^m(y/\sqrt{z_0})(\ln 1/\eta)^{2N-1-m}\Gamma(2N)}{\Gamma(m+1)\Gamma(2N-m)}\\ &= \frac{1}{\eta}\sum_{N=1}^\infty \lambda^N\frac{(-1/2)^{N-1}}{(N-1)!}\sum_{m=0}^{2N-1}\frac{2^m\bar{b}_m(\infty)\Gamma(2N)}{\Gamma(m+1)\Gamma(2N-m)}\left(\ln\frac{1}{\eta}\right)^{2N-1-m}. \end{aligned} \quad (4.22)$$

This result is clearly in the form Eq. (3.2) of the *LA in η space*. The only question is whether the corrections to $b_r(\infty)$ for small z [i.e., if one uses the complete result $b_r(\sqrt{z})$] can lead to nonzero contributions to the η-space LA when transformed. The answer to this question is no, as shown in detail in the Appendix. Thus $\hat{\sigma}(z) = \exp[\nu(z)]$ calculated in the LA, where all terms in $\nu(z)$ which vanish as $z \to \infty$ are dropped, when transformed to η space yields exactly the LA to $(1/\sigma_0)(d\sigma/d\eta)$, where all terms in the perturbation expansion of $\eta(1/\sigma_0)(d\sigma/d\eta)$ which vanish as $\eta \to 0$ are dropped.

Comparison of Eqs. (3.2) and (4.22) yields the relation

$$\mathcal{A}_m^{(N)}(0) = \frac{(-1)^{N-1}2^{m+1-N}\Gamma(2N)}{\Gamma(N)\Gamma(2N-m)\Gamma(m+1)}\bar{b}_m(\infty). \quad (4.23)$$

This equation confirms the results of Eq. (3.5) for the $\mathcal{A}_m^{(1)}(0)$ and $\mathcal{A}_m^{(2)}(0)$ and clearly has the correct form for the coefficients $\mathcal{A}_0^{(N)}(0)$ of the DLLA. Further, since $b_1(\infty) = b_2(\infty) = 0$, this confirms the suggestion of the last section that a precise treatment of transverse-momentum conservation affects only the logarithms three powers down from the double leading logarithms, i.e., $m \geq 3$. This is not surprising when one considers that in the kinematic configuration of Fig. 3(b) there are (at least) two gluons which are neither soft nor collinear, while for the DLLA configurations all gluons are soft and collinear except the one balancing the virtual photon which is essentially soft but not collinear. Hence, by the naive counting that there is one logarithm for the soft and for the collinear characteristic of each gluon, the former configuration is down three logarithms from the latter.

What does all this imply for the behavior of $(1/\sigma_0)(d\sigma/d\eta)$ near $\eta = 0$? Clearly, since $\nu(z)|_{\mathrm{LA}}$ is real (and negative indefinite), $\hat{\sigma}(z)|_{(1),(2),(3),\mathrm{LA}}$ is positive for all z, $0 < z < \infty$. Hence $(1/\sigma_0)(d\sigma/d\eta)$ at $\eta = 0$, which is just the area under $\hat{\sigma}(z)$, is *not* equal to zero[3,8] and, in fact, the first derivative with respect to η at $\eta = 0$ is small but negative. This is illustrated in Fig. 4 where a numerical inversion of $\hat{\sigma}(z)|_{(1),(2),(3),\mathrm{LA}}$ [Eq. (4.15)] is displayed as the long dashed curve along with the one gluon and DLLA cross sections from Fig. 2. The inversion of the full z-space result, which differs mainly in that $\hat{\sigma}(0)|_{\mathrm{full}} \neq 0$ as noted earlier, is almost identical to the curve displayed. For example, the $\eta = 0$ intercept in LA differs from the full z analysis by a factor of order $(1 - (\lambda/2\pi)^{1/2} \times e^{-1/2\lambda})$ where the intercept itself is of order $(2\pi/\lambda)^{1/2}e^{1/2\lambda}$. This tiny difference (for $\lambda \ll 1$) characterizes the tiny difference in z-space areas under the two functions.

