Chiral chains for lattice quantum chromodynamics at $N_c = \infty$

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We study chiral fields [U_l in the group U(N)] on a periodic lattice ($U_l = U_{l+L}$), with action $S = (1/g^2)\Sigma_{l=1}^{L}\mathrm{Tr}(U_l U_{l+1}^\dagger + U_l^\dagger U_{l+1})$, as prototypes for lattice gauge theories [quantum chromodynamics (QCD)] at $N_c = \infty$. Indeed, these chiral chains are equivalent to gauge theories on the surface of an L-faced polyhedron (e.g., $L = 4$ is a tetrahedron, $L = 6$ is the cube, and $L = \infty$ is two-dimensional QCD). The one-link Schwinger-Dyson equation of Brower and Nauenberg, which gives the square of the transfer matrix, is solved exactly for all N. From the large-N solution, we solve exactly the finite chains for $L = 2$, 3, 4, and ∞, on the weak-coupling side of the Gross-Witten singularity, which occurs at $\beta = (g^2N)^{-1} = 1/4$, 1/3, $\pi/8$, and 1/2, respectively. We carry out weak and strong perturbation expansions at $N_c = \infty$ to estimate the singular part for all L, and to show confinement (as $g^2N \to \infty$) and asymptotic freedom ($g^2N \to 0$) in the Migdal β function for QCD. The stability of the location of the Gross-Witten singularity for different-size lattices (L) suggests that QCD at $N_c = \infty$ enjoys this singularity in the transition region from strong to weak coupling.

I. INTRODUCTION

Although the $1/N$ expansion to quantum chromodynamics (QCD) (Ref. 1) is generally acknowledged to be a sensible approximation scheme, progress toward calculating even the first ($N_c = \infty$) term has been confined to severely truncated (or toy) models. Nonetheless, these warm-up exercises are beginning to provide both the arsenal of $1/N$ techniques and a little insight into the special properties of $N_c = \infty$ QCD. Here we wish to consider a class of finite-lattice chiral models, which we argue are the basic ingredient of either the Migdal recursion relations or Kadanoff's blocking transformation.[2,3] Thus, we hope to improve the $1/N$ technology in a direction that will allow at least approximate lattice QCD calculations at $N_c = \infty$.

The basic idea inherent in the Migdal recursion relation is to move bonds (variationally) so the gauge theory in dimension $d = 4$ (or $d = 2$) becomes equivalent to the $d = 2$ (or $d = 1$) chiral model,

$$Z = \int \prod_i dU_i \exp\left[N\beta \sum_{\langle ij\rangle} \mathrm{Tr}(U_i U_j^\dagger + U_j U_i^\dagger)\right]. \tag{1.1}$$

$\beta \equiv 1/\lambda \equiv 1/Ng^2$, where i,j labels the sites, U_i is an $N \times N$ matrix of the adjoint representation for U(N), and dU_i is the Haar measure. The action is the sum over nearest neighbors ($\langle ij\rangle$ = links). Indeed, for $d = 2$ gauge theories (QCD$_2$), this correspondence with the chiral spin model ($d = 1$) is an exact consequence of gauge invariance. Going to the $A_0 = 0$ gauge, QCD$_2$ is equivalent to a product of K chiral chains of length L,

$$Z = \int \prod_{i=1}^{L} dU_i \exp\left[N\beta \sum_{i=1}^{L} \mathrm{Tr}(U_i U_{i+1}^\dagger + U_{i+1} U_i^\dagger)\right], \tag{1.2}$$

for a $K \times L$ rectangular lattice. The study of these chiral chains is the central focus for this paper.

If we impose periodic boundary conditions, $U_{L+1} = U_1$, these chiral chains are amusing in their own right, since for $L = 4, 6, 8, 12, 20$ they are equivalent to lattice gauge theories on the surface of regular polyhedra (tetrahedron, cube, octahedron, etc.). These small *three*-dimensional lattice configurations can serve as basic blocks in Kadanoff's blocking transformation, thus leading to a starting point for an approximate real-space renormalization-group calculation. This equivalence can be established by a particular gauge choice and is shown in Appendix A.

The case of a chiral chain with an infinite length ($L \to \infty$) has been examined by Gross and Witten.[4] They found that at $N_c = \infty$ the free energy was piecewise analytic with a weak (third-order) transition at $\beta = \beta_c = \frac{1}{2}$ between the strong-coupling ($\beta \to 0$) and the weak-coupling ($\beta \to \infty$) domains. We show that, for $L = 2$, 3, and 4, there is a critical point at $\beta_c = \frac{1}{4}$, $\frac{1}{3}$, and $\pi/8$, respectively. ($L = 2$ again corresponds to a Gross-Witten model with β replaced by 2β. Our $L = 3$ result agrees with that of Friedan,[5] obtained from the strong-coupling domain.) These new results are obtained by making use of the single-link integral of Brower and Nauenberg.[6]

A chiral chain is specified by its periodicity L

and its transfer matrix T whose matrix elements are

$$\langle U_2 | T | U_1 \rangle = \exp[N\beta \operatorname{Tr}(U_1 U_2^\dagger + U_2 U_1^\dagger)] . \tag{1.3}$$

Before proceeding to the details of our calculations, we wish to show how the transfer matrix is related to the more general problem of Migdal's recursion relations. A Migdal approximation typically involves two operations: (1) bond moving and (2) decimation. For example, in one variation of this approach we might consider a square lattice first grouped into $\lambda \times \lambda$ blocks (λ an integer). Operation (1) moves all the interior bonds to the block boundary (see Fig. 1, for $\lambda = 2$), which amounts to changing β to $\lambda\beta$, and operation (2) integrates out these sites on the boundary as indicated by a cross in Fig. 1. As explained by Kadanoff, these two operations can be considered as multiplications in two different spaces of coupling parameters related by duality. That is, for the matrix elements given by (1.3), bond moving corresponds to

$$\langle U_2 | T | U_1 \rangle \to (\langle U_2 | T | U_1 \rangle)^\lambda , \tag{1.4}$$

whereas decimation corresponds to

$$\langle U_2 | T | U_1 \rangle \to \langle U_2 | T^\lambda | U_1 \rangle . \tag{1.5}$$

This latter step involves the product of the transfer matrix T, integrated over the internal (crossed) sites. For example, T^2 is calculated by

$$\begin{aligned}\langle U_2 | T^2 | U_1 \rangle &= \int dU \langle U_2 | T | U \rangle \langle U | T | U_1 \rangle \\ &= \int dU \exp[N\beta \operatorname{Tr}(A^\dagger U + U^\dagger A)] ,\end{aligned} \tag{1.6}$$

where $A = U_1 + U_2$. Indeed, this last integral is precisely the single-link integral solved (at $N_c = \infty$) by Brower and Nauenberg by the use of Schwinger-Dyson equations. In Sec. II, we return to this problem for the group U(N) and find an exact solution to T^2 for all N.

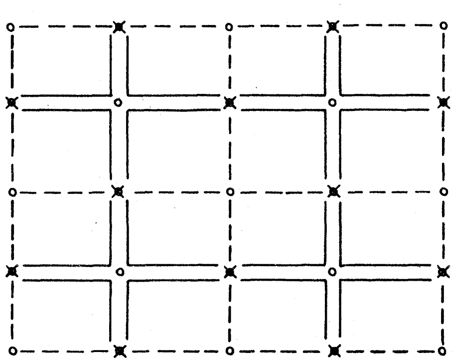

FIG. 1. A simple bond-moving scheme for Migdal's recursion relation of the $d = 2$ chiral model. Dotted bonds have been moved to double the strength on double-line bonds.

