PHYSICAL REVIEW D VOLUME 24, NUMBER 4 15 AUGUST 1981

# Inequivalence of the classes of quantum and classical harmonic potentials: Proof by example

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(Received 27 March 1981)

It has been proven by explicit construction that there exists a quantum harmonic potential (equally spaced energy levels) which does not belong to the class of classical harmonic potentials (frequency independent of energy). In this paper we prove the converse, also by example, that all classical harmonic potentials are not necessarily quantum harmonic potentials. Thus all classical harmonic potentials are not quantum harmonic potentials and vice versa.

## I. INTRODUCTION

As has been reported earlier[1] there exists an infinite number of potentials (called generalized classical harmonic potentials) where the classical frequency $\omega_c$ is independent of the energy. There also exists an infinite number of potentials which support equally spaced (level spacing $=\hbar\omega_q$) quantum eigenenergies. If $\omega_c=\omega_q$ then in the WKB approximation these two classes are equivalent. It has been shown by Nieto and Gutschick[1] that there exists at least one potential where the energy levels are equally spaced but the time period of a classical particle depends slightly on its energy. This is the potential, considered by Abraham and Moses,[2] which is obtained by removing the ground state from the spectrum of the simple harmonic oscillator through the Gel'fand-Levitan formalism.[3] The resulting spectrum is identical to the one for the harmonic oscillator except that the ground state is missing and the corresponding potential, although anharmonic, approaches the harmonic-oscillator potential for large displacements. But whether the converse is true, i.e., whether there exist classical harmonic potentials which are not quantum harmonic potentials, remains an open question. The aim of this paper is to answer this, also with an example, and thus complete the proof of inequivalence to the converse proposition.

## II. THE EXAMPLE

The simplest example of the stated behavior is the asymmetric matched harmonic oscillator

$$V(x)=\begin{cases}\frac{1}{2}m\omega_1^2x^2, & x\geq 0\,,\\ \frac{1}{2}m\omega_2^2x^2, & x\leq 0\,.\end{cases} \tag{1}$$

It is obvious that the time period of a classical particle in such a potential is independent of its energy. In fact, if $T_1$ and $T_2$ are the periods for the two spring constants $\omega_1$ and $\omega_2$, respectively, then the time period in the potential (1) is simply given by the arithmetic mean, i.e.,

$$T=\tfrac{1}{2}(T_1+T_2)\,,\qquad \omega=\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\,. \tag{2}$$

This classical frequency determines the WKB energy levels which are obtained from the Wilson-Sommerfeld quantization rule

$$\sqrt{2m}\int_{x_1}^{x_2}[E-V(x)]^{1/2}dx=(n+\tfrac{1}{2})\pi\hbar\,, \tag{3}$$

where $x_1$ and $x_2$ are the classical turning points determined from $E=\frac{1}{2}m\omega_1^2x_1^2=\frac{1}{2}m\omega_2^2x_2^2$. The solution of Eq. (3), as expected, is

$$E_n=(n+\tfrac{1}{2})\hbar\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\,. \tag{4}$$

The eigenfunctions of the Schrödinger equation for the potential (1) are parabolic cylinder functions and the normalized solutions with the correct asymptotic behavior are

$$\psi(x)=\begin{cases}N_1D_\nu(\xi_1), & x\geq 0\\ N_2D_\mu(-\xi_2), & x\leq 0\,,\end{cases}$$

where $\xi_{1,2}=(2m\omega_{1,2}/\hbar)^{1/2}x$ are dimensionless quantities, and the normalization constants are given by

$$N_1=2\left(\frac{m}{\pi\hbar}\right)^{1/4}\times\left(\frac{\Gamma(-\nu)}{\psi((1-\nu)/2)-\psi(-\nu/2)}\right)^{1/2}(1-\alpha^{-1/2})^{-1/2},$$

$$N_2=2\left(\frac{m}{\pi\hbar}\right)^{1/4}\times\left(\frac{\Gamma(-\mu)}{\psi((1-\mu)/2)-\psi(-\mu/2)}\right)^{1/2}(1-\alpha^{1/2})^{-1/2}\,.$$

Here, $\psi(x)$ is the logarithmic derivative of the gamma function. The quantum numbers $\nu$ and $\mu$ are, of course, related by the condition

$$E=(\nu+\tfrac{1}{2})\hbar\omega_1=(\mu+\tfrac{1}{2})\hbar\omega_2\,,$$
$$\mu=\alpha\nu+\frac{\alpha-1}{2}\,, \qquad (5)$$
$$\alpha=\omega_1/\omega_2\,.$$

Since the potential is asymmetric, the solutions do not have good parity and $\nu$ will not be an integer. The matching conditions for the wave function and its derivative at $x=0$ yield

$$\sqrt{\omega_1}\,\frac{D_\nu'(0)}{D_\nu(0)}=-\sqrt{\omega_2}\,\frac{D_\mu'(0)}{D_\mu(0)}\,, \qquad (6)$$

which reduces to[4]

$$\sqrt{\alpha}\,\frac{\Gamma((1-\nu)/2)\Gamma(-\mu/2)}{\Gamma(-\nu/2)\Gamma((1-\mu)/2)}+1=0\,. \qquad (7)$$

