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Uncountability of the sets of harmonic potentials

Michael Martin Nieto
Theoretical Division, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545
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It has previously been shown that for a given angular velocity ω there exist three distinct sets of potentials with harmonic properties: the set $A(\omega)$ of purely classical harmonic potentials [their classical angular velocity $\omega_c(E)=\omega$ is independent of energy], the set $C(\omega)$ of purely quantum harmonic potentials [their quantum eigenvalues are equally spaced, $(E_{n+1}-E_n)/\hbar=\omega_q(n)=\omega$], and the set $B(\omega)$ of potentials with both of these properties. We show that each of these sets is uncountable.

It has long been known[1] that for a given ω there exists an uncountable number of confining classical potentials with one minimum whose classical angular velocity $\omega_q(E)$ is independent of energy,[2]

$$\omega_c(E)=2\pi\Big/(2m)^{1/2}\int_{x_L(E)}^{x_R(E)}dx[E-V(x)]^{-1/2}\equiv\omega\,, \tag{1}$$

with $x_{L,R}(E)$ being the classical turning points. We call these *classical harmonic potentials*. Similarly, there are an infinite number of potentials[3] whose quantum eigenvalues are equally spaced,

$$\omega_q(n)\equiv(E_{n+1}-E_n)/\hbar=\omega\,. \tag{2}$$

We call these *quantum harmonic potentials*. It has recently been shown[3] that although these two classes are equivalent in the WKB approximation, in fact, they are not exactly equivalent. This was shown by the example of the Abraham-Moses (AM) potential,[4]

$$V_{\rm AM}(x)=\tfrac{1}{2}m\omega^2x^2+\hbar\omega 4\phi(z)[\phi(z)-z]\,, \tag{3}$$

$$\phi(z)=\frac{e^{-z^2}}{\pi^{1/2}\mathrm{erfc}(z)}\,, \tag{4}$$

$$z=(m\omega/\hbar)^{1/2}x\,, \tag{5}$$

which, although it is a quantum harmonic potential,

$$E_n=\hbar\omega(n+\tfrac{1}{2})\,,\quad n=1,2,\ldots$$

is not a classical harmonic potential. $\omega_q(E)$ depends on energy.[3]

There are potentials which are both classical and quantum harmonic potentials. Of course, one of these is the standard harmonic oscillator (HO)

$$V_{\rm HO}(x)=\tfrac{1}{2}m\omega^2x^2\,. \tag{6}$$

Another is the "harmonic oscillator with centripetal barrier" (HOCB) or "isotonic oscillator,"[5]

$$V_0(x)=\frac{\hbar^2a^2}{2m}\nu^2(ax-1/ax)^2 =\tfrac{1}{4}\hbar\omega(y^{1/2}-\nu/y^{1/2})^2\,, \tag{7}$$

$$y=\nu a^2x^2\,, \tag{8}$$

$$\omega=(2\hbar/m)(a^2\nu)=\omega_q=\omega_c\,, \tag{9}$$

$$\nu^2\equiv\lambda(\lambda+1)\,, \tag{10}$$

$$E_n=\hbar\omega[n+\tfrac{1}{2}(\tfrac{3}{2}+\lambda-\nu)]\,,\quad n=0,1,2,\ldots\,. \tag{11}$$

Ghosh and Hasse[6] have also shown that there exists a potential which is a classical harmonic potential but not a quantum harmonic potential. This is the potential of two half-harmonic oscillators (HHO) of different widths:

$$V_{\rm HHO}=\begin{cases}\tfrac{1}{2}m\omega_1{}^2x^2\,, & x\geq 0\\ \tfrac{1}{2}m\omega_2{}^2x^2\,, & x\leq 0\,,\end{cases} \tag{12}$$

such that

$$\frac{2\omega_1\omega_2}{\omega_1+\omega_2}=\omega\,. \tag{13}$$

Thus we have three sets of potentials for a given ω: the set $A(\omega=\omega_q)$, which contains classical harmonic potentials that are not quantum harmonic potentials; the set $C(\omega=\omega_q)$, which contains quantum harmonic potentials that are not classical harmonic potentials; and the set $B(\omega=\omega_c=\omega_q)$, whose potentials are both. In this note we demonstrate that all three sets are uncountable.

First consider the V_0 and $V_{\rm HHO}$ potentials of Eqs. (7) and (12). They both are potentials with two parameters: (a and ν) and (ω_1 and ω_2), respectively. From Eqs. (9) and (12), the two pairs of two parameters can be varied continuously in such a way as to maintain the value of ω, even though these variations continuously change the shapes of V_0 and $V_{\rm HHO}$, respectively. This trivally demonstrates the uncountability of the sets $B(\omega)$ and $A(\omega)$.

