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Physical consequences of the choice of the Lagrangian

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The physical consequences of the existence of inequivalent Lagrangians associated with a given equation of motion are examined in quantum mechanics. It is shown that in the case of conservative systems one additional condition is sufficient to select the physically correct Lagrangian. It is remarked that in the case of dissipative systems the situation is quite different, and no conditions are known to single out a unique Lagrangian.

I. INTRODUCTION

It is known in classical mechanics that a given equation of motion may be generated by many inequivalent Lagrangians.[1] Classes of such Lagrangians have been explicitly constructed by several authors.[1,2] Recently, Okubo[3] considered the influence of this phenomenon on the canonical quantization of classical systems. He investigated only if, given a Lagrangian, the quantization is possible and consistent. He did not, however, discuss what, to our mind, is the most important issue: Is it true that the choice of inequivalent Lagrangians implies contradictory physical results after quantization? If this is indeed true, is there any criterion to select the physically correct Lagrangian?

This last question is exactly what we want to dwell upon here. We shall analyze this matter both in the case of conservative and dissipative systems. We shall find out that the answer is quite different in each case. For conservative systems it will be shown that the requirement that the Hamiltonian be equal to the energy is sufficient to define a class of equivalent Lagrangians which describe a physically unique quantum system. For dissipative systems the same condition cannot be fulfilled, and even a similar but weaker condition is restrictive enough to rule out all Lagrangians. Thus the lack of any physically reasonable justification for choosing a particular Lagrangian makes the canonical quantization of dissipative systems necessarily ambiguous.

II. CONSERVATIVE SYSTEMS

Although less discussed, it is known that even in the case of conservative systems the canonical quantization starting from inequivalent Lagrangians leads to inequivalent physical results. An eloquent example of this fact has been produced by Kennedy and Kerner.[4] They constructed a Hamiltonian for the one-dimensional harmonic oscillator which generates its equation of motion but equals the square root of energy. The outcome of the subsequent quantization was a continuous spectrum for the energy, in conflict with the usual evenly spaced spectrum of the harmonic oscillator. This indicates that there must exist some criterion to extract the physically correct Lagrangian. Let us study this now.

Consider a single particle whose motion is governed by the equation (in Cartesian coordinates x_1, x_2, x_3)

$$m\ddot{\vec{x}}+\vec{\nabla}V(\vec{x})=0\,. \tag{1}$$

In order to give a Lagrangian description of Eq. (1) the first requirement on the Lagrangian, of course, is that it furnish an equation of motion equivalent to Eq. (1). At least in one dimension infinitely many inequivalent Lagrangians that lead to an equation of motion equivalent to the one-dimensional version of Eq. (1) are available.[2,3] Nontrivial higher-dimensional examples have not been discovered, to the best of the author's knowledge.

The work of Kennedy and Kerner strongly suggests that the condition that the Hamiltonian be equal to the energy is of fundamental importance as regards quantization. So our additional condition on the Lagrangian is that its corresponding Hamiltonian be equal to the total energy of the system. This implies the following differential equation for the Lagrangian L:

$$\dot{\vec{x}}\cdot\frac{\partial L}{\partial\dot{\vec{x}}}-L=\frac{m}{2}\dot{\vec{x}}^2+V(\vec{x})\,. \tag{2}$$

Its general solution is easily found to be

$$L(\vec{x},\dot{\vec{x}},t)=\frac{m}{2}\dot{\vec{x}}^2-V(\vec{x})+\dot{\vec{x}}\cdot\vec{f}(\vec{x},t)\,, \tag{3}$$

where $\vec{f}(\vec{x},t)$ is an arbitrary vector field. If the Lagrangian (3) is inserted into Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_i}\right)-\frac{\partial L}{\partial x_i}=0\,, \tag{4}$$

one gets

$$m\ddot{x}_i + \frac{\partial V}{\partial x_i} + \dot{x}_j \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}\right) + \frac{\partial f_i}{\partial t} = 0\,, \qquad (5)$$

where Einstein's summation convention for repeated indices is being employed. Equation (5) is equivalent to Eq. (1) if and only if

$$\frac{\partial f_i}{\partial t} = 0, \quad \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0\,. \qquad (6)$$

In other words, the Lagrangians (3) all lead to Eq. (1) if and only if there exists a scalar function $\phi(\vec{x})$ such that $\vec{f} = \vec{\nabla}\phi(\vec{x})$. Thus in the case of conservative systems all Lagrangians of the form

$$L(\vec{x}, \dot{\vec{x}}) = \frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x}) + \dot{\vec{x}} \cdot \vec{\nabla}\phi(\vec{x})\,, \qquad (7)$$

and only these, satisfy both requirements.