The conclusion is then that the *subleading logarithms* which arise from a correct treatment of transverse-momentum conservation can play a major role in filling in the zero at $\eta = 0$ and obscuring the maximum which was present near $\ln 1/\eta \sim 1/\lambda$ in the DLLA. It is informative to directly understand this result in terms of the LA expression in Eq. (4.22). Consider the contribution arising from m fixed while N is summed over, corresponding to a fixed number of powers of logarithms down from the leading contribution for each order of λ. This corresponds to a fixed number of nonsoft, noncollinear ($k_{T_i} > Q_T$) gluons. The

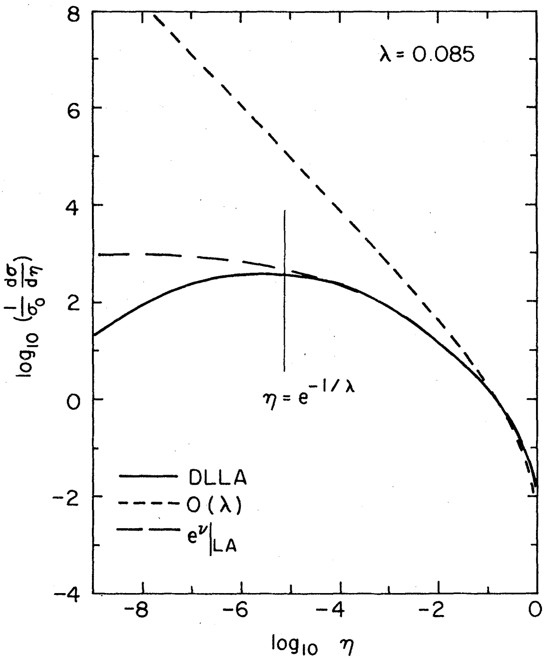


FIG. 4. Theoretical approximations to the cross section defined in the text. The long-dashed line is the soft logarithmic approximation [LA, (1), (2), (3)]. The solid line is the DLLA Eq. (2.12). The dashed line is the corresponding one-gluon contribution.

sum over N still yields a damping factor and the fixed-m contribution has the form (leading contribution in powers of $\lambda \ln 1/\eta$)

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\bigg|^{m}_{(1),(2),(3),\mathrm{LA}} \cong C_m \frac{\lambda^{m+1}}{\eta}\left(\ln\frac{1}{\eta}\right)^{m+1} e^{-(\lambda/2)\ln^2 1/\eta}, \qquad (4.24)$$

where C_m is an m-dependent constant of order 1. Such a term will still exhibit a maximum in η but for $m > 1/\lambda$ the maximum occurs at $\ln 1/\eta \simeq \sqrt{m/\lambda}$ instead of at $\ln 1/\eta \simeq 1/\lambda$ as it does for the $m < 1/\lambda$ contributions. Hence it is the successively larger m, more subleading, contributions, which peak at smaller and smaller η, which serve to fill in the dip at small η. At any finite m there remains a zero at $\eta = 0$ but this is finally filled in, in a limiting-distribution sense, as $m \to \infty$. This limiting behavior is the reminder that the exponential has been expanded inside the infinite y integral of Eq. (4.21) and the resulting integrals carefully defined in a distribution sense. It is important that for $\lambda \ll 1$ there is a sizable range of η, $\eta \gtrsim e^{-1/\lambda}$ where the DLLA is a good approximation and connects smoothly to the perturbative region at $\eta \sim 1$ as illustrated in Fig. 4.

Since the filling of the zero at $\eta = 0$ requires *all* subleading logarithms to act in concert, one must ask whether there are further corrections, associated with approximations (1), (2), (3), which are now relevant. Note in particular that in LA the cross section $(1/\sigma_0)\, d\sigma/d\eta$ remains nonzero for all $0 \le \eta \le \infty$ while in DLLA it vanished for $\eta > 1$. The required θ functions have been dropped in making approximation (2) and including correctly the conservation of transverse momentum. Whereas the DLLA *underestimates* k_{T_i} phase space, the z-space analysis of the LA *overestimates* it.

V. z-SPACE CORRECTIONS

As demonstrated in the previous section, in the approximation where (1) only the most infrared singular piece of the matrix element is kept, (2) the corrections due to energy conservation are ignored, and (3) only the simplest upper limit on the individual gluon-energy integrals is kept, but transverse momentum is conserved exactly, the sum of *all* subleading logarithms serves to fill in the zero at $\eta = 0$ exhibited by the double-leading logarithms alone.

The specific approximation studied in detail, the LA, corresponds to keeping only those contributions, order by order in λ, which are nonvanishing as $z \to \infty$ or $\eta \to 0$ (modulo the kinematic $1/\eta$ factor), approximations which turned out to be equivalent. The nonvanishing of $(1/\sigma_0)\, d\sigma/d\eta$ at $\eta = 0$ was immediately obvious in this analysis since the transformed cross section $\hat{\sigma}(z)$ was greater than zero for all $0 < z < \infty$. In fact in this approximation $(1/\sigma_0)$ $d\sigma/d\eta$ is nonzero for all η $(0 \le \eta < \infty)$. In the LA in η-space language the absence of the zero at $\eta = 0$ was traced to the contributions of kinematic configurations where arbitrarily large numbers of gluons are emitted which are *neither* soft *nor* collinear. It is necessary to now discuss to what extent these conclusions are dependent on approximations (1), (2), and (3).

