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SU(3) gauge field configurations in static, external sources

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Classical SU(3) gauge theory with a static external source is studied numerically for finite-energy non-Coulombic solutions. Spherically symmetric, time-independent solutions are found for a class of sources described by two parameters. We have found a number of solutions that bifurcate at a critical source strength, as well as solutions that end at a critical point. Stability of these solutions is briefly discussed.

I. INTRODUCTION

Recently, several authors[1-4] have discussed solutions to classical Yang-Mills equations in the presence of static external sources. Jackiw, Jacobs, and Rebbi,[2] and Jackiw and Rossi[3] have studied spherically symmetric solutions in the presence of a static source of the form $\rho_a(\vec{x}) = (x^a/r)q(r)$ for the SU(2) theory. The most interesting features of the solutions obtained by them numerically are the following: (a) These solutions have finite-energy content. (b) There are two types of solutions which are both non-Coulombic. (c) One type of solution has a critical source strength above which the solutions bifurcate and no solutions exist below the critical point. (d) The upper branch with higher-energy content can be shown to be unstable while the lower branch shares the stability properties of the solution at the bifurcating point. (e) The solution at the bifurcating point is absolutely stable under radial oscillations. While these results have been obtained for a special choice of the source strength $g(r) = Q\delta(r - r_0)$ (δ-shell source) their qualitative features are characteristic of the nonlinear equations of motion. In spite of the very interesting features these solutions possess, their quantum significance is as yet unclear.

In this article, we study SU(3) gauge fields in the presence of static external sources. There are two good reasons for such a study. (1) If these solutions have quantum significance, then, SU(3) being the gauge group of quantum chromodynamics (QCD), its solutions may correspond to the *reality* of strong interaction. (2) SU(3) gauge theory should possess a larger class of solutions than SU(2) and hence many more bifurcating solutions may exist. This expectation is confirmed by our study.

II. SU(3) GAUGE FIELDS IN THE PRESENCE OF STATIC EXTERNAL SOURCES

The Yang-Mills equations in the presence of an external source $J^\mu(x)$ are given by

$$D_\mu F^{\mu\nu} = J^\nu \,, \tag{1}$$

where

$$D^\mu = \partial^\mu + ig[A^\mu, \quad] \tag{2}$$

is the covariant derivative and $J_\nu(x) = (\rho(x), 0, 0, 0)$ is the static external source. We use the matrix notation

$$\rho = \sum_{a=1}^{8} \rho_a \left(\frac{\lambda_a}{2i}\right) \tag{3}$$

and

$$F_{\mu\nu} = \sum_{a=1}^{8} F_{\mu\nu,a} \left(\frac{\lambda_a}{2i}\right) , \tag{4}$$

where $\{\lambda_a\}$ are the SU(3) generators.

Following several authors[5,6] we define a source to be maximally spherically symmetric if

$$[\vec{J} + \vec{T}, \rho] = 0 \,, \tag{5}$$

where $\vec{J}$ generates spatial rotations and $\vec{T}$ generates the SO(3) subgroup of SU(3). In other words, of the two possible embeddings of SU(2) in SU(3) we choose the isospin-1 embedding, namely, the SO(3) subgroup generated by λ_7, $-\lambda_5$, and λ_2. It can be shown that the most general source that is maximally spherically symmetric is of the form

$$\rho_{ab} = i\epsilon_{abc} \frac{x^c}{r} \frac{q_1(r)}{g^2} + \left(\frac{x_a x_b}{r^2} - \frac{\delta_{ab}}{3}\right) \frac{q_2(r)}{g^2} . \tag{6}$$

Our notation is as follows. ρ is a 3×3 traceless Hermitian matrix and thus the antisymmetric part corresponds to λ_7, $-\lambda_5$, and λ_2 while the remaining term arises due to the fact that with respect to these three generators, the remaining λ matrices transform as a quadrupole moment operator. From Eq. (1) it follows that A_0 satisfies the condition of parallelism:

$$[A_0, \rho] = 0 \,. \tag{7}$$

The general ansatz for A_0 thus turns out to be

$$A_{0,ab} = i\epsilon_{abc} \frac{x^c}{r} \frac{f_1(r)}{g^2} + \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right) \frac{f_2(r)}{g^2} . \tag{8}$$

