Brief Reports

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Note on the Schwinger-Goto-Imamura term and the Jacobi identity

Yasushi Takahashi
Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2J1
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It is pointed out that the presence of the Schwinger-Goto-Imamura term destroys the Jacobi identities of currents. It is shown that in order to save the Jacobi identity, a new c-number term must be accounted for in double commutators of currents.

According to a naive canonical calculation, the equal-time commutator between the time and space components of the vector current

$$J_\mu(x) = i\bar{\psi}(x)\gamma_\mu\psi(x) \tag{1}$$

vanishes, i.e.,

$$[J_0(x), J_i(x')] = 0\,. \tag{2}$$

However, the vanishing of the commutator contradicts the positivity of the energy eigenvalues as has been pointed out by several authors,[1] and the careful calculation by, for example, the point-separation technique[2] gives, instead of (2),

$$[J_0(x), J_i(x')] = -iK^2\partial_i\delta(\vec{x}-\vec{x}')\,, \tag{3}$$

where K^2 is a diverging c number. The effect of the point separation appears only in the commutators of vector (or axial vector) current. We thus obtain, for example, the set of equations

$$[J_0(x), J_0(x')] = 0\,, \tag{4a}$$

$$[J_0(x), J_i(x')] = -iK^2\partial_i\delta(\vec{x}-\vec{x}')\,, \tag{4b}$$

$$[J_i(x), J_j(x')] = 2i\epsilon_{ijk}J_{5k}(x)\delta(\vec{x}-\vec{x}')\,, \tag{4c}$$

$$[J_0(x), J_{5j}(x')] = 0\,, \tag{4d}$$

$$[J_i(x), J_{5j}(x')] = 2i\epsilon_{ijk}J_k(x)\delta(\vec{x}-\vec{x}')\,, \tag{4e}$$

$$[J_{5i}(x), J_{5j}(x')] = 2i\epsilon_{ijk}J_{5k}(x)\delta(\vec{x}-\vec{x}')\,, \tag{4f}$$

where

$$J_{5\mu}(x) = i\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)\,. \tag{5}$$

The above set of equations then gives

$$[[J_0(x), J_i(x')], J_{5j}(x'')] = 0\,, \tag{6a}$$

$$[[J_{5j}(x''), J_0(x)], J_i(x')] = 0\,, \tag{6b}$$

$$[[J_i(x'), J_{5j}(x'')], J_0(x)] = -2K^2\epsilon_{ijk}\partial_k\delta(\vec{x}-\vec{x}')\,. \tag{6c}$$

Hence, the Jacobi identity between J_0, J_i, and J_{5j} is destroyed. We are thus confronted with a mathematical difficulty forced by the physical requirement that the energy should be positive. As has been said, the Schwinger-Goto-Imamura term (3) is required by the positivity of the energy, but also it has been speculated that the very presence of the term may play a vital role in the treatment of the U(1) problem.[3]

In view of the above situation, the double commutators (6) are recalculated using the point separation for x' and x''. Limiting ourselves to the spacelike point separation, we obtain

$$[[J_0(x), J_i(x',\epsilon')], J_{5j}(x'',\epsilon'')] = -[B_{ij}^{(+)}\delta(\vec{x}'_- - \vec{x}''_+) + B_{ij}^{(-)}\delta(\vec{x}'_+ - \vec{x}''_-)][\delta(\vec{x}-\vec{x}'_+) - \delta(\vec{x}-\vec{x}'_-)]\,, \tag{7a}$$

$$[[J_{5j}(x'',\epsilon''), J_0(x)], J_i(x',\epsilon')] = [B_{ij}^{(+)}\delta(\vec{x}'_- - \vec{x}''_+) + B_{ij}^{(-)}\delta(\vec{x}'_+ - \vec{x}''_-)][\delta(\vec{x}-\vec{x}''_-) - \delta(\vec{x}-\vec{x}''_+)]\,, \tag{7b}$$

$$[[J_i(x',\epsilon'), J_{5j}(x'',\epsilon'')], J_0(x)]$$
$$= -B_{ij}^{(+)}\delta(x'_- - x''_+)[\delta(x - x''_-) - \delta(x - x'_+)] - B_{ij}^{(-)}\delta(x'_+ - x''_-)[\delta(x - x'_-) - \delta(x - x''_+)]\,, \tag{7c}$$