Consider in particular the effect of the θ function corresponding to energy conservation in Eq. (2.2) (literally $Q^2 > 0$) and long ago discarded. As an illustration, the exact result at order λ, except for the approximation (1) of using the simplified matrix element, is given by

$$\frac{1}{\sigma_0}\frac{d\sigma^{(1)}}{d\eta}\bigg|_{(1)} = \frac{\lambda}{\eta}\ln\left[\frac{1+(1-4\eta)^{1/2}}{1-(1-4\eta)^{1/2}}\right]\theta(1-4\eta). \qquad (5.1)$$

Note both that the proper limit for η is $\eta \le \frac{1}{4}$, not $\eta \le 1$ as assumed in approximation (3), and that the cross section actually vanishes as $2\lambda(1-4\eta)^{1/2}/\eta$ as $\eta \to \frac{1}{4}$. Furthermore, if energy conservation is treated properly this one-gluon contribution is the dominant contribution as $\eta \to \frac{1}{4}$, both by powers of λ *and* powers of $(1-4\eta)^{1/2}$. This last result does *not* obtain in approximation (2).

It is straightforward to show that a cross section $(1/\sigma_0)d\sigma/d\eta$, which is nonzero only over a finite range of η (has only finite support), will have a z-space transform which exhibits oscillations in the asymptotically large-z regime. As a simple example consider a function which vanishes as a power of $(1-4\eta)$ as $\eta\to\frac{1}{4}$:

$$f(\eta)\underset{\eta\to\frac{1}{4}}{\longrightarrow}C(1-4\eta)^r . \tag{5.2}$$

The leading term as $z\to\infty$ of the transform of this function is then

$$\hat{f}(z)\underset{z\to\infty}{\longrightarrow}C\Gamma(r+1)\left(\frac{4}{\sqrt{z}}\right)^{r+1}J_{r+1}\left(\frac{\sqrt{z}}{2}\right) . \tag{5.3}$$

Thus the expected truly asymptotic behavior of $\hat{\sigma}(z)$ when energy conservation is also treated correctly is

$$\hat{\sigma}(z)\Big|_{(1)}\underset{z\to\infty}{\longrightarrow}\frac{32\lambda\sqrt{\pi}}{z^{3/4}}J_{3/2}\left(\frac{\sqrt{z}}{2}\right)$$
$$\longrightarrow -\frac{64\lambda}{z}\cos\left(\frac{\sqrt{z}}{2}\right) , \tag{5.4}$$

where the remaining approximation of simplifying the matrix element is explicitly noted. Such behavior will arise when the appropriate θ functions are kept. Note, however, that this large-z "tail" was *not* kept in the LA of the previous section. It exhibits pure power, not logarithmic, dependence on z. Such "finite-support" corrections are presumably important only for values of z such that

$$\lambda z^{-1}\gtrsim e^{(-\lambda/2)\ln^2 z} , \tag{5.5a}$$

i.e.,

$$\ln z\gtrsim O(1/\lambda) . \tag{5.5b}$$

It is only in this region that the LA result has damped sufficiently for these "small" corrections to appear. Note that, while at any given z this tail is small, it extends over an infinite range with very slowly decreasing magnitude and accounts for the transformed cross section vanishing over an *infinite* range. Thus arguments based purely on the positivity of $\hat{\sigma}(z)$ or exponential behavior near $z=0$ must be considered suspect.

To further illustrate the importance of such a tail at large z, consider the z-space transform of the DLLA η-space result:

$$\bar{\sigma}(z)_{\text{DLLA}}\equiv\int_0^1 d\eta\, J_0(\sqrt{z\eta})\frac{\lambda}{\eta}\ln\frac{1}{\eta}\,e^{-(\lambda/2)\ln^2\eta} , \tag{5.6a}$$

or using techniques similar to those in Eq. (4.22),

$$\bar{\sigma}(z)_{\text{DLLA}}=\sum_{N=0}^{\infty}\frac{(-\lambda/2)^N}{N!}\sum_{m=0}^{2N}\frac{2^m\bar{b}_m(\sqrt{z})(\ln z_0/z)^{2N-m}\Gamma(2N+1)}{\Gamma(m+1)\Gamma(2N+1-m)}+J_0(\sqrt{z}) .$$
$$=e^{-(\lambda/2)\ln^2(z/z_0)}+J_0(\sqrt{z})(1-e^{-(\lambda/2)\ln^2(z/z_0)})$$
$$+\sum_{N=1}^{\infty}\frac{(-\lambda/2)^N}{N!}\sum_{m=1}^{2N}\frac{2^m\bar{b}_m(\sqrt{z})\Gamma(2N+1)(\ln z_0/z)^{2N-m}}{\Gamma(m+1)\Gamma(2N+1-m)} . \tag{5.6b}$$

