Quark model for heavy baryons

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The first $L=0$ lowest states of baryons formed by heavy quarks c, b, and t are studied. For this purpose a nonrelativistic treatment is justified and the K-harmonic method is used to solve the three-body Schrödinger equation, for a power-law confining potential between quark pairs.

I. INTRODUCTION

The radial-excitation spectra of heavy-quark–antiquark bound states (quarkonia) by assuming a confinement potential between the quark and antiquark, together with a Coulomb-type potential [important at very short distances and arising from a massless-gluon exchange, expected on the basis of quantum chromodynamics (QCD)], have been studied rather extensively[1]; the general features of quarkonium are found to be well reproduced. The q_iq_j interaction considered in this paper obtained from the knowledge of the $q_i\bar{q}_j$ interaction (we should notice that[2] $V_{qq}=\frac{1}{2}V_{q\bar{q}}$) is given by a power-law confinement potential[3]

$$V_{q_i\bar{q}_j}=A_{ij}r_{ij}^{n}-\frac{4}{3}\frac{\alpha_s}{r_{ij}}+C_{ij},\quad n>0. \tag{1}$$

The observed spectrum of the charmonium system indicates that the exponent in the potential must lie within $0<n<2$.[1] The linear potential ($n=1$) has been studied rather extensively.[1,4] Careful studies[1] of charmonium show that $n\simeq\frac{1}{2}$ gives best fits with observable quantities. This result depends to some extent on the choice of the c-quark mass.[4]

In this report we study the radial-excitation spectrum of baryons composed of heavier (charm, bottom, and top) quarks using a nonrelativistic treatment with a pairwise power-law confinement potential[1] following Eq. (1). The constant C_{ij} is included to take into account the fact that because of confinement one cannot calculate the absolute energies of the system but only the energy differences. The nonrelativistic approximation for the heavy-quark system may be justified thanks to the magnitude of the quark masses.

One cannot use the Faddeev equations[5] in the t-matrix form for the purpose because the Schmidt norm of the kernel of these equations diverges for positive power-law potentials. We use here the K-harmonic method to solve the three-body Schrödinger equation. We find, as we will point out later, that this method is a very powerful one for quark nuclear physics. Faddeev equations may be used as well if they are written in the configuration space and differential form.[6]

The total wave function for the three-heavy-quark system may be written as a product of the space, spin, isospin, and color wave functions, respectively. The color wave function must be totally antisymmetric in color indices. Hence, for either two or three identical quarks, according to the Pauli principle, the product of space, spin, and isospin wave functions must be symmetric. Since the heavy quarks are isosinglets, the isospin part is always symmetric in our context and therefore spin and configuration wave-function symmetries become correlated. They should be both symmetric or both antisymmetric. For example, a baryon formed by three identical quarks has the space part completely symmetric and hence should have spin $\frac{3}{2}$. In the case of nonidentical quarks the symmetry is mixed and the resultant spins may be $\frac{1}{2}$ or $\frac{3}{2}$. The spin-dependent forces or mass-spin relations determine the level ordering.

II. CALCULATION METHOD

The nonrelativistic Schrödinger equation for three particles may be written as

$$\left[-\frac{\hbar^2}{2m}(\nabla_{\vec{\xi}_i}^{\,2}+\nabla_{\vec{\eta}_i}^{\,2})+V_{123}(\vec{\xi}_i,\vec{\eta}_i)\right]\Psi(\vec{\xi}_i,\vec{\eta}_i)=E\,\Psi(\vec{\xi}_i,\vec{\eta}_i), \tag{2}$$

if c.m. motion is removed and the following Jacobi coordinates are used:

$$\vec{\eta}_i=\left[\frac{m_jm_k}{m(m_j+m_k)}\right]^{1/2}(\vec{r}_j-\vec{r}_k),$$
$$\vec{\xi}_i=\left[\frac{mM}{m_i(m_j+m_k)}\right]^{1/2}\left(\frac{m_j\vec{r}_j+m_k\vec{r}_k}{m_j+m_k}-\vec{r}_i\right), \tag{3}$$

where $M=m_1+m_2+m_3$, $m=(1/M)\sum_{i<j}m_im_j$, (i,j,k) is a cyclic permutation of $(1,2,3)$, and V_{123} is the

interaction potential among the three particles. For pairwise interactions

$$V_{123}=\sum_{i<j=1}^{3} V_{ij}(\vec{r}_i-\vec{r}_j)\equiv\sum_{k=1}^{3} V_k. \tag{4}$$