The essential ingredient of a Migdal approximation is to alternate computing T^λ with $(\langle U_2 | T | U_1 \rangle)^\lambda$. In Kadanoff's treatment for finite N, this is facilitated by the character expansion for

$$\langle U_2 | T | U_1 \rangle = \sum_\nu \lambda_\nu d_\nu \chi_\nu (U_2^\dagger U_1) , \tag{1.7}$$

where the sum over the primitive characters χ_ν completely diagonalizes the transfer matrix. In (1.7), λ_ν are eigenvalues of T and the characters are normalized by

$$\int dU \, d_{\nu_2} \chi_{\nu_2} (U_2^\dagger U) d_{\nu_1} \chi_{\nu_1} (U^\dagger U_1) = \delta_{\nu_1 \nu_2} d_{\nu_1} \chi_{\nu_1} (U_2^\dagger U_1) . \tag{1.8}$$

It thus follows that

$$\langle U_2 | T^2 | U_1 \rangle = \sum_\nu \lambda_\nu{}^2 d_\nu \chi_\nu (U_2^\dagger U_1) . \tag{1.9}$$

For $N \to \infty$, this approach is not very promising since the νth term $\sim (\beta N^2)^{r_\nu}$, where r_ν is the rank of the tensor product which defines the νth irreducible representation. Since $\lambda_\nu \sim (\beta)^{r_\nu}$, the convergence of the series improves rapidly as $\beta \to 0$ with N fixed, but deteriorates for $N \to \infty$ at fixed β. From the arguments of Brower and Nauenberg, we know that the actual limit is

$$\langle U_2 | T^\lambda | U_1 \rangle = \exp[N^2 f_\lambda (U_1 U_2^\dagger + U_2 U_1^\dagger)] , \tag{1.10}$$

where $f_\lambda \sim O(1)$ and $f_1 = \beta(1/N) \operatorname{Tr}(U_1 U_2^\dagger + U_2 U_1^\dagger)$.

The full Migdal recursion relation at $N_c = \infty$ is therefore still more difficult than our exactly soluble examples. However, we can see some qualitative properties by looking at strong- and weak-coupling expansions. As $g^2 N \to \infty$, we recall from Ref. 6, that $f_2 \simeq (1/N)(1/g^2 N)^2 \operatorname{Tr}[(U_1^\dagger + U_2^\dagger) \times (U_1 + U_2)]$. Comparing with f_1, this corresponds to, under a decimation, $1/g^2 N \to (1/g^2 N)^2$. We can generalize this to show that a λ-fold decimation ($f_1 \to f_\lambda$) gives

$$(1/g^2 N) \to (1/g^2 N)^\lambda .$$

Now by alternating bond moving and decimation in the manner described in Ref. 3 for a d-dimensional chiral-spin model or $2d$-dimensional gauge model, we obtain

$$(1/g^2 N)' = \lambda^{(d-n)} (1/g^2 N)^{\lambda^n} , \tag{1.11}$$

where $n = 1$ and 2 for spin and gauge models, respectively. The differential β function for $\lambda = 1 + \epsilon$, $\beta(g) = -\Delta g/\epsilon$ is then given by

$$-\beta(g)/g = \frac{n}{2} [\ln(g^2 N) - 1] + \frac{2n - d}{2} \tag{1.12}$$

for $g^2N \to \infty$. This exhibits the expected confinement phase for large coupling.

The weak-coupling expansion can be done by using the calculation of T^λ in Sec. IV for $g^2N \to 0$. The salient feature is that the one-loop piece $\mathrm{Tr}(\ln D)$ is essential at the critical dimension $d = 2N$ to give the right sign for asymptotic freedom,

$$-\beta(g)/g = \frac{(2n-d)}{2} + \frac{n}{8}(g^2N) . \tag{1.13}$$

In our calculation, we have arbitrarily truncated the action to $(1/g^2)' \mathrm{Tr}(U_p + U_p^\dagger)$ after each iteration. As pointed out by Kadanoff,[3,7] a proper calculation would keep higher terms, $\mathrm{Tr}(U_p^2)$, etc., which is equivalent to truncation to the quadratic terms in the field strenghts $(U_p + U_p^\dagger \simeq 2 - F^2)$. This procedure gives a coefficient of $\frac{1}{24}$ [instead of $n/8 = \frac{1}{4}$ in Eq. (13)] in much better accord with the exact result of $11/48\pi^2$.

In the present context, we use this Migdal approximation as merely an illustration of how our results on chiral chains might be extended to QCD_4 at $N_c = \infty$. Our main objective which we return to in the Conclusion is to make it plausible that QCD_4 enjoys the same third-order crossover singularity encountered in our finite-lattice examples.

II. SINGLE-LINK INTEGRAL REVISITED

The representation, Eq. (1.6), for the square of the transfer matrix belongs to the general class of single-link integrals

$$Z_0(A^\dagger A, \beta) = \int dU \exp[N\beta\, \mathrm{Tr}(A^\dagger U + AU^\dagger)], \tag{2.1}$$

where U is an element of the group $\mathrm{U}(N)$ and A is an arbitrary $N \times N$ matrix. This integral was evaluated at $N_c = \infty$ for the weak-coupling phase in Ref. 6,

$$f_2 = \frac{1}{N^2} \ln Z_0 \to 2\beta \frac{1}{N} \sum_j \sqrt{x_j} - \frac{1}{2N^2} \sum_{i,j} \ln\beta(\sqrt{x_i} + \sqrt{x_j}) - \tfrac{3}{4} , \tag{2.2}$$

where x_j are eigenvalues of the Hermitian matrix $AA^\dagger$. This result will be used repeatedly later. In this section, we review and generalize the result of Ref. 6 to the case where both N and β can be arbitrary.

The $\mathrm{U}(N)$ invariance of the Haar measure implies that the one-link integral, (2.1), depends only on the eigenvalues of the Hermitian matrix $AA^\dagger$ and that the function $Z_0(x_i, \beta)$ satisfies a Schwinger-Dyson equation. Restricting to the $\mathrm{U}(N)$ singlet subspace, the Schwinger-Dyson equation was shown to be equivalent to a partial differential equation,

$$\frac{1}{N}\sum_k x_k \frac{\partial Z_0}{\partial x_k} + \frac{1}{N^2}\sum_k x_k^2 \frac{\partial^2 Z_0}{\partial x_k^2} + \frac{1}{N^2}\sum_{s\neq k} \frac{x_k x_s}{x_k - x_s}\left(\frac{\partial Z_0}{\partial x_k} - \frac{\partial Z_0}{\partial x_s}\right) = \beta^2 \sum_k x_k Z_0 , \tag{2.3}$$

with the boundary condition $Z_0(x_i, 0) = 1$ and it is completely symmetric in x_k. Equation (2.3), at $N_c \to \infty$, can be either solved perturbatively in β (strong-coupling expansion) or solved exactly, leading to (2.2), valid in the weak-coupling region.