An explicit numerical evaluation of these expressions for $\lambda=0.25$ is displayed by the solid line in Fig. 5. (This value of λ was chosen to facilitate numerical computation.) The long dashed curve in this figure is the cross section $\hat{\sigma}(z)=\exp[\nu(z)]$ using the complete expression of Eq. (4.8). Note that both cross sections have intercept 1 corresponding to a properly normalized $(1/\sigma_0)d\sigma/d\eta$. The two curves are similar ($\sim\exp[(-\lambda/2)\ln^2(z/z_0)\theta(z-z_0)]$) until the region where $\bar{\sigma}(z)$ exhibits its first zero and the magnitudes of both curves are much less than 1. This similarity reflects the fact that in η space (see Fig. 4) the cross sections are essentially identical in the region from the peak in the DLLA (very small η) out to η of order 1. The similarity is also suggested by Eq. (5.6b).

Note that since

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\Big|_{\text{DLLA}}\underset{\eta\to 1}{\longrightarrow}\frac{\lambda}{\eta}\ln\frac{1}{\eta}\theta(1-\eta)$$
$$\longrightarrow\lambda(1-\eta)\theta(1-\eta) , \tag{5.7}$$

it follows from Eq. (5.3) that $(4\eta\to\eta,\ \sqrt{z}/2\to\sqrt{z})$

$$\bar{\sigma}(z)_{\text{DLLA}}\underset{z\to\infty}{\longrightarrow}\frac{4\lambda}{z}J_2(\sqrt{z})$$
$$\longrightarrow -\frac{4\lambda(\sqrt{2/\pi})}{z^{5/4}}\cos(\sqrt{z}-\pi/4) . \tag{5.8}$$

While the range of z exhibited is not yet asymptotic [Eq. (5.8) becomes a good approximation for $z\gtrsim e^{2/\lambda}$], the magnitude of the envelope of the tail of $\bar{\sigma}(z)_{\text{DLLA}}$ is clearly larger than the tail of $\hat{\sigma}(z)$.

The dashed curve in Fig. 5 is the transform of a "hybrid" cross section defined by

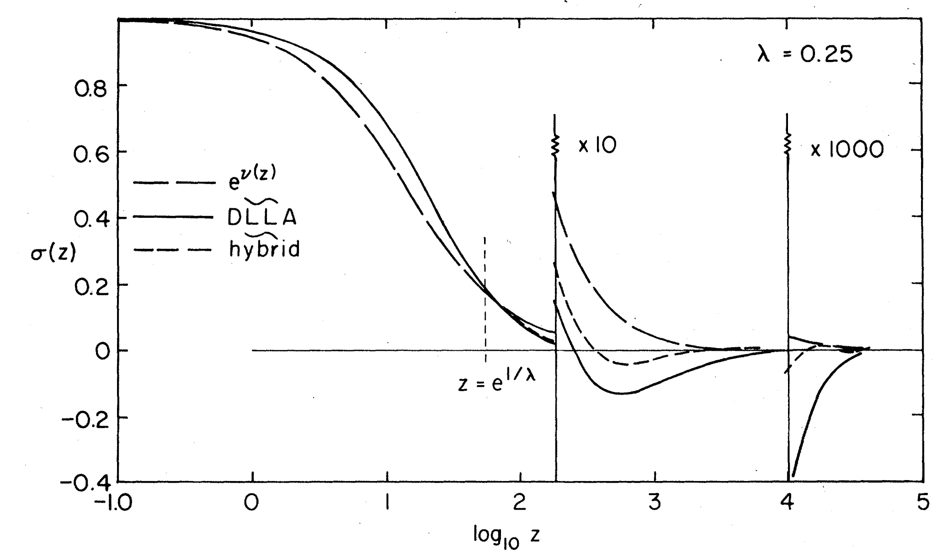


FIG. 5. Theoretical approximations to the impact-parameter-space transform of the cross section defined in the text. The long-dashed line is the exponentially falling form defined by Eq. (4.6). The solid line is the transform of the DLLA result of Eq. (2.12). The short-dashed line is the transform of a hybrid cross section, defined in Eq. (5.9), which resembles the DLLA but is not damped at $\eta = 0$.