Equation (7) can be solved numerically to find the quantum numbers $\nu$, $\mu$ as a function of $\alpha$, see Fig. 1. As expected for $\alpha=1$, the quantum numbers become integers. For $\alpha>1$ the quantum numbers are simply obtained by the replacement $\alpha\to\alpha^{-1}$ and $\nu\rightleftarrows\mu$. As $\alpha$ deviates more and more from unity, the levels become more unequally

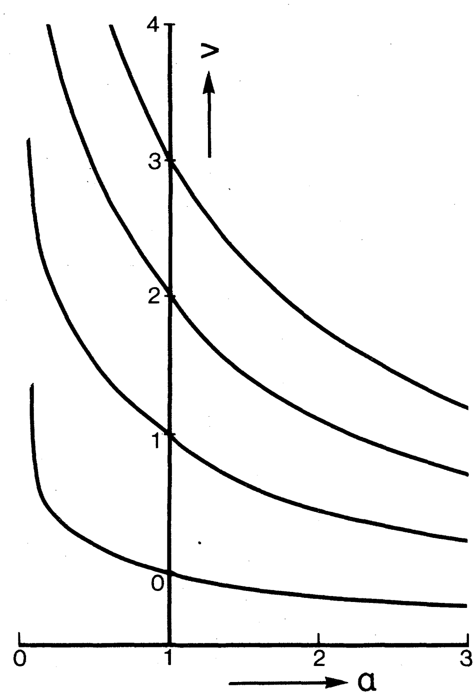


FIG. 1. Quantum number $\nu$ plotted as a function of $\alpha$. For $\alpha\to\infty$, $\nu\to-\frac{1}{2}$, and for $\alpha\to 0$, $\nu\to\infty$. The intersections of a vertical line, at a particular value of $\alpha$, with the different curves give the quantum numbers of the system at that value of $\alpha$.

TABLE I. Quantum numbers $\nu$ and quantum number differences $\Delta\nu$ for two values of $\alpha$.

| $\alpha=0.5$ $\nu$ | $\Delta\nu$ | $\alpha=0.8$ $\nu$ | $\Delta\nu$ |
|---|---|---|---|
| 0.1787 | | 0.0567 | |
| | 1.3147 | | 1.1092 |
| 1.4934 | | 1.1659 | |
| | 1.3414 | | 1.1125 |
| 2.8348 | | 2.2784 | |
| | 1.3327 | | 1.1102 |
| 4.1675 | | 3.3886 | |
| | 1.3315 | | 1.1117 |
| 5.4990 | | 4.5003 | |
| | 1.3347 | | 1.1107 |
| 6.8337 | | 5.6110 | |
| | 1.3337 | | 1.1114 |
| 8.1670 | | 6.7224 | |
| | 1.3327 | | 1.1109 |
| 9.4997 | | 7.8333 | |
| | 1.3328 | | 1.1112 |
| 10.8335 | | 8.9443 | |
| | 1.3333 | | 1.1111 |
| 12.1668 | | 10.0556 | |

spaced for small quantum numbers but the spacing approaches the WKB value for large quantum numbers, see Table I. To show this analytically we convert the $\Gamma$ functions with negative arguments in Eq. (7) to those with positive arguments and use the asymptotic form of these functions for large arguments[5] to arrive at

$$\frac{\sin[\frac{1}{2}\pi(\nu-\mu)]+\sin[\frac{1}{2}\pi(\nu+\mu)]}{\sin[\frac{1}{2}\pi(\nu-\mu)]-\sin[\frac{1}{2}\pi(\nu+\mu)]}=1\,, \qquad (8)$$

which is satisfied whenever $(\nu+\mu)$ is an even integer. Using Eq. (5) we obtain, for large $\nu$ and $\mu$,

$$\nu\to\frac{2n}{1+\alpha}\,,$$

where $n$ is an integer.

The level spacing for large quantum numbers comes out to be

$$\Delta E=\hbar\omega_1\Delta\nu\to\hbar\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\,,$$

which is precisely the WKB value.

In conclusion, we want to point out that the simple example provided here completes the proof of the inequivalence of the classes of classical and quantum harmonic potentials. We see that although the two sets overlap, neither is contained within the other. After submission of the manuscript the authors were informed that one can also obtain nonequal level spacing in half an oscillator potential by imposing nonstandard boundary conditions at the origin.[6]

One of us (G.G.) would like to thank Professor Y. Gambhir and H. Wolter for helpful discussions on this subject. He also thanks the Alexander von Humboldt foundation for a fellowship. The other (R.W.H.) acknowledges fruitful discussions with Professor M.M. Nieto and Professor G. Süssmann.

[1]M. M. Nieto and V. P. Gutschick, Phys. Rev. D 23, 922 (1981).
[2]P. B. Abraham and M. E. Moses, Phys. Rev. A 22, 1333 (1980).
[3]I. M. Gel'fand and B. M. Levitan, Izv. Akad. Nauk SSSR Ser. Mat. 15, 309 (1951) [Am. Math. Soc. Transl. 1, 253 (1955)].
[4]W. Magnus, F. Oberhettinger, and R. P. Soni, *Special Functions of Mathematical Physics* (Springer, Berlin, 1966), p. 329.
[5]Reference 4, p. 12.
[6]T. E. Clark, R. Menikoff, and D. H. Sharp (unpublished).