Let us define a new function by multiplying Z_0 by the Vandemonde determinant

$$W_0(x_k, \beta) \equiv Z_0(x_k, \beta)\left[\prod_{i<j} (x_i - x_j)\right] \tag{2.4}$$

so that W_0 is completely antisymmetric in x_k. As shown in Appendix B, this leads to a partial differential equation

$$\left[\sum_k z_k^2 \frac{\partial^2}{\partial z_k^2} + (3-2N)\sum_k z_k \frac{\partial}{\partial z_k} - \sum_k z_k^2 + \tfrac{2}{3}N(N-1)(N-2)\right] W_0 = 0 , \tag{2.5}$$

where $z_k \equiv 2N\beta\sqrt{x_k}$. For $N = 1$, this corresponds to the Bessel's equation; for general N, the completely antisymmetric solution is given by (see Appendix B)

$$W_0 = (\mathrm{const})\, \det[z_j^{\,i-1} I_{i-1}(z_j)] , \tag{2.6}$$

where $I_i(z)$ is the Bessel function. By imposing the boundary condition at $\beta = 0$, we obtain the exact result valid for all N

$$Z_0 = \left\{(2^{N(N-1)/2}\left[\prod_{k=0}^{N-1} (k!)\right]\right\} \frac{\det(z_j^{\,i-1} I_{i-1}(z_j))}{\det((z_j^{\,2})^{i-1})} . \tag{2.7}$$

The structure of Eq. (2.7) is illustrated in Appendix B by considering various special choices of the matrix A where the single-link integral is previously known.[8] To obtain the large-N limit, we can rewrite (2.7) as

$$Z_0 = \frac{2^{N(N-1)/2}(\prod_{k=0}^{N-1} k!)}{\prod_{i<j}(z_i + z_j)} \frac{\det(z_j^{\,i-1} I_{i-1}(z_j))}{\det(z_j^{\,i-1})} . \tag{2.8}$$

Since $z_i = O(N)$ we can make use of the asymptotic behavior

$$I_k(z) \simeq \frac{e^z}{(2\pi z)^{1/2}}[1 + O(1/z)] \tag{2.9}$$

to obtain

$$\det(z_j^{\,i-1} I_{i-1}(z_j)) \simeq \exp\left[\sum_j z_j - \tfrac{1}{2}\sum_j \ln(2\pi z_j)\right] \det(z_j^{\,i-1}) . \tag{2.10}$$

It then follows that

$$f_2 = \frac{1}{N^2}\ln Z_0 - \frac{1}{N^2}\sum_j z_j - \frac{1}{2N^2}\sum_{i,j}\ln\left(\frac{z_i+z_j}{2N}\right) - \frac{3}{4}\,. \tag{2.11}$$

If one next substitutes $2\beta N\sqrt{x_i}$ for z_i, Eq. (2.2) is then obtained. Since (2.9) has been used, (2.2) is the exact $N_c=\infty$ solution only for the weak-coupling phase.

We are hopeful that the integral for SU(N) and the $N_c=\infty$ solution for the strong-coupling side may also be solved by exploiting the fermionic character of the Schwinger-Dyson equation.

III. EXACT RESULTS FOR CHIRAL CHAINS

In this section, we present exact $N_c=\infty$ solutions for chiral chains with periods $L=2$, 3, and 4 (Fig. 2). Our strategy is to first perform group integrations in (1.2) with the help of the single-link integral result for all U_i except two, leading to a representation for Z_L in the form

$$Z_L = \int dU\,dV \exp[N^2 S_{\rm eff}^{(L)}(UV^\dagger)] \tag{3.1}$$

suitable for a large-N steepest-descent analysis. In particular, since the integral depends only on the combination $UV^\dagger$, we are able to change variable to θ_j, $e^{i\theta_j}$ being eigenvalues of $UV^\dagger$,

$$Z_L = (\text{const})\int\prod d\theta_i \Delta^2(\theta_j)\exp[N^2 S_{\rm eff}^{(L)}(\theta_k)]\,, \tag{3.2}$$

where $-\pi\le\theta_j\le\pi$, $\Delta=\det(\Delta_{jk})$, $\Delta_{jh}=\exp[i(j\theta_k)]$. In the limit $N\to\infty$, Z_L is dominated by a stationary configuration, with θ_j distribution specified by a density function $\rho_L(\theta)$, which is given by the solution of

$$P\int d\phi\,\rho_L(\phi)\cot\left(\frac{\theta-\varphi}{2}\right) + \frac{\delta}{\delta\theta}S_{\rm eff}^{(L)}(\theta,\rho_L) = 0\,. \tag{3.3}$$

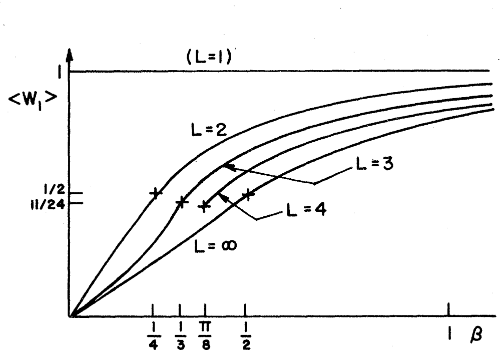

FIG. 2. The behavior of the single-plaquette expectation value as a function of β and L. At the critical point (marked by a cross), $\partial\langle w_1\rangle/\partial\beta$ is still continuous, but the second derivative shows a discontinuity.

A. Two-link chiral chain

For $L=2$, Eq. (1.2) is already in the desired form with $S_{\rm eff}^{(2)}(UV^\dagger) = (2\beta)(1/N)\,\mathrm{Tr}(UV^\dagger+VU^\dagger)$, i.e., the large-$N$ density $\rho_2(\theta)$ is given by the solution of the Gross-Witten equation

$$P\int d\phi\,\rho(\phi)\cot\left(\frac{\theta-\varphi}{2}\right) - 4\beta\sin\theta = 0\,, \tag{3.4}$$

which differs from the infinite-chain model only in replacing β by 2β. It then follows that a third-order phase transition exists between the strong- and weak-coupling domains at $\beta_c=\frac{1}{4}$. For $\beta>\frac{1}{4}$, the density function is

$$\rho_2(\theta) = \frac{4\beta}{\pi}\cos\frac{\theta}{2}\left(\frac{1}{4\beta}-\sin^2\frac{\theta}{2}\right)^{1/2}, \tag{3.5}$$

with ρ_2 nonvanishing only in the range $|\theta|\le\theta_c \equiv 2\sin^{-1}(\frac{1}{4}\beta)\le\pi$, and satisfying the normalization condition

$$\int_{-\pi}^{\pi}\rho_L(\theta)\,d\theta = 1\,. \tag{3.6}$$

B. Three-link chiral chain

Choosing $U=U_1, V=U_2$, $S_{\rm eff}^{(3)}$ is then given by

$$e^{N^2 S_{\rm eff}^{(3)}} = e^{N\beta\mathrm{Tr}(UV^\dagger+VU^\dagger)}\int dU_3\, e^{N\beta\mathrm{Tr}(A^\dagger U_3+U_3^\dagger A)}\,, \tag{3.7}$$

where $A=U+V$. Since the integral (3.7) is precisely a single-link integral, it follows from (2.8) that it is a function of $AA^\dagger = 2+UV^\dagger+VU^\dagger$ only. In the limit $N\to\infty$, we substitute the limit (2.2) for $S_{\rm eff}^{(3)}$ into (3.3), obtaining the equation for the spectral density $\rho_3(\theta)$,

$$2\beta\left(\sin\theta+\sin\frac{\theta}{2}\right) - P\int_{-\theta_c}^{\theta_c} d\phi\,\rho(\phi)\left[\cot\left(\frac{\theta-\phi}{2}\right)+\frac{1}{2}\,\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\cos\frac{\phi}{2}}\right], \tag{3.8}$$

where $0\le\theta_c\le\pi$, and it is to be determined by the normalization condition (3.6).