$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{\text{hybrid}} \equiv \frac{\lambda C}{\eta_p}\ln\frac{1}{\eta_p}\, e^{-(\lambda/2)\ln^2\eta_p}, \quad 0 \le \eta \le \eta_p$$

$$\equiv \frac{\lambda C}{\eta}\ln\frac{1}{\eta}\, e^{-(\lambda/2)\ln^2\eta}, \quad \eta_p \le \eta \le 1$$

$$\equiv 0, \quad \eta \ge 1, \qquad (5.9)$$

where η_p is the location of the peak in the DLLA $\ln 1/\eta_p = [1 + (\sqrt{1+4\lambda})^{1/2}]/2\lambda$ and the normalization constant is chosen to give unit area, $C \simeq 1/(1+\lambda e^{-1/2\lambda})$. Thus for small η the hybrid cross section has no peak and no zero at $\eta = 0$, while at larger η it follows the DLLA and vanishes at $\eta = 1$. The transformed hybrid, $\tilde{\sigma}_{\text{hybrid}}(z)$, is essentially identical to the other two curves up to the neighborhood of the first zero. The similarity of $\tilde{\sigma}_{\text{hybrid}}$ and $\tilde{\sigma}_{\text{DLLA}}$, one of which has a zero at $\eta = 0$ and one of which does not, is understandable because, as noted in Sec. II, the maximum in the DLLA occurs at such small η that the η range over which the two cross sections differ discernibly is tiny, of order $e^{-1/\lambda}$. For the remaining η range the cross sections differ by a term of order $\lambda e^{-1/2\lambda}$. The difference only becomes apparent in z space when the bulk contribution to all the $\sigma(z)$'s, the LA result $\exp[-(\lambda/2)\ln^2 z]$, has decreased to this same order ($e^{-1/2\lambda}$). This again is for $\ln z \geq 1/\lambda$, just when the finite-support zeros become important. The presence of a zero at $\eta = 0$, implying zero total area under the transformed cross section, is, in fact, realized in z space not by any gross structure at small z but rather by a small negative displacement over the entire range of the large-z tail. Thus the fact that $\sigma(z)$ behaves as $\exp[\nu(z)]$ for $z \le e^{1/\lambda}$ does *not* guarantee that the corresponding $d\sigma/d\eta$ has no zero at $\eta = 0$. The argument in z space must rather hinge on the structure of the large-z tail where nonlogarithmic corrections can be important. This in turn suggests that small-η structure cannot be precisely analyzed without an adequate treatment of energy conservation, which affects this large-z regime. This is consistent with the earlier k_T-space observation that small-η behavior was related to the contributions of arbitrarily large numbers of *energetic* gluons which are also affected by energy conservation.

It is instructive to consider a simple model which implements at least some of the features of energy conservation, while still being amenable to simple analysis. Consider the following generalization of Eq. (4.5):

$$\left.\tilde{\sigma}(\vec{b})\right|_{(1),(\tilde{2}),(3)} \equiv \sum_{N=0}^{\infty}\frac{\lambda^N}{N!}\prod_{i=1}^{N}\left\{\frac{1}{\pi}\int d^2k_{Ti}\,\theta\left(1-\frac{k_{Ti}^2}{s}\right)\left[\frac{\ln s/k_{Ti}^2}{k_{Ti}^2}\right]_+ e^{-i\vec{b}\cdot\vec{k}_{Ti}}\right\}\theta\left(\sqrt{s}-\sum_{j=1}^{N}|\vec{k}_{Tj}|\right), \qquad (5.10)$$

where approximation (3), $k_{Ti}^2 \leqslant s$, is maintained and energy conservation is included only crudely, approximation ($\bar{2}$). Since

$$\sum_i |\vec{k}_T| \geqslant |\sum_i \vec{k}_{Ti}| = |\vec{Q}_T|, \tag{5.11}$$

the final θ function ensures a cross section which vanishes for $\eta > 1$ (instead of $\frac{1}{4}$) and bounded total energy in the gluons. Such a model does not treat correctly the boundary of phase space and the full coupling of integrals implied by the θ function in Eq. (2.2). However, it will give some indication of how the large-z behavior of the LA is changed by the inclusion of energy conservation.