The most general ansatz consistent with spherical symmetry for A_i can be shown to be of the form

$$A_{i,ab}=i\left(\delta_{ib}\frac{x_a}{r}-\delta_{ia}\frac{x_b}{r}\right)\frac{G(r)-1}{g^2}. \tag{9}$$

The equations of motion reduce to coupled ordinary differential equations in the radial coordinate:

$$-f_1''(r)+\frac{2}{r^2}G^2(r)f_1(r)=2rq_1(r), \tag{10}$$

$$-f_2''(r)+\frac{6}{r^2}G^2(r)f_2(r)=2rq_2(r), \tag{11}$$

$$-G''(r)+\frac{1}{r^2}[G^2(r)-1]G(r)-\frac{1}{r^2}(f_1{}^2+f_2{}^2)G(r)=0. \tag{12}$$

Here $f''(r)\equiv d^2f/dr^2$ etc. The energy of these solutions takes the form

$$\begin{aligned}\epsilon&=\tfrac{1}{2}\int d^3x[\mathrm{Tr}(E^2+B^2)]\\&=\frac{4\pi}{g^2}\int_0^\infty dr\bigg\{\tfrac{1}{2}[f_i'(r)]^2+\tfrac{1}{6}[f_2'(r)]^2+[G'(r)]^2\\&\qquad+\frac{1}{r^2}(f_1{}^2+f_2{}^2)G^2(r)\\&\qquad+\frac{1}{2r^2}(G^2-1)^2\bigg\}.\end{aligned} \tag{13}$$

Regularity at the origin requires that

$$f_1(0)=f_2(0)=0,\quad G(0)=1, \tag{14}$$

while finiteness of energy implies that

$$f_1(\infty)=f_2(\infty)=0 \tag{15}$$

and

$$G(\infty)=\pm 1. \tag{16}$$

$G(\infty)=+1$ implies $(A_{i,ab})\to 0$ at infinity while $G(\infty)=-1$ implies $(A_{i,ab})$ goes to a pure gauge at spatial infinity. We have not succeeded in obtaining analytic solutions to the above equations for any choice of the source strength described by two functions $q_1(r)$ and $q_2(r)$. We have, however, obtained several interesting numerical solutions below. We have chosen for the source strengths q_1 and q_2 a δ-shell,

$$\begin{aligned}q_1(r)&=Q\delta(r-r_0),\\q_2(r)&=R\delta(r-r_1).\end{aligned} \tag{17}$$

δ-shell potentials are easy to handle when doing numerical calculations and are nonpathological. Our source is characterized by two parameters Q and R. For obvious reasons we will call Q and R "monopole" and "quadrupole" parts of the spherically symmetric SU(3) source, although, in the decomposition (8), the two terms are not separately gauge invariant. Nevertheless, the Casimir invariants of SU(3) for the particular

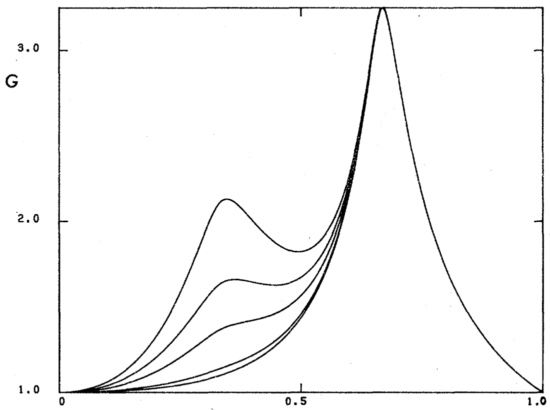

FIG. 2. $G(x)$ for type-I family, for various values of Q. R is fixed at 12.0. Starting from the lowest curve these correspond to $Q=0$, 4.0, 12.0, 20.0, and 36.0. Here $x=r/(1+r)$, where r is the radial variable. Q is located at $x=\frac{1}{3}$ and R is located at $x=\frac{2}{3}$.