A systematic procedure for solving (3.8) is described in Appendix C. Here we simply state that the solution is

$$\rho_3(\theta) = \frac{\beta}{\pi}\cos\frac{\theta}{4}\left[2\cos\frac{\theta}{2}+\left(1-\frac{1}{3\beta}\right)^{1/2}\right] \times\left\{2\left[\cos\frac{\theta}{2}-\left(1-\frac{1}{3\beta}\right)^{1/2}\right]\right\}^{1/2}, \tag{3.9}$$

with $|\theta|\le\theta_c = 2\cos^{-1}(1-1/3\beta)^{1/2}$. This result is valid in the region $\beta_c\ge\equiv\frac{1}{3}$, where $\rho_3(\theta)$ is manifestly non-negative. In view of the result previously obtained by Friedan[5] for the strong-coupling domain, it follows that a critical point exists at $\beta_c = \frac{1}{3}$. Our result for $\rho_3(\theta)$ agrees with that of Friedan at $\beta = \beta_c$ and we shall shortly demonstrate that this again corresponds to a third-order phase transition.

C. Four-link chiral chain

With $L=4$, we choose $U=U_1$, $V=U_3$; $S_{\rm eff}^{(4)}$ is then given by

$$e^{[N^2 S_{\rm eff}^{(4)}]} = \left[\int dU_2 e^{N\beta \operatorname{Tr}(A^\dagger U_2 + U_2^\dagger A)}\right] \times \left[\int dU_4 e^{N\beta \operatorname{Tr}(A^\dagger U_4 + U_4^\dagger A)}\right], \tag{3.10}$$

where again $A = U+V$. It follows from (2.2), (3.3), and (3.14) that the spectral density $\rho_4(\theta)$ is determined by

$$4\beta \sin\frac{\theta}{2} - P\int_{-\theta_c}^{\theta_c} d\phi\, \rho(\phi)\left[\cot\left(\frac{\theta-\phi}{2}\right) + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\cos\frac{\phi}{2}}\right]. \tag{3.11}$$

The solution is (see Appendix C for the details)

$$\rho_4(\theta) = \frac{2\beta}{\pi}\left(\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}, \tag{3.12}$$

with $0 \leq \theta_c \leq \pi$ determined by the normalization condition

$$\frac{2\beta}{\pi}\int_{-\theta_c}^{\theta_c} d\theta \left(\sin^2\frac{\theta_c}{2} - \sin\frac{\theta}{2}\right)^{1/2} = 1. \tag{3.13}$$

At $\beta = \pi/8$, $\theta_c = \pi$ where it corresponds to a critical point for the transition from the weak- to the strong-coupling domain.

D. The weak-strong transition for $L=3$

When our weak-coupling result for the three-link chiral chain is compared with that for the strong-coupling region the nature of the critical point at $\beta_c = \frac{1}{3}$ can be determined by comparing the derivatives of the free energy $F(\beta)$ from both sides of β_c. From Ref. 5, if we define the free energy by

$$F(\beta) = \frac{1}{N^2}\ln Z_L(\beta) \tag{3.14}$$

we find, for $\beta \leq \frac{1}{3}$,

$$F_-(\beta) = 3\beta^2 - \tfrac{9}{8}\beta^2 z_2{}^4 + \tfrac{1}{2}\ln\frac{-z_2}{2\beta}, \tag{3.15}$$

where $z_2 = -2(3\beta)^{-1/2}\sin\frac{1}{3}\sin^{-1}(3\beta)^{3/2}$. At $\beta=\beta_c$, $F_-(\beta_c) = \frac{5}{24} + \frac{1}{2}\ln\frac{3}{2}$. To find $F_+(\beta)$ for $\beta \geq \beta_c$, we note that a single-link expectation value $w_1(\beta) \equiv (1/N)\langle \operatorname{Tr}(U_1 U_2^\dagger)\rangle$ is related to $F(\beta)$ by

$$w_1(\beta) = \frac{1}{2L}\frac{\partial F(\beta)}{\partial\beta}. \tag{3.16}$$

At $N_c \to \infty$, $w_1(\beta)$ can be easily obtained from (3.13)

$$w_1(\beta) = \int d\theta\, \rho_3(\theta)\cos\theta = \beta + \frac{1}{2} - \frac{1}{8\beta} - \beta\left(1 - \frac{1}{3\beta}\right)^{3/2}. \tag{3.17}$$

From (3.19) and (3.21), we find both $w_1(\beta)$ and $\partial w_1(\beta)/\partial\beta$ are continuous at β_c, taking on values $\frac{11}{24}$ and $\frac{17}{8}$, respectively. However, $\partial^2 w_1/\partial\beta^2$ diverges when approaching β_c from above and is finite from below, therefore explicitly demonstrating that the critical point for the three-link chiral chain is again third order.

Finally, by integrating $w_1(\beta)$, for $\beta > \beta_c$,

$$F_+(\beta) = (2L)\int_{\beta_c}^{\beta} d\beta\, w_1(\beta) + \tfrac{5}{24} + \tfrac{1}{2}\ln\tfrac{3}{2} = 6\beta - \ln 2\beta - \tfrac{1}{2}\ln 3 - \tfrac{3}{2} + F_{\rm reg}(1/\beta), \tag{3.18}$$

where

$$F_{\rm reg}\left(\frac{1}{\beta}\right) = -\frac{1}{2}\ln\frac{1+(1-1/3\beta)^{1/2}}{2} + \left\{\left(3\beta^2 - \tfrac{5}{2}\beta\right)\left[1 - \left(1 - \frac{1}{3\beta}\right)^{1/2}\right] - \frac{\beta}{2} + \frac{3}{8}\right\}$$

vanishes as $1/\beta \to 0$. We shall show in Sec. IV that the "singular" part of $F(\beta)$ can be obtained by considering Z_L to one-loop order.

Thus we know that the chiral chains have third-order transitions for $L = 2$, 3, and ∞ whose location is determined dynamically. It seems very likely that all the chiral chains have a third-order transition with a crossover singularity which moves monotonically from $\beta_c = \frac{1}{4}$ at $L=2$ to $\beta_c = \frac{1}{2}$ at $L=\infty$, in accord with our subsequent analysis in Sec. IV.

Finally, we wish to emphasize that the case $L=2$ is unique in that the one-loop correction (see Sec. IV) gives the exact result for $N_c = \infty$ on the weak-coupling side. For $L=3$ and 4 this is not the case, and for $L \geq 5$ we are unable to find analytical solutions, since this requires computing f_λ in Eq. (1.10) for $\lambda \geq 3$. The possible role of some kind of semiclassical configurations or effective action for even these finite cyclic chains is still obscure. Their solution might give deeper insight into the $N_c = \infty$ QCD problem.

IV. BEYOND EXACT SOLUTIONS

The exact solutions we have presented so far support the supposition that the presence of a critical point of a higher order is a general feature of lattice gauge systems in the large-N limit. However, in the absence of analytic solutions for systems more involved than the four-link chain, we must resort to expansion schemes for gaining further insight. While strong-coupling expansions can be performed by means of rather standard techniques, the weak-coupling expansion is much more involved. In order to provide an estimate of the first nontrivial contributions to the functional integral in the limit $\beta\to\infty$, we take advantage of the observation that the weak coupling is dominated by the classical solutions.