Not surprisingly, even dealing with the "simplified" form of Eq. (5.10) is technically difficult. However, the argument of the θ function was chosen to allow a *factorizing* integral representation in analogy to the impact-parameter analysis. One can write

$$\theta(x) = \frac{1}{2\pi i}\int_C \frac{dy}{y} e^{+iyx}, \tag{5.12}$$

where the contour C in the complex y plane runs along the real axis $-\infty < \text{Re}y < \infty$, passing *below* the origin. Thus in terms of the usual scaled variables

$$\hat{\sigma}(z)\Big|_{(1,\bar{2},3)} = \frac{1}{2\pi i}\int_C \frac{dy}{y} e^{iy} \exp[\nu(z,y)] \tag{5.13}$$

with the definition

$$\nu(z,y) \equiv 4\lambda \int_0^1 \frac{dx}{x} \ln\frac{1}{x}[J_0(x\sqrt{z})e^{-ixy} - 1]. \tag{5.14}$$

Again the integral representation has allowed factorization and thus exponentiation. Integrating as before by parts yields

$$\begin{aligned}\nu(z,y) &= -2\lambda \int_0^1 dx \ln^2\frac{1}{x}[\sqrt{z}J_1(x\sqrt{z}) - iyJ_0(x\sqrt{z})]e^{-ixy} \\ &= -2\lambda \int_0^{\sqrt{z}} dt \ln^2(t/\sqrt{z})\left[J_1(t) - i\left(\frac{y}{\sqrt{z}}\right)J_0(t)\right] \\ &\quad \times e^{-it(y/\sqrt{z})},\end{aligned} \tag{5.15}$$

which is clearly a function of z and $y/\sqrt{z}$. In the limit $y \ll \sqrt{z}$ this expression approaches Eq. (4.10) and

$$\nu(z,0) = \nu(z). \tag{5.16}$$

In the limit $y \gg \sqrt{z}$ the factor $e^{-it(y/\sqrt{z})}$ oscillates more rapidly than the other functions in the integrand and

$$\nu(z,y) \underset{y \ll \sqrt{z}}{\longrightarrow} 0. \tag{5.17}$$

In the intermediate ranges of y ($y \approx \sqrt{z}$) $\nu(z,y)$ has an imaginary part. Unfortunately a simple closed-form expression for Eq. (5.15) has not yet been found. For the present purposes the essential feature of Eq. (5.15) is the appearance of an imaginary part for $\nu(z,y)$ for y of order $\sqrt{z} > 0$. Consider now evaluating the y integral in Eq. (5.13). For small $z < 1$, $\nu(z,y)$ is essentially zero for all y and the y integral is essentially 1. Thus as before one finds

$$\hat{\sigma}(z)|_{(1),(\bar{2}),(3)} \simeq 1, \quad z < 1. \tag{5.18}$$

As z increases the range of y over which $\nu(z,y)$ differs from zero also increases. The region y near zero, where e^{iy}/y is peaked, produces a result $\approx e^{\nu(z)}$ so that for z of order 1 $\hat{\sigma}(z)|_{(1),(\bar{2}),(3)}$ will be similar to the previous results, exhibiting a damped behavior $\exp[-(\lambda/2)\ln^2(z/z_0)]$. However as z is further increased the effective $\nu(z)$, integrated over an ever-increasing range of y, will have a phase. Eventually $\hat{\sigma}(z)$ will have a zero and then slowly approach the power-bounded oscillatory behavior expected at truly asymptotic z. Clearly the details of this transition in behavior will depend on the details of the approximations made. Energy conservation affects both the subleading logarithms and the "power" corrections in z space. As was just illustrated, both of these contributions can be relevant to the question of behavior near $\eta = 0$. Hence one must return to Eq. (2.2) and not make either approximation (2) or (3).

At some level of precision the approximation (1) to the matrix element must also be improved. As noted in Sec. III, the correct inclusion of a single energetic gluon and the running coupling constant in the matrix element influences the LA *one* logarithm down from the double-leading-logarithm result. While these contributions do not qualitatively affect the appearance of the zero, they certainly have quantitative effects. Also the contributions from diagrams other than the ladder diagrams kept here apparently[4] appear *two* logarithms down from the leading result.

Thus it is not obvious to what degree the question of small-η behavior is simplified by transforming to z space. This question, sensitive to *all* nonleading logarithms and, presumably, constants in η space, depends on nonleading logarithms and, at least quantitatively, on power corrections when viewed in z space.