Now the question is the following: Do all of these classical Lagrangians conduce to the same physical properties in quantum mechanics? It is quite easy to show that they do. The canonical momentum derived from the Lagrangian (7) is

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{x}}} = \vec{p} + \vec{\nabla}\phi(\vec{x})\,, \qquad (8)$$

where $\vec{p} = m\dot{\vec{x}}$ is the usual linear momentum. Notice that Eq. (8) can be viewed as part of a transformation to new canonical variables

$$\vec{X} = \vec{x}, \quad \vec{P} = \vec{p} + \vec{\nabla}\phi(\vec{x})\,. \qquad (9)$$

In classical mechanics this is a canonical transformation with a generating function[5] $F_3(\vec{p}, \vec{X}) = -\vec{p}\cdot\vec{X} - \phi(\vec{X})$. In quantum mechanics it is a unitary transformation performed by the unitary operator $U = \exp[-(i/\hbar)\phi(\vec{x})]$, as shown by Dirac.[6] Therefore, all Lagrangians of the form (7) yield the same physics in the quantum theory. Incidentally, this is an illustration of the general result that to every time-independent canonical transformation of a classical system there corresponds a unitary transformation of its quantum counterpart.[7] The final conclusion is that for conservative systems the additional condition $H = E$ defines a physically unique quantum system. It is clear that this result can be generalized without any changes for systems containing any number of particles. There is also ample empirical support to the assumption that the unique quantum system thus obtained gives a correct description of nature within the domain of validity of quantum mechanics.

The above result deserves a few comments. First of all, one cannot refrain from pointing out that there seems to be no *a priori* physical reason to require $H = E$ in classical mechanics. The derivation of Hamilton's equations in any modern book on analytical mechanics reveals that the Hamiltonian is constructed directly from the Lagrangian, and its possible connection with the energy is discussed only *a posteriori*. Besides, the value of the energy depends on the equations of motion alone, so that its value is altogether insensitive to the choice of the Hamiltonian. In quantum mechanics, however, the Hamiltonian plays a double role: it is responsible for the time evolution of the system and is also the energy observable. Contrary to what happens in classical mechanics, in quantum mechanics the energy eigenvalues are drastically dependent on the choice of the Hamiltonian.[4] Therefore, the condition $H = E$ acquires a physical meaning only in quantum mechanics, and notably enough it is sufficient to characterize a unique physical system.

III. DISSIPATIVE SYSTEMS

For our present purposes it will be enough to deal with one-dimensional systems. So let us consider the same equation of motion analyzed by Okubo, namely

$$\ddot{x} + \gamma\dot{x} = 0\,, \qquad (10)$$

γ being a positive constant. This is the equation of motion for a particle of unit mass submitted to the frictional force $F = -\gamma\dot{x}$. Bateman's Lagrangian[8] for this equation of motion is

$$L(x, \dot{x}, t) = \tfrac{1}{2}e^{\gamma t}\dot{x}^2\,. \qquad (11)$$

Kobussen[2] has shown that Eq. (10) is also generated by all time-independent Lagrangians of the form

$$L(x, \dot{x}) = \dot{x}\int^{\dot{x}} \frac{dy}{y^2}\, g(y + \gamma x) \qquad (12)$$

with $g(u)$ an arbitrary real function such that $g'(u) \neq 0$. The Lagrangian given by Okubo's Eq. (2.4) is a particular case of Eq. (12) with $g(u) = u$ and a total time derivative neglected. It is very easy to verify that any two Lagrangians of the form (12) with different g's are inequivalent. So we have to face the problem of which Lagrangian does one choose.