The matrix elements of an L-fold transfer matrix can be represented by a function integral

$$\begin{aligned}\langle U|T^L|V\rangle &= \int dw_1 dw_2 \cdots dw_{L-1} \\ &\quad\times e^{N\beta \mathrm{Tr}(Uw_1^\dagger + w_1 w_2^\dagger + \cdots + w_{L-1}V^\dagger + \mathrm{H.c.})} \\ &= \int \prod_{i=1}^{L-1} dw_i\, e^{N\beta\mathrm{Tr}(w_1^\dagger + w_1 w_2^\dagger + \cdots + w_{L-1}UV^\dagger + \mathrm{H.c.})}.\end{aligned} \tag{4.1}$$

Denoting $Z_L \equiv (UV^\dagger)^{1/L}$, the classical solution which maximizes the action in (4.1) can easily be found:

$$w_i = (Z_L^\dagger)^i, \quad i = 1, \ldots, L-1. \tag{4.2}$$

We next carry out the semiclassical approximation by integrating the Gaussian fluctuation about the solution (4.2). Define

$$v_k \equiv w_k (Z_L)^k \equiv e^{i(g/2)\lambda_\alpha \theta_k^\alpha} \tag{4.3}$$

and expand v_k around the identity, keeping only the quadratic terms in θ_j^α for the exponent in (4.1) so that

$$\begin{aligned} S &\equiv N\beta\,\mathrm{Tr}[w_1 + w_1 w_2^\dagger + \cdots + w_{L-1}(Z_L)^L + \mathrm{H.c.}] \\ &\simeq NL\beta\,\mathrm{Tr}(Z_L + Z_L^\dagger) - \tfrac{1}{2}\sum_{\alpha\beta}\sum_{i,j} \theta_i^\alpha M_{ij} D_{\alpha\beta}\theta_j^\beta, \end{aligned} \tag{4.4}$$

where $D_{\alpha\beta} = \frac{1}{4}\mathrm{Tr}[\lambda_\alpha\lambda_\beta(z+z^\dagger)]$ and M_{ij} is a matrix with $\det(M_{ij}) = L$. After performing the Gaussian integrations, we obtain for f_L defined by Eq. (1.11), in the limit $N\to\infty$ and β large,

$$\begin{aligned} f_L(UV^\dagger) &= \frac{L}{N}\mathrm{Tr}\beta(Z_L + Z_L^\dagger) - \frac{L-1}{2N^2}\mathrm{Tr\,Ln}[D_{\alpha\beta}(Z)] \\ &\quad - \tfrac{1}{2}(L-1)(\ln 2\beta + \tfrac{3}{2}) - \tfrac{1}{2}\ln L, \end{aligned} \tag{4.5}$$

where $Z_L = (UV^\dagger)^{1/L}$ and we have adopted the normalization $\mathrm{Tr}\lambda_\alpha\lambda_\beta = 2\delta_{\alpha\beta}$. Note, in particular, for $L=2$, Eq. (4.5) can be shown to agree with our exact large-N result valid in the weak-coupling region with β finite.

The above result can be used to find the behavior of the differential Migdal β function as $g^2N\to 0$. Note that in this limit, $UV^\dagger$ is close to the identity matrix; by expanding about the identity, we find, under a λ-fold decimation ($f_1\to f_\lambda$),

$$\frac{1}{g^2N} \to \frac{1}{\lambda}\left(\frac{1}{g^2N}\right) - \frac{\lambda-1}{4\lambda^2}. \tag{4.6}$$

Now, by alternating bond moving and decimation for a d-dimensional chiral chain in the manner described in Ref. 3, we obtain

$$\left(\frac{1}{g^2N}\right)' = \lambda^{d-1}\left[\frac{1}{\lambda}\left(\frac{1}{g^2N}\right) - \frac{\lambda-1}{4\lambda^2}\right]. \tag{4.7}$$

Similar analysis can also be carried out for a $2d$-dimensional gauge model. The differential β function is then given by

$$-\frac{\beta(g)}{g} = \frac{2n-d}{2} + \frac{n}{8}(g^2N) \tag{4.8}$$

as promised in the Introduction.

Our result (4.5) also allows us to immediately obtain the free energy of the closed chain to the one-loop order. Setting $U=V$, thus $Z_L = I$, we obtain

$$F_L(\beta) = 2L\beta - \tfrac{1}{2}(L-1)\ln 2\beta - \tfrac{3}{4}(L-1) - \tfrac{1}{2}\ln L. \tag{4.9}$$

For $L=3$, $F_3(\beta) = 6\beta - \ln 2\beta - \frac{3}{2} - \frac{1}{2}\ln 3$. This should be compared with Eq. (3.18); we notice that they agree exactly at $1/\beta\to 0$.

Equation (4.9) also allows us to obtain an estimate for the dependence of the critical coupling β_c on L by matching it from the results from the strong-coupling domain. Both from our three-link analysis and from that of Ref. 4, we expect that the transition would be of the third order; we therefore concentrate on the second derivative of the free energy with respect to β. From Eq. (4.9), we have

$$\frac{1}{2L}\frac{\partial^2 F}{\partial\beta^2} = \frac{\partial W_1}{\partial\beta} = \frac{L-1}{4L}\frac{1}{\beta^2}. \tag{4.10}$$

From a strong-coupling analysis, we have

$$W_1 \cong \beta + \beta^{L-1} + \cdots \tag{4.11}$$

so that

$$\frac{1}{2L}\frac{\partial^2 F}{\partial\beta^2} \cong 1 + (L-1)\beta^{L-2}. \tag{4.12}$$

By matching (4.11) with (4.12), we obtain

$$\frac{L-1}{4L}\frac{1}{\beta_c{}^2} \cong 1 + (L-1)\beta_c{}^{L-2}. \tag{4.13}$$

Equation (4.13) yields $\beta_c = 0$, $\frac{1}{4}$, 0.319, 0.366, $\frac{1}{2}$ for $L = 1$, 2, 3, 4, and ∞, respectively, whereas our corresponding exact values are 0, $\frac{1}{4}$, $\frac{1}{3}$, $\pi/8$, $\frac{1}{2}$. For large L, we obtain

$$\beta_c(L) \cong \frac{1}{2} - \frac{1}{4L}. \tag{4.14}$$

Thus, by keeping only two terms for the free energy in the strong- and weak-coupling expansions, we are able to determine the crossover singularity to 5–10% accuracy (see Fig. 3).

V. CONCLUSIONS

In this paper, we have extended the analysis of the one-link or external-field problem of Ref. 6, and applied it to several finite chiral chains as a bridge to QCD_4 at $N_c=\infty$. For the single-link problem, we have recast the Schwinger-Dyson equation of Brower and Nauenberg[6] in terms of N independent fermionic coordinates. This allowed a general solution of the $U(N)$ integral. We are also hopeful that by exploiting this fermionic character along the lines of Brezin *et al.*,[9] the SU(N) problem and the strong-coupling phase at $N=\infty$ can also be solved.

Already, with our weak-coupling solution and the power-series solution at strong coupling, we see the necessity for a phase boundary (see note added) in the general external-field problem at $N_c=\infty$. What is not obvious is whether or not the subsequent integrals over the sources (or other links in $A^\dagger A$) will smear out the singularity, so that no hint of the Gross-Witten[4] singularity persists in QCD_4.