VI. SUMMARY AND CONCLUSION

The double-leading-logarithmic contributions summed to all orders in perturbation theory provide an approximation to $(1/\sigma_0)d\sigma/d\eta$ which vanishes at $\eta = 0$ and exhibits a peak at small η. The evaluation of these contributions involves (1) approximating the matrix elements, (2) ignoring energy conservation, (3) choosing specific independent limits for the various phase-space integrals, and (4) keeping only the strong-ordered regions of transverse-momentum space.

The approximation (4) clearly underestimates the k_T-space contribution. It can be eliminated by going to impact-parameter space where the expression in z space still factorizes order by order and hence, when summed to all orders, yields an exponential form. Since the exponent in this expression is real (and negative) it is clear that at this level of approximation the area under the transformed cross section, and thus $(1/\sigma_0)\,d\sigma/d\eta$ at $\eta=0$, is nonzero. In η space this is understood as a filling in of the zero at $\eta=0$ by a summation over *all* subleading logarithmic contributions to all orders in this approximation. Such an unconstrained contribution from *any* number of nonsoft, noncollinear gluons is clearly an artifact of approximation (2) and overestimates the contribution of such gluons. While it is difficult to impose energy conservation directly in momentum space, the finiteness of the total energy available to the gluons does guarantee that the transformed cross section exhibits an oscillating, power-bounded tail at truly asymptotic z. This simple result renders naive arguments based solely on positivity in z space suspect. Furthermore, since the region of η space under study is $\eta \lesssim e^{-1/\lambda}$, the analyses in z space must be reliable for z at least as large as $O(e^{1/\lambda})$. Again this is precisely the region in z space where the present considerations suggest that the transformed cross section is sensitive to approximations involving energy conservation. Hence, whether viewed in momentum space or transform space, a precise treatment of the small-η behavior would seem to require a careful study of the effects of these approximations.

The summary of these considerations must be that there exists at the moment no reliable predictions for the small-η behavior in quantum chromodynamics. Given the number of candidate contributions capable of filling in the zero at $\eta=0$ in the DLLA, it seems unlikely that this feature is a true prediction. However the questions of the precise magnitude at $\eta=0$ and of whether the derivative at $\eta=0$ is positive or negative, i.e., whether $(1/\sigma_0)\,d\sigma/d\eta$ has a peak at some $\eta>0$, are much less obvious. The considerations discussed here suggest that these are very delicate questions to be answered clearly only after a highly detailed analysis is performed. Finally the reader is reminded that such an analysis concerns the behavior at such small η that at foreseeable energies any detailed structure predicted by the perturbation theory is almost certainly obscured by the smearing due to the nonperturbative, confinement affects mentioned earlier. Any meaningful comparison to data must include these effects.

Note added. After the work discussed here was completed, the authors received a paper by P. E. L. Rakow and B. R. Webber [Cavendish Laboratory Report No. HEP 81/4, 1981 (unpublished)] wherein several of the results of Sec. IV are obtained by slightly different methods than discussed here.

ACKNOWLEDGMENTS

We wish to thank our colleagues at the University of Washington and elsewhere for many helpful discussions. Particularly stimulating were conversations with R. Petronzio, L. S. Brown, J. Collins, and D. Soper. This research was supported in part by the U.S. Department of Energy under Contract No. DE-AC06-76ER01388.

APPENDIX

Consider the effect of the corrections to Eq. (4.15) on the transformed cross section. To understand the role of power corrections to $\nu(z)$ rewrite Eq. (4.8), using Eq. (4.11) as

$$\nu(z) = -\frac{\lambda}{2}\ln^2\frac{z}{z_0} + \frac{\lambda}{2}\Delta b_0 \ln^2 z - 2\lambda\,\Delta b_1 \ln z + 2\lambda\,\Delta b_2\,. \qquad (\mathrm{A1})$$

The terms of interest are those containing powers of Δb_r ($r=0,1,2$) in the expansion of $\exp[\nu]$ in Eq. (4.20). The general form of such corrections is given by (where the kinematic $1/\eta$ is taken to the left-hand side),

$$\eta\,\Delta\sigma(\eta) = \lambda\int_0^\infty dy\,J_1(y)\ln^n y\,\ln^m\eta\,[\Delta b_r(y/\sqrt{\eta})]^k\,, \qquad (\mathrm{A2})$$

where n, m, k are integers and Δb_r from Eq. (4.11) is bounded by, for $0\leqslant z\leqslant\infty$,

$$|b_r(\infty)| > |\Delta b_r(\sqrt{z})| \geqslant \left|\left(\frac{2}{\pi\sqrt{z}}\right)^{1/2}\ln^r\sqrt{z}\,\cos\left(\sqrt{z}-\frac{\pi}{4}\right)\right|\,. \qquad (\mathrm{A3})$$