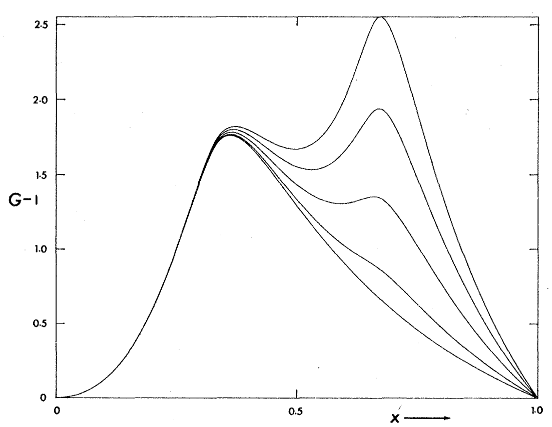

FIG. 1. $G(x)$ for type-I solutions for various R. Q is fixed at 12.0. Starting from the lowest curve these correspond to $R=0$, 0.5, 1.0, 1.5, and 2.0.

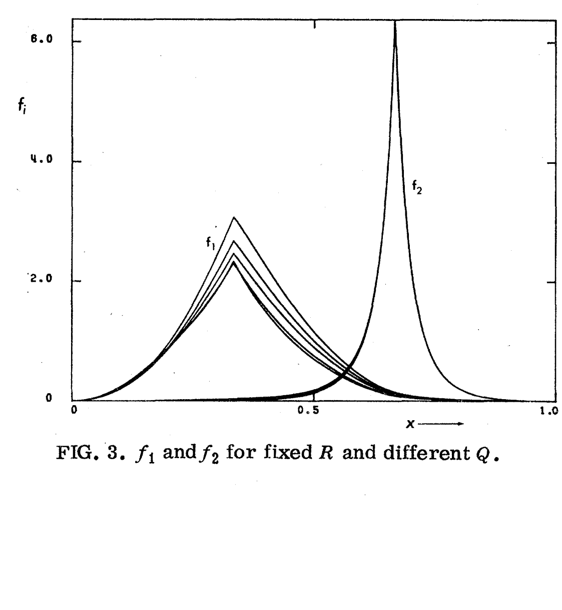

FIG. 3. f_1 and f_2 for fixed R and different Q.

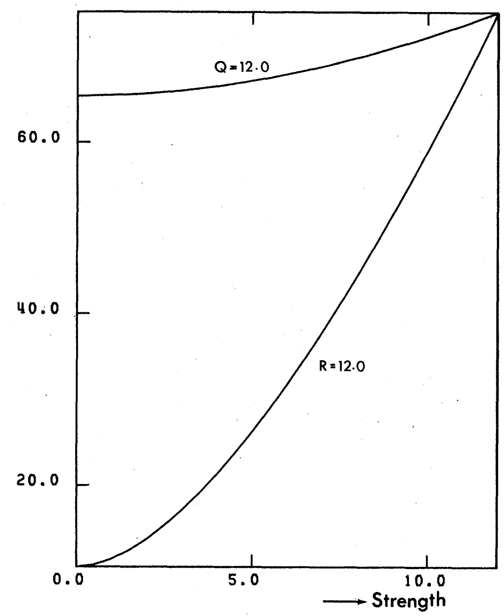

FIG. 4. Energy vs strength for type-1 solutions.

choice of q_1 and q_2 can be expressed in terms of Q and R and since there are two Casimir invariants for SU(3), Q and R can in fact be used instead of gauge invariants and we shall present energy as a function of Q and R.