Another way to raise this question is to note that the singularity found by Gross and Witten in QCD_2 actually occurs in the first term of the character expansion (1.7). Indeed, the free energy per plaquette $[f=F/V\,d(d-1)]$ is given exactly by the first character,

$$f=\frac{1}{g_0^{\,2}}\;\frac{\int du\,\mathrm{Tr}(U)\exp[(1/g_0^{\,2})\mathrm{Tr}(U+U^\dagger)]}{\int du\,\exp[(1/g_0^{\,2})\mathrm{Tr}(U+U^\dagger)]}$$

$$\equiv\beta\lambda_N(\beta)N^2\,. \qquad (5.1)$$

Thus, in QCD_2 the Gross-Witten singularity is essentially a "kinematical" feature of the first term in the character expansion. In any other many-plaquette model the character expansion is an infinite series, and it might seem miraculous if the singularity at $\beta=\frac{1}{2}$ in each term were to cancel and move to another location. However, as seen by our finite lattice examples just such a "miracle" occurs. The neighboring plaquettes do not wash out the kinematical singularity, but rather shift it to a dynamically determined location, while (at least in the $L=3$ chiral chain) the nature of the singularity (i.e., its third-order property) is unaffected. We feel that this is a persuasive illustration of how this singularity may be generalized to QCD_4. The fluctuations in the N^2 group manifold cause a sharp transition, which is only gently perturbed by the nearby plaquettes. We know from Creutz's[10] Monte Carlo calculations that the correlations are very short-ranged at $g^2N\simeq 2$, so the effect is essentially a local one.

Moreover, the recent Monte Carlo calculations for SU_2 QCD show a sharp peak in the specific heat.[11] This appears to be rather consistent with a developing cusp in $F''(\beta)$ at $N_c=\infty$, but somewhat difficult to understand in terms of surface roughening[12] since it appears in a bulk quantity. Finally, recent efforts to estimate the crossover point in $N_c=\infty$ from series expansions indicate a value for $(g^2N)_{\text{crossover}}$ at a little more than 2, in remarkable agreement with our most "realistic" finite lattice, the cube.[13] A glance at Fig. 3 gives $(g^2N)_{\text{crossover}}\simeq 2.4$ in this case. While these arguments are still quite qualitative, the evidence is encouraging.

Efforts are underway to see if the presence of a pinch effect where complex singularities in g^2N collapse at real $g^2N\simeq 2$ can be related to instantonlike lattice effects. Clearly, if rather small lattices are to be used via Monte Carlo calculations to obtain information on the continuum theory at $g_0^{\,2}=0$, a firm understanding of the crossover region is imperative. Indeed, if at $N_c=\infty$, the third-order singularity occurs and its width is order $1/N$ the extrapolation to $g_0^{\,2}=0$ may not be hurt badly for any calculations just inside the transition point. Owing to the many degrees of freedom in the group, small spacetime lattices may not be so small after all.

Note added. On completing this work, we were informed by Brezin and Gross[14] of an elegant extension of the solution of Ref. 6 at $N_c=\infty$ to the strong-coupling phase, where they show that the phase boundary occurs at $g^2N=(1/2N)\,(1/\sqrt{x_i})$. This allows us to calculate the free energy in the strong-coupling region for $L=3$—corroborating Friedan's solution—and for $L=4$, thus completely solving the tetrahedron problem. In both cases

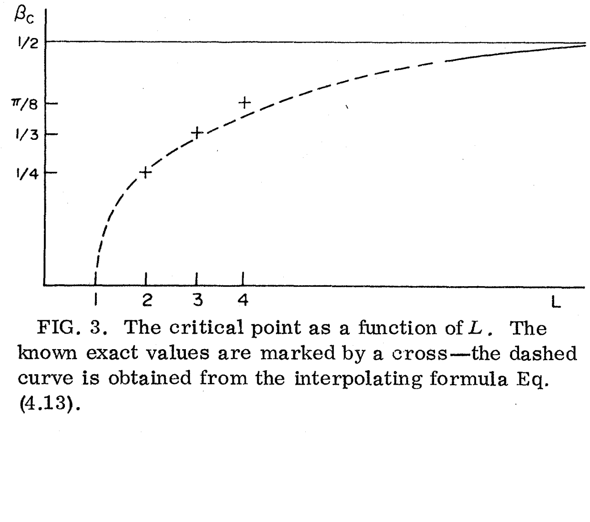

FIG. 3. The critical point as a function of L. The known exact values are marked by a cross—the dashed curve is obtained from the interpolating formula Eq. (4.13).

the free energy exhibits a third-order phase transition. Details are given in the following paper.[15]

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APPENDIX A: GAUGE THEORIES ON POLYHEDRA

To prove the equivalence of gauge theories on three-dimensional polyhedra with one-dimensional periodic chiral chains, we must pick a particular gauge. Consider for example the cube of Fig. 4(a) flattened to the surface of a plane as in Fig. 4(b). Any polyhedron can be similarly drawn. Now pick a closed path C on the surface that starts on one face and intersects each face once and only once. This divides the surface of F faces (or plaquettes) into two regions, inside and outside of the closed curve. Now we attempt to perform gauge transformations at V vertices to try to set all the links not intersected by the path C to $U=1$. Clearly, if this is possible, the gauge action which is a sum over F plaquettes,

$$S=\frac{1}{g^2}\sum_{p=1}^{L}\mathrm{Tr}(U_p+U_p^\dagger)\,, \tag{A1}$$

U_p being product of U's around a plaquette p, becomes an L-link periodic chain.

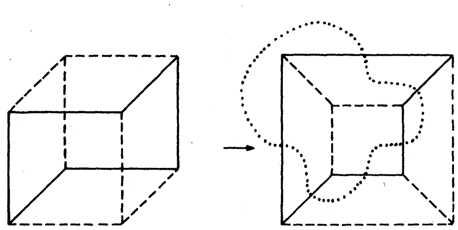

FIG. 4. (a) Cube with dashed lines gauged to $U=1$, leaving a six-link chiral chain for the gauge field theory on the surface. (b) Cube flattened to plane—the surfaces are paired by considering a closed path (dotted) which covers the surface by entering and exiting the faces across the *non*gauged lines.

To begin with let us count the degrees of gauge freedom. Given E edges, and F faces, we must gauge away $E-F$ ($F=L$) link variables ($U=1$). From Euler's theorem $E-F=V-2$, we see for a closed surface (no handles) there are two extra gauge transformations left over. However, a global gauge transformation of all the vertices inside or outside the curve C cannot affect the links we wish to set to $U=1$. Hence the number is exactly right.

Now to see that such a gauge choice is indeed possible we note that the links on the inside (outside) of the closed path form a tree graph. By working from the "center" at one ungauged vertex to the edges, gauging the link you pass, all the links can be gauged except the ends, which intersect C. (Incidentally, for higher topologies with H handles we cannot choose such a gauge. There are $V-2+2H$ links to gauge, and of the V gauge choices, one overall gauge is useless. Hence, it is not possible to set $V-2+2H$ gauges except for $H=0$.) In the case $H=0$, we have one gauge left, to set any link in the remaining chiral chain to $U=1$, getting a complete fixed gauge.

As an application of the general gauge-fixing procedure we have devised, we want to show how to reduce the problem of a single plaquette propagating in time to a chiral chain problem.