with

$$\vec{x}'_{\pm} = \vec{x}' \pm \tfrac{1}{2}\vec{\epsilon}', \quad \vec{x}''_{\pm} = \vec{x}'' \pm \tfrac{1}{2}\vec{\epsilon}'', \tag{8}$$

$$B_{ij}^{(+)} = \epsilon_{ijk}\bar{\psi}(x'_+)\gamma_k\psi(x''_-) + \delta_{ij}\bar{\psi}(x'_+)\gamma_4\gamma_5\psi(x''_-), \tag{9a}$$

$$B_{ij}^{(-)} = \epsilon_{ijk}\bar{\psi}(x''_+)\gamma_k\psi(x'_-) - \delta_{ij}\bar{\psi}(x''_+)\gamma_4\gamma_5\psi(x'_-), \tag{9b}$$

$$J_\mu(x',\epsilon') \equiv i\bar{\psi}(x'_+)\gamma_\mu\psi(x'_-), \tag{10a}$$

$$J_{5\mu}(x'',\epsilon'') \equiv i\bar{\psi}(x''_+)\gamma_5\gamma_\mu\psi(x''_-). \tag{10b}$$

If we add all three equations (7), we find that the result vanishes before the limit is taken. Namely, the Jacobi identity holds for arbitrary $\vec{\epsilon}'$ and $\vec{\epsilon}''$. In order for the point separation to be effective, it should be assumed that $\vec{\epsilon}' + \vec{\epsilon}'' \neq 0$. The actual values of the double commutators (7) depend on how $\vec{\epsilon}'$ and $\vec{\epsilon}''$ approach zero, though as Eqs. (7) imply, the Jacobi identity holds true for arbitrary $\vec{\epsilon}'$ and $\vec{\epsilon}''$. Having confirmed this, we illustrate the following two cases.

(i) If we take the limit in the order of $\vec{\epsilon}' \to 0$ and $\vec{\epsilon}'' \to 0$, we obtain

$$[[J_0(x), J_i(x')], J_{5j}(x'')] = 0, \tag{11a}$$

$$[[J_{5j}(x''), J_0(x)], J_i(x')] = 2K''^2\epsilon_{ijk}\partial_k\delta(\vec{x}-\vec{x}')\delta(\vec{x}'-\vec{x}''), \tag{11b}$$

$$[[J_i(x'), J_{5j}(x'')], J_0(x)] = -2K''^2\epsilon_{ijk}\partial_k\delta(\vec{x}-\vec{x}')\delta(\vec{x}'-\vec{x}''), \tag{11c}$$

instead of Eqs. (6), where $K''^2 \equiv 2/3\pi^2\epsilon''^2$.

(ii) On the other hand, if we take the limit $\vec{\epsilon}'' \to 0$ first and then take $\vec{\epsilon}' \to 0$, we obtain instead of (11)

$$[[J_0(x), J_i(x')], J_{5j}(x'')] = 2K'^2\epsilon_{ijk}\partial_k\delta(\vec{x}-\vec{x}')\delta(\vec{x}'-\vec{x}''), \tag{12a}$$

$$[[J_{5j}(x''), J_0(x)], J_i(x')] = 0, \tag{12b}$$

$$[[J_i(x'), J_{5j}(x'')], J_0(x)] = -2K'^2\epsilon_{ijk}\partial_k\delta(\vec{x}-\vec{x}')\delta(\vec{x}'-\vec{x}''), \tag{12c}$$

with $K'^2 = 2/3\pi^2\epsilon'^2$.

The lesson we have learned from the above exercise is that Eq. (4) cannot be used naively, but if the point-separation technique is used again in calculating the double commutators, the result differs from that obtained from (4), depending on ways in which the limits $\vec{\epsilon}' \to 0$ and $\vec{\epsilon}'' \to 0$ are taken, and, nevertheless, the Jacobi identity holds true.

We have here used spacelike point separation in order to avoid the problem of dynamics. It is interesting to see how the dynamics affects the double commutators.[4]

We point out finally that the above problem does not simply arise in field theory in 1+1 dimensions, because the space component of the current commutes with itself.

Note added in proof. The author thanks Professor Johnson for letting him know of the paper by K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. 37–38 (1966), where the failure of the Jacobi identity in the presence of the Schwinger-Goto-Imamura term was noted.

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