III. RESULTS

Equations (10), (11), and (12) are solved by numerical methods using the general purpose code

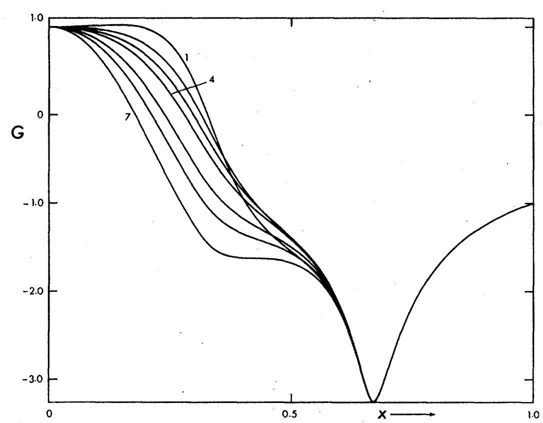

FIG. 5. $G(x)$ for the group-one, type-II family. R is fixed at 12.0. Q_{critical} is 11.236. Curve IV corresponds to Q_{critical}. Bifurcating solutions lie on either side of this curve. These correspond to the following values of Q. ϵ is the energy in units of $4\pi/g^2$. Curve I: $Q=20.0$ ($\epsilon=116.774$). Curve II: $Q=14.0$ ($\epsilon=95.377$). Curve III: $Q=1.20$ ($\epsilon=89.603$). Curve IV: $Q=11.3$ ($\epsilon=87.794$). Curve V: $Q=12.0$ ($\epsilon=89.417$). Curve VI: $Q=14.0$ ($\epsilon=94.047$). Curve VII: $Q=20.0$ ($\epsilon=108.803$).

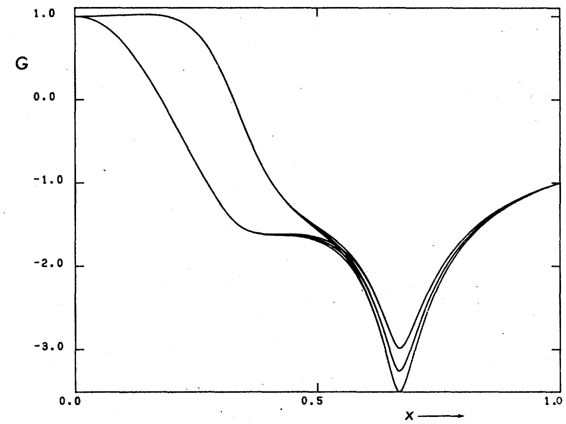

FIG. 6. $G(x)$ for the group-one, type-II family with Q fixed at 12.

COLSYS (Ref. 7) (collocation for systems). The semi-infinite interval in $r(0 \leq r < \infty)$ is mapped on to $(0 \leq x < 1)$ in x. We exhibit the solutions for f_1, f_2, and G as functions of x. The radii of the δ shells were taken to be $x_0=\frac{1}{3}$ and $x_1=\frac{2}{3}$ [see Eq. (17)]. This choice was purely a matter of convenience. As in the SU(2) case, we have obtained two basically different types of solutions, corresponding to the two different boundary conditions (15) for $G(r)$.

a. Type-I solutions $[G(\infty)=+1]$. In Figs. 1 and 2 we exhibit $G(x)$ for various values of Q and R while in Fig. 3, $f_1(x)$ and $f_2(x)$ are plotted. These solutions are analytic in Q and R. Note that $f_2(x)=0$ for $R=0$. This case corresponds to the SU(2) case studied in Ref. 2. Our numerical results for this case are in excellent agreement with those of Ref. 2. The energy of these solutions is plotted for various combinations of Q and R in Fig. 4.

b. Type-II solutions $[G(\infty)=-1]$. As in SU(2) this case turned out to be very interesting and

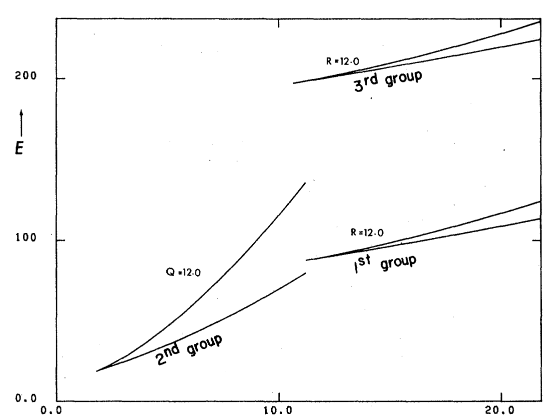

FIG. 7. Energy vs strength for type-II solutions.