The lattice version of the problem corresponds to the gauge theory of an infinitely long "tower" of elementary cubes. However, by projecting the external surface of the tower on a plane and gauging the links as in Fig. 5, we observe that the "timelike" plaquettes are described by the product of two links and the "spacelike" plaquettes are described by a single link—the overall action may then assume the form

$$S=\beta N\left[\sum_i \mathrm{Tr}(U_i U_{i+1}^\dagger+U_{i+1}U_i^\dagger)+\sum_i \mathrm{Tr}(U_{4i}+U_{4i}^\dagger)\right] \tag{A2}$$

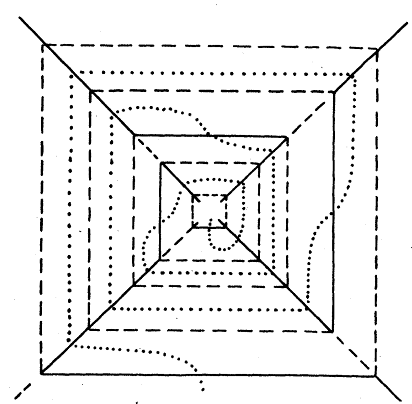

FIG. 5. Helical gauge for the propagating square.

corresponding to an infinite one-dimensional chain with potential terms. It is worth mentioning that the Hamiltonian form of this problem has been solved by Jevicki and Sakita (Ref. 16) and their weak-coupling solution is exactly the same as our $L=4$ weak-coupling solution, with critical point $\beta_c=\pi/8$.

APPENDIX B

We want to find out the effect of replacing Z_0 with $W_0/\Pi_{i<j}(x_i-x_j)$ in the Schwinger-Dyson equation, Eq. (2.3). By the use of

$$\frac{\partial Z_0}{\partial x_k}=\left(\sum_{s\neq k}\frac{1}{x_s-x_k}\right)Z_0+\frac{1}{\prod_{i<j}(x_i-x_j)}\frac{\partial W_0}{\partial x_k} \tag{B1}$$

and

$$\frac{\partial^2 Z_0}{\partial x_k^2}=-\left[\sum_{s\neq k}\frac{1}{(x_s-x_k)^2}\right]Z_0+\left[\sum_{s\neq k}\frac{1}{(x_s-x_k)}\right]^2 Z_0+\frac{1}{\prod_{i<j}(x_i-x_j)}\left(2\sum_{s\neq k}\frac{1}{x_j-x_k}\frac{\partial W_0}{\partial x_k}+\frac{\partial^2 W_0}{\partial x_k^2}\right), \tag{B2}$$

one finds that W_0 obeys the equation

$$\left[\frac{1}{N}\sum_{k,s}\frac{x_k}{x_s-x_k}+\frac{1}{N^2}\sum_{k,s}\frac{x_k^2}{(x_s-x_k)^2}+\frac{1}{N^2}\sum_{k,s,t}\frac{x_k}{(x_s-x_k)}\frac{x_k}{(x_t-x_k)}-\frac{2}{N^2}\sum_{kst}\frac{x_kx_t}{(x_t-x_k)}\frac{1}{(x_s-x_k)}\right]W_0$$
$$+\sum_k\left[\frac{1}{N}x_k+\frac{2}{N^2}\sum_s\frac{x_k^2}{x_s-x_k}+\frac{2}{N^2}\sum_s\frac{x_sx_k}{x_k-x_s}\right]\frac{\partial W_0}{\partial x_k}+\frac{1}{N^2}\sum_k x_k^2\frac{\partial^2 W_0}{\partial x_k^2}=\beta^2\left(\sum_k x_k\right)W_0\,. \tag{B3}$$

However, it is easy to prove the following algebraic identities:

$$\sum_{k,s}\frac{x_k}{x_s-x_k}=-\frac{N(N-1)}{2}, \tag{B4}$$

$$\frac{1}{N}x_k+\frac{2}{N^2}\sum_s\frac{x_k^2}{x_s-x_k}+\frac{2}{N^2}\frac{x_sx_k}{x_k-x_s}=\frac{2-N}{N^2}x_k, \tag{B5}$$

$$\frac{1}{N^2}\sum_{k,s,t}\frac{x_k}{(x_s-x_k)}\frac{x_k}{(x_t-x_k)}-\frac{2}{N^2}\sum_{k,s,t}\frac{x_kx_t}{x_t-x_k}\frac{1}{x_s-x_k}=\frac{(N-1)^2}{N}-\frac{1}{N^2}\sum_{k,s,t}\frac{x_sx_t}{(x_s-x_k)}(x_t-x_k), \tag{B6}$$

Moreover,

$$\sum_{k,s}\frac{x_k^2}{(x_s-x_k)^2}-\sum_k\left[\sum_s\frac{x_s}{x_s-x_k}\right]^2=-2\sum_{\substack{k,s,t\\k\neq s\neq t}}\frac{x_sx_t}{(x_s-x_k)(x_t-x_k)}=-\frac{2N(N-1)(N-2)}{3!} \tag{B7}$$

as one may easily check by evaluating explicitly the sum for the permutations of an arbitrary set of three indices k, s, t. Collecting all results and rescaling the variables to $z_k=2N\beta\sqrt{x_k}$ it is straightforward to obtain Eq. (2.5):

$$\left[\sum_k z_k^2\frac{\partial^2}{\partial z_k^2}+(3-2N)\sum_k z_k\frac{\partial}{\partial z_k}-\sum_k z_k^2+\tfrac{2}{3}N(N-1)(N-2)\right]W_0=0\,. \tag{B8}$$

Equation (B8) with the constraint that W_0 be a completely antisymmetric function of the z_k looks like a Schrödinger-type equation for a gas of N noninteracting fermions in external potential field. Its solution will then have the form of a Slater determinant involving the single-particle wave functions for the first N energy levels.

Let us show that such a solution may assume the form

$$W_0=(\text{const})\det(z_j^{\,i-1}I_{i-1}(z_j))\,. \tag{B9}$$

By an extensive use of the properties of Bessel functions one finds that

$$\frac{\partial W_0}{\partial z_k}=\det(z_1^{\,i-1}I_{i-1}(z_1),\ldots,z_k^{\,i-1}I_{i-2}(z_k),\ldots,z_N^{\,i-1}I_{i-1}(z_N))\,, \tag{B10}$$

$$\frac{\partial^2 W_0}{\partial z_k^2}=W_0+\det(z_1^{\,i-1}I_{i-1}(z_1),\ldots,(2i-3)z_k^{\,i-2}I_{i-2}(z_k),\ldots,z_N^{\,i-1}I_{i-1}(z_N))\,, \tag{B11}$$

$$z_k^2\frac{\partial^2 W_0}{\partial z_k^2}+(3-2N)z_k\frac{\partial W_0}{\partial z_k}-z_k^2W_0=\det(z_1^{i-1}I_{i-1}(z_i),\ldots,2(i-N)z_k^iI_{i-2}(z_k),\ldots,z_N^{i-1}I_{i-1}(z_N))\,. \tag{B12}$$

The determinant in Eq. (B12) can be split into a sum of determinants in which only one element in the kth column is different from zero.

This allows us to eliminate all the other elements in the same row. Now we can perform the sum on k by first grouping together all the determinants having the nonzero element of the kth column in the same row (say the ith):

$$\sum_k \det(z_1^{i-1}I_{i-1}(z_1),\ldots,2(i-N)z_k^iI_{i-2}(z_k),\ldots,z_N^{i-1}I_{i-1}(z_N))=\sum_m \det\begin{bmatrix} z_j^0I_0(z_j)\\ \cdots \\ 2(m-N)z_j^mI_{m-2}(z_j)\\ \cdots \\ z_j^{N-1}I_{N-1}(z_j)\end{bmatrix}\,. \tag{B13}$$

Now the properties of Bessel function imply that

$$z^mI_{m-2}(z)=z^mI_m(z)+2(m-1)z^{m-1}I_{m-1}(z) \tag{B14}$$

and after elimination of the linearly dependent vectors the previous sum simply amounts to

$$\sum_m 2(m-N)2(m-1)\det(z_j^{i-1}I_{i-1}(z_j)) = -4\frac{N(N-1)(N-2)W}{6} \tag{B15}$$

and this concludes our proof.