Equation (3) describes three equivalent coordinate systems for the three-body problem. A relation among them exists.[7] Equation (2) can be solved using the hyperspherical-harmonics approach.[7,8] It consists of selecting the most important variables and then expanding the configuration-space wave function ψ in a complete orthonormal set of functions, depending on all variables, in the following way[8]:

$$\psi(\vec{\xi}_i,\vec{\eta}_i)=\sum_{K,\alpha_i}\frac{\Phi_{K\alpha_i}(\rho)}{\rho^{5/2}}\,u_{K\alpha_i}(\hat{\xi}_i,\hat{\eta}_i,\theta_i), \tag{5}$$

where $\rho^2=\xi_i^{\,2}+\eta_i^{\,2}=\xi_j^{\,2}+\eta_j^{\,2}=\xi_k^{\,2}+\eta_k^{\,2}$, $\eta_i=\rho\cos\theta_i$, and $\xi_i=\rho\sin\theta_i$ $(0\leq\theta_i\leq\pi/2)$. The notation $\hat{\xi}_i$ and $\hat{\eta}_i$ means $\hat{\xi}_i\equiv(\theta_{\xi_i},\phi_{\xi_i})$ and $\hat{\eta}_i\equiv(\theta_{\eta_i},\phi_{\eta_i})$. Complete orthonormal sets of angular functions $\{u_{K\alpha_i}\}$ have been derived.[7-10] They are the angular part of homogeneous polynomials $P_{K\alpha_i}$ of degree K which satisfy the Laplace equation in the six-dimensional space, and α_i stands for all other necessary quantum numbers of the configuration space.

One should observe that for the important $L=0$ case, the angular functions have a simple dependence of the angles, namely θ_i and ϕ_i, where ϕ_i is defined by $\cos\phi_i=\hat{\xi}_i\cdot\hat{\eta}_i$. Furthermore, only one extra quantum number besides K is necessary to specify the states in the configuration space.[9]

If we substitute Eq. (5) into the Schrödinger equation, we find that the partial wave functions $\Phi_{K\alpha_i}(\rho)$ should satisfy the following system of coupled differential equations[8]:

$$\left[\frac{d^2}{d\rho^2}-\frac{\Lambda(\Lambda+1)}{\rho^2}-k^2\right]\Phi_{K\alpha_i}(\rho)+\sum_{K'\alpha_i'}V_{K\alpha_i,K'\alpha_i'}(\rho)\Phi_{K'\alpha_i'}(\rho)=0, \tag{6}$$

where $\Lambda=K+\frac{3}{2}$, $k^2=(2m/\hbar^2)|E|$, E being the energy of the bound state, and

$$V_{K\alpha_i,K'\alpha_i'}(\rho)=-\frac{2m}{\hbar^2}\langle u_{K\alpha_i}|V_{123}|u_{K'\alpha_i'}\rangle \tag{7}$$

(integration is over the angles).

We can obtain[11] closed expressions for the matrix elements [Eq. (7)] for the interactions given by Eq. (1). Hence Eq. (7) can be written as

$$V_{K\alpha_i,K'\alpha_i'}(\rho)=-\frac{2m}{\hbar^2}\rho^n C_{KK'}^{\alpha_i\alpha_i'}(n), \tag{8}$$

where the $C_{KK'}^{\alpha_i\alpha_i'}(n)$'s are coefficients depending only on the quantum numbers and n. Given V_{123}, Eq. (7) can be calculated and then the system given by Eq. (6) solved numerically.[12] That is done by truncating the sum over K by an upper value $K_{\max}$. For our purpose, the $K_{\max}=4$ was accurate enough (error being much less than 1 MeV) although the numbers quoted in our tables were obtained with $K_{\max}\sim 10$. Another advantage of the K-harmonic approach is that excited states can be obtained easily.[12]

TABLE I. Masses of states and values of $\langle r^2\rangle^{1/2}$ for $L=0$, $J=\frac{3}{2}$ $(i=j=k)$. Units $\hbar=c=1$. Numerical errors of order of 3%.

Baryons	n	Ground state (GeV)	$\langle r^2\rangle^{1/2}$ (fm)	First excited state (GeV)	Second excited state (GeV)
ccc	$\frac{1}{2}$	4.77	0.27	5.09	5.47
	1	4.77	0.28	5.08	5.48
	2	4.81	0.25	5.48	5.96
bbb	$\frac{1}{2}$	15.97	0.16	16.35	16.65
	1	14.98	0.21	15.32	15.61
	2	15.08	0.16	15.73	16.38
ttt	$\frac{1}{2}$	49.55	0.15	49.41	50.17
$m_t=16$ GeV	1	48.62	0.16	48.89	49.11
	2	48.54	0.15	48.89	49.24
ttt	$\frac{1}{2}$	60.98	0.14	61.32	61.89
$m_t=20$ GeV	1	60.30	0.16	60.64	60.96
	2	60.48	0.14	60.79	61.10

TABLE II. Same as Table I, but $L=0$, $J=\frac{3}{2}$ $(i\neq j\neq k)$.

Baryons	n	Ground state (GeV)	$\langle r^2\rangle^{1/2}$ (fm)	First excited state (GeV)	Second excited state (GeV)
cbt	$\frac{1}{2}$	24.33	0.12	24.46	24.49
$m_t=16$ GeV	1	23.17	0.17	13.65	23.79
	2	23.53	0.14	24.09	24.84
cbt	$\frac{1}{2}$	28.17	0.11	28.49	28.65
$m_t=20$ GeV	1	27.16	0.15	27.44	27.60
	2	27.31	0.13	27.85	28.25

TABLE III. Same as Table I, but $L=0$, $J=\frac{1}{2}$, $J_{12}=0$ $(i=j\neq k)$.