Since $|\Delta b_r|$ has a finite upper bound as $z\to 0$, the region of the y integral $0\leqslant y\leqslant N\sqrt{\eta}$, for $N\gg 1$, yields a contribution bounded by $\lambda/\eta(N\sqrt{\eta}/2)^2\,|b_r(\infty)|^k\ln^n(N\sqrt{\eta})\ln^m\eta$ so that this contribution vanishes as $\eta\to 0$. This region does *not* contribute to the LA. For $y\geqslant N\sqrt{\eta}$ the asymptotic expansion of Δb_r is appropriate ($y/\sqrt{\eta}\geqslant N\gg 1$). For $k\geqslant 2$ this yields a contribution of the form $\lambda|\eta^{k/4}|$ times an absolutely convergent y integral so again there is no LA contribution. The final and most interesting case is $k=1$. When the asymptotic expression is substituted and the lower limit $(N\sqrt{\eta})$

is set to zero, which, as noted above, does not affect the $\eta \to 0$ behavior, one has [see Eq. (4.11)] the form

$$\eta\, \Delta\sigma(k=1) = \lambda \ln^m\eta \int_0^\infty dy\, J_1(y) \ln^{r+n}(y) J_0(y/\sqrt{\eta})\,. \tag{A4}$$

As in Eq. (4.12) this integral can be related to the $(\partial/\partial t)^{r+n}$ derivative of an integral with a factor y^t instead of the logarithms. The resulting form can be evaluated[12] with a result similar to Eq. (4.12) ($t<1$),

$$\int_0^\infty dy\, J_1(y) J_0(y/\sqrt{\eta}) y^t = 2^t \eta^{1+t/2}\left(-\frac{t}{2}\right)\frac{\Gamma(1+t/2)}{\Gamma(1-t/2)} F\left(1+\frac{t}{2}, 1+\frac{t}{2}; 2, \eta\right), \tag{A5}$$

appropriate for $\eta<1$. Thus this integral vanishes identically as $t\to 0$ while all higher derivatives with respect to t, at $t=0$, vanish as a power as $\eta \to 0$.

[1] Yu. L. Dokshitzer, D. I. Dyakonov, and S. I. Troyan, Phys. Rep. 58, 269 (1980).

[2] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, Phys. Lett. 85B, 297 (1979); G. C. Fox and S. Wolfram, Caltech Rep. No. CALT 68-723 1979 (unpublished); A. V. Smilga, Nucl. Phys. B161, 449 (1979); W. Marquardt and F. Steiner, Phys. Lett. 93B, 480 (1980); G. Curci and M. Greco, *ibid.* 79B, 406 (1978); C. Y. Lo and J. D. Sullivan, *ibid.* 86B, 327 (1979); J. B. McKitterick, University of Illinois Report No. ILL-(TH)-80-22, 1980 (unpublished); H. F. Jones and J. Wyndham, Nucl. Phys. B176, 466 (1980).

[3] G. Parisi and R. Petronzio, Nucl. Phys. B154, 427 (1979); J. C. Collins and D. E. Soper, University of Oregon Report No. OITS 155, 1981 (unpublished); R. Baier and K. Fey, Z. Phys. C 2, 339 (1979); G. Curci, M. Greco, and Y. Srivastava, Phys. Rev. Lett. 43, 834 (1978); Nucl. Phys. B159, 451 (1979).

[4] S. D. Ellis and W. J. Stirling, Phys. Rev. D 23, 214 (1981).

[5] A. J. Buras, Rev. Mod. Phys. 52, 199 (1980); see also C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, Phys. Rev. D 17, 2298 (1978); 19, 2018 (1979); G. Altarelli, R. K. Ellis, and G. Martinelli, Nucl. Phys. B143, 521 (1978); B157, 461 (1978).

[6] Ch. Berger *et al.*, Phys. Lett. 99B, 292 (1981); G. Matthiae, Report No. CERN EP/80-183, 1980 (unpublished).

[7] V. V. Sudakov, Zh. Eksp. Teor. Fiz. 30, 87 (1956) [Sov. Phys.—JETP 3, 65 (1956)].

[8] H. F. Jones and J. Wyndham, Report No. ICTP/79-80/48, 1980 (unpublished).

[9] G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).

[10] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (U. S. Department of Commerce, Washington, D. C., 1972), Eq. (23.2.18).

[11] Reference 10, Eqs. (11.4.16) and (6.1.33).

[12] Reference 10, Eq. (11.4.33).