We may compare our result with the previously known limiting situations discussed by Bars and Green.[8] When $A=a\,I$ one must take the limit $z_i \to 2\beta Na$ for each i—this amounts to taking the proper derivatives of Bessel functions and the final result is

$$Z_0(z_i\to 2\beta Na)=\det(I_{i-j}(2\beta Na))\,.$$

Analogous arguments work for the case when only one eigenvalue of A is different from zero—the result is

$$z_0(z_1\to 2\beta Na, z_{i\neq 1}\to 0) =(N-1)!(\beta Na)^{1-N}I_{N-1}(2\beta Na)\,. \tag{B16}$$

Finally, as we have shown in Sec. II, the large-N-limit result of Brower and Nauenberg is recovered from the present general result.

APPENDIX C

With variable changes $z=e^{i\theta}$, $z=e^{i\theta/2}$, $z=e^{i\theta/2}$ for $L=2,3,4$, respectively, Eqs. (3.4), (3.8), (3.11) can be restated as, on the unit circle between $z_c^*(L)$ and $z_c(L)$,

$$R_L(z)=\mathrm{Re}[F_L(z,\rho_L)] \tag{C1}$$

where

$$R_2(z)=\left(\frac{2\beta}{i}\right)\left(z-z^{-1}\right), \tag{C2a}$$

$$R_3(z)=\left(\frac{\beta}{i}\right)[(z^2-z^{-2})+(z-z^{-1})], \tag{C2b}$$

$$R_4(z)=\left(\frac{2\beta}{i}\right)(z-z^{-1}) \tag{C2c}$$

and

$$F_2(z,\rho_2)=\int_{-\theta c(2)}^{\theta c(2)} d\phi\,\cot\left(\frac{\theta-\phi}{2}\right)\rho_2(\phi)=\int_{z_c^*(2)}^{z_c(2)}\left(\frac{dz'}{z'}\frac{z+z'}{z-z'}\right)\rho_2(z')\,, \tag{C3a}$$

$$F_3(z,\rho_3)=\int_{-\theta_c(3)}^{\theta_c(3)} d\phi\left[\cot\left(\frac{\theta-\phi}{2}\right)+\frac{1}{2}\,\frac{\sin\theta/2}{\cos\theta/2+\sin\theta/2}\right]\rho_3(\phi)=\int_{z_c^*(3)}^{z_c(3)}\frac{dz'}{z'}\left(\frac{z+z'}{z-z'}\right)\rho_3(z') \tag{C3b}$$

$$F_4(z,\rho_4)=\int_{-\theta_c(4)}^{\theta_c(4)} d\phi\left[\cot\left(\frac{\theta-\phi}{2}\right)+\frac{\sin\theta/2}{\cos\theta/2+\sin\theta/2}\right]\rho_4(\phi)=4\int_{z_c^*(4)}^{z_c(4)} dz'\,\frac{z}{(z^2-z'^2)}\rho_4(z')\,. \tag{C3c}$$

In (C3), $0\leq\theta_c(L)\leq\pi$ and will eventually be determined by the normalization condition on $\rho_L(\theta)$.

For these three cases, it is not difficult to determine directly that $F_L(\theta)$ must be given by

$$F_2(\theta)=4\beta\sin\theta-4\sqrt{2}\beta\cos\frac{\theta}{2}[\cos\theta_{c(2)}-\cos\theta]^{1/2}\,, \tag{C4a}$$

$$F_3(\theta)=2\beta\left(\sin\theta+\sin\frac{\theta}{2}\right)+2\sqrt{2}\beta\left(\cos\frac{\theta}{4}\right)\left[2\cos\frac{\theta}{2}+\left(1-\frac{1}{3\beta}\right)^{1/2}\right]\left(\cos\frac{\theta_c}{2}(3)-\cos\frac{\theta}{2}\right)^{1/2}, \tag{C4b}$$

$$F_4(\theta) = 4\beta \sin\frac{\theta}{2} - 2\sqrt{2}\beta[\cos\theta_c(4) - \cos\theta]^{1/2} . \quad \text{(C4c)}$$

However, a more systematic procedure exists which we describe in what follows. We shall first drop the subscript L.

It follows from (C1) and (C3) that

$$G(z) = -\frac{F^2}{2R} + F \quad \text{(C5)}$$

is a meromorphic function with possible poles at zeros of $R(z)$. Since the asymptotic behavior of $F(z)$ and $R(z)$ and their behavior at $z = 0$ are known, $G(z)$ can be parametrized in terms of a few parameters. Once this is done, going back to the cut on the unit circle, one finds

$$\rho^2(t) = \frac{1}{4\pi^2}[2R(z)G(z) - R^2(z)] . \quad \text{(C6)}$$

The remaining unknown parameters can be uniquely specified by (i) the positivity of $\rho(\theta)$, and (ii) $F(z)$ is analytic inside the unit circle, where $F(z) = R(z) - 2\pi i\rho(z)$. We now return to individual cases.

(a) $L = 2$: In this case, we can parametrize

$$G_2(z) = (-i)\left[\left(\frac{1+z}{1-z}\right) - \left(\frac{4bz}{1-z^2}\right)\right] \quad \text{(C7)}$$

leading to the representation

$$\rho_z^2 = \left(\frac{16}{\pi^2}\beta^2\right)\left(\cos\frac{\theta}{2}\right)^2\left(\frac{1}{4\beta} - \sin^2\frac{\theta}{2}\right) - \frac{4}{\pi^2}\beta b . \quad \text{(C8)}$$

For $\beta \geq \frac{1}{4}$, the positivity and the analyticity conditions force one to have $b = 0$. This, in turn, leads to Eq. (3.5).

(b) $L = 3$: We parametrize $G_3(z)$ by

$$G_3(z) = i(a + b + \tfrac{1}{2})\frac{(z - z^{-1})}{(z + z^{-1} + 1)} - ia\left(\frac{z+1}{z-1}\right) - ib\left(\frac{z-1}{z+1}\right) \quad \text{(C9)}$$

so that

$$4\pi^2\rho_3^2 = (\beta^2)(z^{1/2} + z^{-1/2})^2\left[(z^3 + z^{-3}) + \frac{1}{\beta}(z + z^{-1}) - \frac{(2\beta - 6a + 2)}{\beta}\right] + 2b\beta[(z^{1/2} - z^{-1/2})^2] . \quad \text{(C10)}$$

The positivity and the analyticity requirements dictate that, of the eight zeros of $\rho^2(z)$, a double zero exists at $z = -1$, two simple zeros at z_c and z_c^* on the right-half unit circle, and one additional pair of complex conjugate double zeros on the unit circle. This forces us to have $b = 0$ and $a = (\frac{1}{3})\beta(1 - 1/3\beta)^{3/2} - \frac{1}{3}\beta - \frac{1}{3}$, which in turn leads to (3.9).

(c) $L = 4$: We can parametrize $G_4(z)$ as

$$G_4(z) = \frac{2iaz}{z^2 - 1} \quad \text{(C11)}$$

which directly leads to

$$4\pi^2\rho_4^2 = 16\beta^2\left(\frac{a}{2\beta} - \sin^2\frac{\theta}{2}\right) . \quad \text{(C12)}$$

Writing $\sin^2(\frac{1}{2}\theta_c) \equiv a/2\beta < 1$, we arrive at (3.12) with $\theta_c(4)$ determined by the normalization condition (3.13).

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