Baryons	n	Ground state (GeV)	$\langle r^2\rangle^{1/2}$ (fm)	First excited state (GeV)	Second excited state (GeV)
ccb	$\frac{1}{2}$	10.41	0.22	10.73	10.82
	1	9.35	0.34	9.70	9.79
	2	10.23	0.23	11.09	11.38
bbc	$\frac{1}{2}$	23.43	0.20	13.69	13.93
	1	12.37	0.31	12.64	12.97
	2	13.00	0.22	13.65	14.01
cct	$\frac{1}{2}$	21.64	0.17	21.94	22.06
$m_t=16$ GeV	1	20.58	0.24	20.89	21.03
	2	21.37	0.16	22.10	22.53
bbt	$\frac{1}{2}$	27.57	0.20	27.83	27.89
$m_t=16$ GeV	1	26.53	0.22	26.78	26.84
	2	26.82	0.21	27.29	27.47
ttc	$\frac{1}{2}$	35.84	0.20	36.06	36.25
$m_t=16$ GeV	1	34.78	0.22	34.99	35.20
	2	35.16	0.21	35.51	36.13
ttb	$\frac{1}{2}$	38.75	0.20	38.97	39.09
$m_t=16$ GeV	1	37.42	0.23	37.62	38.10
	2	38.36	0.21	38.22	38.44
cct	$\frac{1}{2}$	25.63	0.13	25.93	26.05
$m_t=20$ GeV	1	24.57	0.16	24.88	25.02
	2	25.35	0.14	26.07	26.51
bbt	$\frac{1}{2}$	31.55	0.17	31.81	31.87
$mm_t=20$ GeV	1	30.52	0.22	30.74	30.83
	2	31.00	0.18	31.46	31.82
ttc	$\frac{1}{2}$	43.83	0.21	44.04	44.24
$m_t=20$ GeV	1	42.77	0.21	42.99	43.20
	2	43.10	0.21	43.42	44.38
ttb	$\frac{1}{2}$	46.72	0.19	46.92	47.07
$m_t=20$ GeV	1	45.04	0.21	45.87	46.32
	2	45.81	0.20	46.13	46.37

For any quark pair, constants A_{ij}, α_s, and C_{ij} of Eq. (1) here take the same values as those of Ref. 4 (unit $\hbar = c = 1$). In that reference these parameter values were determined by fitting the charmonium data. In the calculations it was observed that the contributions coming from the Coulomb-type term in Eq. (1) are relatively small. Hence it can be neglected in the first approximation. Because of lack of experimental data and for definiteness, calculations were only done here for the $L = 0$ (ground, first, and second excited) states.

III. RESULTS

Our results are summarized in Tables I–III.

The baryons formed by two identical quarks and a distinct one should have spin $\frac{1}{2}$ in the ground state according to the spin-mass relation. Our results are close to the bag-model results[13]; similar work was also done by Hasenfratz *et al.*[14]

In order to improve our calculations, it seems important to consider many effects, such as spin-dependent potential, relativistic corrections, and three-body force. The latter appears to play an important role in the results. Some recent serious efforts in this direction have been done by different groups.[15] They could derive directly from QCD a three-body force proportional to the hyperradius ρ which could easily be added to Eq. (4), leaving the calculation of the matrix elements, given by Eq. (7), still exact.

A word should be said about light hadrons. It seems that a nonrelativistic approach can still be used for light-hadron calculations if a suitable spin-dependent force[16] is added to the local interaction given by Eq. (1).

An important feature of this problem consists of comparing molecular structures (such as H_2^+) with quark systems of very different masses, for example, the baryon ttc. Let us consider the $n = 2$ (harmonic-oscillator) case in Eq. (1) which renders the problem analytically solvable:

$$V_{123} = \beta\left(r - \frac{R}{2}\right)^2 + \beta\left(r + \frac{R}{2}\right)^2 + \beta' R^2, \tag{9}$$

where $A_{12} = A_{13} = \beta$ and $A_{23} = \beta'$, $\vec{R}$ refers to the relative coordinate between the t quarks, and $\vec{r}$ is between c and the tt center of mass. Assuming that the spring constants (β and β') are of the same order of magnitude, we find that the radial oscillation frequency for the heavier quark (t) is much smaller than that for the lighter quark (c), the ratio being essentially controlled by the reduced mass ratio

$$\omega_r = \left[\frac{2\mu_{tt}}{\mu_{c(tt)}(1 + \beta'/2\beta)}\right]^{1/2} \omega_R. \tag{10}$$

Analysis of the radial wave functions with $m_c = 1.5$ GeV, $m_b = 4.7$ GeV, $m_t = 16$–20 GeV, $\mu_{c(tt)} = 1.45$ GeV, and $\mu_{tt} = 10$ GeV and probability densities show that in configuration space, heavier quarks (t) are relatively far apart and the lighter quark (c) moves analogous to the electron in the H_2^+ molecule. In this respect, for such baryons one can use simple approximation techniques (such as Born-Oppenheimer) known in molecular physics instead of using the exact hyperspherical-harmonic approach.

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