The AM potential of Eq. (3) is a member of the set $C(\omega)$. It was obtained by AM[4] from the harmonic-oscillator potential by using a procedure based on the Gel'fand-Levitan formalism.[7] This procedure yields a new potential with the same quantum eigenvalues as the old one except with any particular eigenvalue removed, in this case, that of the ground state. However, since the harmonic oscillator is a potential with one parameter (ω itself), the AM potential is also a one-parameter potential. Thus, it only gives us one member in the set $C(\omega)$, not an uncountable number.[8]

But the discussion of the past two paragraphs indicates how one might obtain an uncountable number of potentials which belong to the set $C(\omega)$. Consider a potential which belongs to set B (both classical and quantum harmonic), and which has an ω which depends continuously on two parameters. Using the AM procedure, we obtain a new potential with the same eigenvalues, except with the ground state removed, so the new potential is still a quantum harmonic potential. We then demonstrate that the new potential has a classical angular velocity which depends on energy, $\omega_c(E)$.

We now do this with the HOCB set B potential of Eq. (7). The reader can consult AM[4] for the details of the procedure. Here it is enough to state the results. With some slightly involved algebra the reader can verify for himself that the potential

$$V(x)=V_0(x)+V_1(x)\,, \tag{14}$$

$$V_1(x)=2\hbar\omega[(\lambda+1-y)\Phi(\lambda,y)+y\Phi^2(\lambda,y)]\,, \tag{15}$$

$$\Phi(\lambda,y)\equiv\frac{y^{\lambda+1/2}e^{-y}}{\Gamma(\lambda+\frac{3}{2},y)}\,, \tag{16}$$

$$\Gamma(b,y)=\Gamma(b)-\int_0^y dt\,e^{-t}t^{b-1}\,, \tag{17}$$

has Schrödinger eigenvalues

$$E_n=\hbar\omega[n+\tfrac{1}{2}(\tfrac{3}{2}+\lambda-\nu)]\,,\quad n=1,2,3,\ldots \tag{18}$$

with orthonormal eigenfunctions

$$\chi_n(x)=\psi_n(x)+n^{-1}N_n\Phi(\lambda,y)e^{-\lambda/2}y^{(\lambda+3)/2}L_{n-1}^{(\lambda+3/2)}(y)\,, \tag{19}$$

where the ψ_n are the orthonormal eigenfunctions for the unperturbed V_0 potential,

$$\psi_n(x)=N_ne^{-y/2}y^{(\lambda+1)/2}L_n^{(\lambda+1/2)}(y)\,, \tag{20a}$$

$$N_n=\left(\frac{2a\nu^{1/2}\Gamma(n+1)}{\Gamma(\lambda+\frac{3}{2}+n)}\right)^{1/2}. \tag{20b}$$

Since $y=(\nu a^2)x^2$, the variations of ν as a^{-2} which keeps ω constant will not change the scale of y, but will change the shape of V.

The properties of the incomplete gamma function $\Gamma(\lambda+\frac{3}{2},y)$ mean

$$\lim_{y\to 0}\{\Phi(\lambda,y),V_1(x)\}=0\,,$$
$$\lim_{y\to\infty}\{\Phi(\lambda,y),V_1(x)/\hbar\omega\}=1\,. \tag{21}$$

We now show analytically that $\omega_c(E)$ is dependent on energy for all $a-\nu$ that satisfy Eq. (9). For large classical energy E, Eq. (21) shows that $V_1(x)$ is a vanishingly small perturbation on $V_0(x)$. Therefore,

$$\lim_{E\to\infty}\omega_c(E)=\omega\,. \tag{22}$$

For energies close to the minimum of the potential, V varies as (prime is d/dx)

$$\begin{aligned}V(x)&=V(x_m)+\tfrac{1}{2}V''(x_m)(x-x_m)^2+\ldots\\&=V_m+\tfrac{1}{2}m\omega_c^2(V_m)(x-x_m)^2+\ldots,\end{aligned} \tag{23}$$

where x_m is determined by the condition $V'(x_m)=0$.

With the aid of (subscript y means d/dy)

$$\Phi_y(\lambda,y)=\left(\frac{\lambda+\frac{1}{2}}{y}-1\right)\Phi(\lambda,y)+\Phi^2(\lambda,y)\,, \tag{24}$$

x_m or y_m is determined as the solution to

$$\nu^2=y_m^2[1+12\Phi_y(\lambda,y_m)+8y_m\Phi_{yy}^2(\lambda,y_m)]\,. \tag{25}$$

Putting (25) into (23)

$$\begin{aligned}\omega_c^2(V_m)=\omega^2[&1+12\Phi_y(\lambda,y_m)+18y_m\Phi_{yy}(\lambda,y_m)\\&+4y_m^2\Phi_{yyy}(\lambda,y_m)]\,.\end{aligned} \tag{26}$$