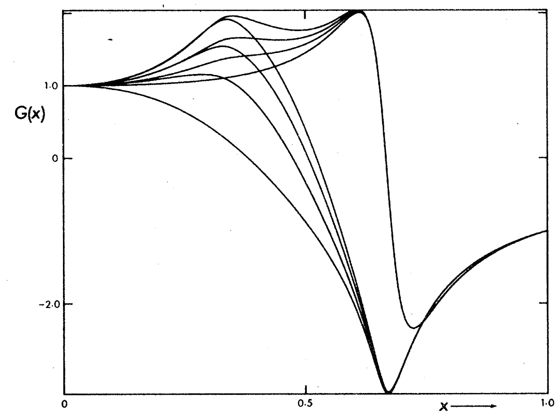

FIG. 8. $G(x)$ for the type-II, group-two family. R is fixed at 12. $Q = 0$, 12.0, 20.0, and 30.0.

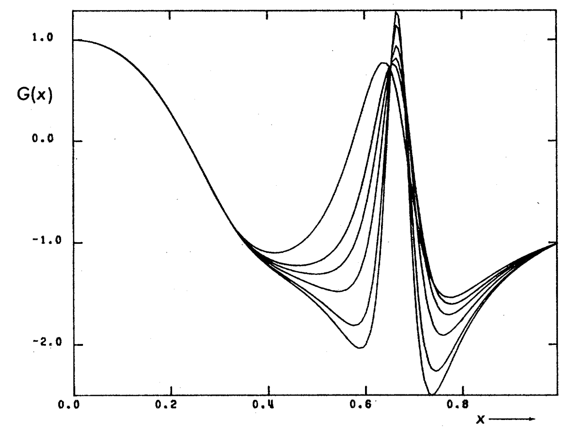

FIG. 10. $G(x)$ for the type-II, group-three family. $Q = 12.0$ and different values of R ranging from $R = 15.0$ to $R_c = 4.2$. These solutions belong to the lower branch.

more complex than the type-I solutions. Numerical studies yielded six different sets of solutions for a given combination of Q and R. Type-II solutions exist only for sufficiently strong sources. Below a critical value of Q or R, or both Q and R, solutions cease to exist. For Q and R greater than critical values Q_c and R_c, solutions bifurcate. We divide the type-II solutions into three groups. (i) The first group contains two solutions for a given value of Q and R. They reduce to the known SU(2) solutions as $R \to 0$. These solutions are characterized by the existence of a Q_c which is a function of R. Significant differences exist in the behavior of $G(x)$ as a function of x between the type-I and all groups of type-II solutions. $f_1(x)$ and $f_2(x)$ are qualitatively alike for both types of solutions. We therefore exhibit $G(x)$ only for type-II solutions. $G(x)$ vs x is shown in Figs. 5 and 6 for the first group of solutions. There is an odd number of curves in these figures. Bifurcating solutions lie on either side of the solution at the bifurcating point. In Fig. 7 energy of these solutions is plotted as a function of Q for a fixed value of R. (ii) The second group also contains two solutions for a given Q and R but they differ from the previous case not only in their energy content but also in the fact that there exists a critical value of R, R_c, which varies smoothly with Q, namely $R_c\ (Q=0) \neq 0$. These are the more general solutions allowed by the SU(3) symmetry and so do not exist in the SU(2) case. Again for $R > R_c$ solutions bifurcate. In Figs. 8 and 9 the solutions are shown which demonstrate the above-mentioned behavior. E vs (Q, R) is again shown in Fig. 7. (iii) In this group, solutions cease to exist for $(Q, R) < (Q_c, R_c)$ in contrast to the previous two groups where there exists only one critical strength, either in Q or in R. The criti-