The Lagrangian (11) was deemed "unsuitable for the standard quantization procedure" by Okubo because it depends explicitly on time. This statement appears to be open to criticism for the following reasons: Let us take it for granted that an unambiguous canonical quantization of dissipative systems is actually possible. Then for dissipative systems, that is, those whose energy is a decreasing function of time, one must choose a Lagrangian that depends explicitly on time. In this instance the Hamiltonian is also explicitly time dependent so that its associated Schrödinger equation does not admit stationary states. On the contrary, if the Lagrangian is time independent so is the Hamiltonian, and the Schrödinger equation does possess stationary states, in contradiction to the original hypothesis that the system was dissipative. This is also a further simple example which shows that inequivalent Lagrangians conduce to inequivalent physical predictions in the quantum theory. In addition to the previous arguments let us recall that the Lagrangian (11), or its generalized version including an additional conservative force, has been employed as a starting point for canonical quantization with mathematically consistent results,[9] even though this attitude has been criticized on physical grounds.[10]

In view of this situation we want to search for some criterion to select the physically correct Lagrangian, if the latter exists at all. First we try, as was done in the conservative case, to require $H=E$. Some words of clarification are necessary. What we call the energy is simply the sum of kinetic plus potential energy, where the potential energy includes all conservative forces but none of the nonconservative ones.[11] Then we have as before [consider the one-dimensional version of Eq. (3) with $m=1$ and $V=0$]

$$L(x,\dot{x},t)=\frac{\dot{x}^2}{2}+\dot{x}f(x,t), \qquad (13)$$

where f is arbitrary. It is exceedingly easy to ascertain that no Lagrangian of this form can lead to Eq. (10), no matter what f is. Therefore, the condition $H=E$ is too stringent for dissipative systems. This is no surprise whatsoever. Observe that no vestige of the dissipative force was left in the Lagrangian (13). Owing to the fact that the energy fully ignores the presence of the dissipative force, the existence of a Hamiltonian which is equal to the energy and at the same time capable of giving rise to an equation of motion in which the dissipative force is present becomes highly improbable.

Clearly the above condition has to be weakened in order to permit the existence of Lagrangians. In the author's opinion the simplest weaker condition one can think of is $dH/dt=dE/dt$, since this allows H to differ from E by an arbitrary function of any constants of the motion. This enlarges considerably the set of admissible Lagrangians. For the equation of motion (10), the two independent constants of the motion are

$$C_1(x,\dot{x},t)=\dot{x}+\gamma x, \quad C_2(x,\dot{x},t)=e^{\gamma t}\dot{x}\,. \qquad (14)$$

So the most general consequence of $\dot{H}=\dot{E}$ is

$$\dot{x}\,\frac{\partial L}{\partial\dot{x}}-L=\frac{\dot{x}^2}{2}+F(C_1(x,\dot{x},t),\,C_2(x,\dot{x},t))\,, \qquad (15)$$

where $F(\mu,\nu)$ is an arbitrary function of two real variables. The general solution to Eq. (15) is

$$L(x,\dot{x},t)=\frac{\dot{x}^2}{2}+\dot{x}\int^{\dot{x}}\frac{dy}{y^2}\,F(C_1(x,y,t),\,C_2(x,y,t)) + \dot{x}f(x,t)\,, \qquad (16)$$

where f is arbitrary. Observe that this is a possible way to take into account the dissipative force at least through the constants of the motion C_1 and C_2, for they carry the seal of the dissipative force, to wit, the constant γ. Whence the optimistic hope that out of the family of Lagrangians determined by Eq. (16) only one of them, or at most a subset of equivalent ones, might produce an equation of motion equivalent to Eq. (10). This hope is untenable, however, because unfortunately even within this enlarged class of Lagrangians there is none capable of generating an equation of motion equivalent to Eq. (10). This is a special case of a result that has been proved by the present author[12] for somewhat more general dissipative systems. Since no other conditions have been found to get rid of the ambiguity of the canonical quantization of dissipative systems, it seems that one must resort to alternative methods for quantizing these systems.[13] Is the problem of looking for a unique prescription for quantizing dissipative systems well posed and does it make sense from the physical standpoint? This is still an open question.

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[5]See, for example, H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, Mass., 1980), Chap. 9, whose notation we follow.

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