Using (24), Φ_{yy} and Φ_{yyy} can be calculated yielding powers of Φ up to Φ^4. The powers of Φ multiplied by powers of y in the square brackets of (26) cannot cancel each other for the continuous range of allowed λ (or ν). This shows that there are an

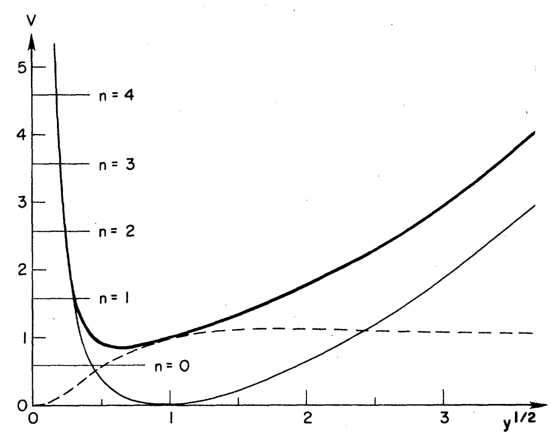

FIG. 1. For the special case $\lambda=\frac{1}{2}$, V_0, V_1, and V are plotted in units of $\hbar\omega$, as a function of $y^{1/2}$. V_0 is the thin continuous line, V_1 is the dashed line, and V is the thick line. The eigenenergies (n) are indicated. $n=0$ shows the ground-state energy $(1-3^{1/2}/4)$ for V_0, and $n=1$ shows the ground-state energy for V.

uncountable number of members of the set $C(\omega)$.

This result can be more easily understood by considering the special case $\lambda = \frac{1}{2}$ ($\nu = 3^{1/2}/2$). Then the incomplete gamma function is particularly simple and

$$\Phi(\tfrac{1}{2}, y) = \frac{y}{1+y}, \tag{27}$$

$$V = \tfrac{1}{4}\left(y - \sqrt{3} + \frac{3}{4y}\right) + \frac{3y + y^2}{(1+y)^2}. \tag{28}$$

In Fig. 1, V_0, V_1, and V are plotted in units of $\hbar\omega$ as a function of $y^{1/2}$. The ground state of V_0 is at $E_0 = (1 - 3^{1/2}/4) = 0.566\,99$. V_1 goes through unity at $y = 1$, reaches a slight maximum of $\frac{9}{8}$ at $y = 3$, and then asymptotically approaches unity as y goes to infinity. The minimum and location of V_m are

$$V_m = 0.829\,52, \quad y_m = 0.392\,87. \tag{29}$$

Since Eqs. (26) and (27) tell us that

$$\omega^2(V_m, \lambda = \tfrac{1}{2}) = \omega^2\left(1 + \frac{12(1 - y_m)}{(1 + y_m)^4}\right), \tag{30}$$

the combination of Eqs. (29) and (30) shows us that $\omega_c(V_m) \neq \omega$.

Now make a small, positive but finite change ϵ in the value of λ. Obviously we still have $\omega_c(V_m) \neq \omega$. Since there are an uncountable number of values λ between $\frac{1}{2}$ and $(\frac{1}{2} + \epsilon)$, we have shown the uncountability of potentials in the set $C(\omega)$. Note that $\omega_c(V_m) > \omega$ because V_1 makes the minimum of V narrower than that of V_0.

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[1]L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1960), pp. 25–29.

[2]We ignore trivial shifts of origin and are concerned only with different shapes of $V(x)$ as a function of x.

[3]M. M. Nieto and V. P. Gutschick, Phys. Rev. D 23, 922 (1981). The factor $\frac{1}{2}$ in Eq. (16) is for the harmonic oscillator but in general is a constant.

[4]P. B. Abraham and H. E. Moses, Phys. Rev. A 22, 1333 (1980).

[5]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1332 (1979); Y. Weissman and J. Jortner, Phys. Lett. 70A, 177 (1979); V. P. Gutschick, M. M. Nieto, and L. M. Simmons, Jr., *ibid.* 76A, 15 (1980). Also see the interesting related paper, V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica (Utrecht) 72, 597 (1974).

[6]G. Ghosh and R. W. Hasse, preceding paper, Phys. Rev. D 24, 1027 (1981).

[7]I. M. Gel'fand and B. M. Levitan, Izv. Akad. Nauk SSSR, Ser. Mat. 15, 309 (1951) [Am. Math. Soc. Transl. 1, 253 (1955)].

[8]As observed in Ref. 3, the AM procedure can be continued, removing each $n = 1, 2, 3, \ldots$ eigenstate in succession, to yield a countable number of potentials which are almost certainly members of the set $C(\omega)$.