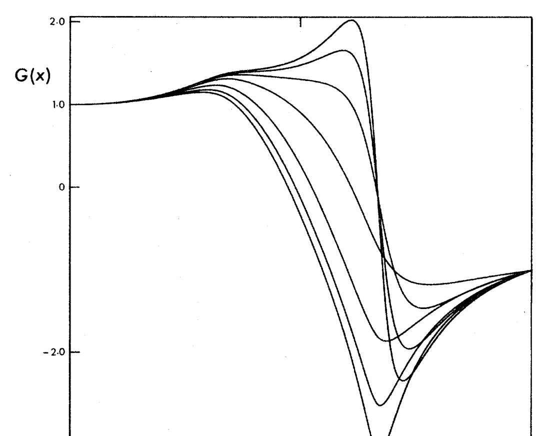

FIG. 9. $G(x)$ for the type-II, group-two family. R is fixed at 12 and for different values of R.

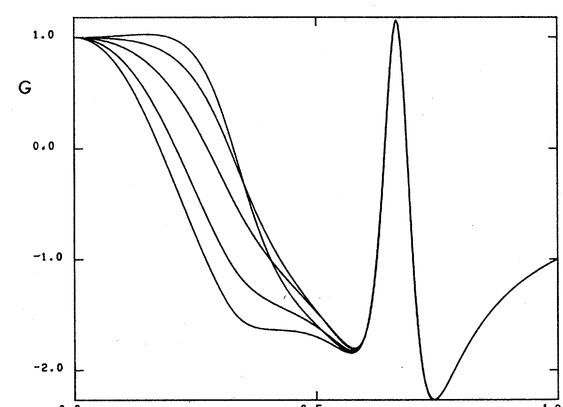

FIG. 11. $G(x)$ for the type-II, group-three family. $R = 12.0$ and different values of Q.

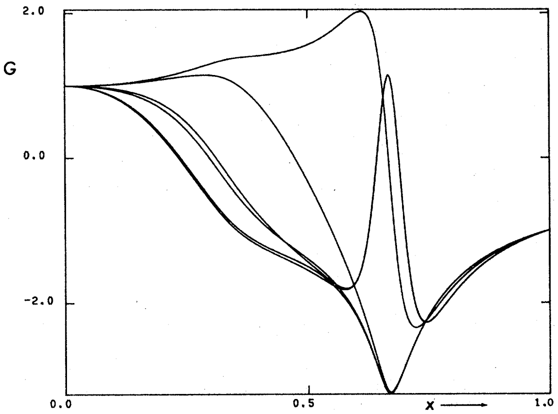

FIG. 12. All six solutions for $G(x)$ belonging to the type-II family. All six are for the same combination of Q and R; $R=Q=12.0$.

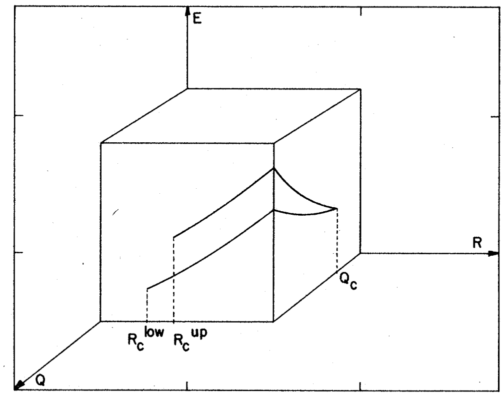

FIG. 13. Schematic drawing of energy of the third group of type-II solutions.

cal R values are different for the two branches and solutions simply terminate as $R=R_c$ (see Figs. 10, 11, and 12). In groups I and II we came across a bifurcating point called in the mathematical literature as a secondary branching point.[8] But in the present case there exists both a primary and secondary branching point.[8] Q_c is a secondary branch point (bifurcation point), but R_c is a primary branching point. Solutions do not exist for $R<R_c$. At R_c, however, the upper and lower branches do not come together. Moreover, R_c's are different for the upper and lower branches. Figure 13 is a schematic drawing that illustrates these features. Energy vs (Q,R) is plotted in Fig. 7 for all three groups of type-II solutions.

IV. STABILITY

Jackiw and Rossi[3] showed that in the case of the bifurcating solutions, the upper branch is unstable under linear deformations. Further, the lower branch shares the stability properties of the solution at the bifurcation point. Their analysis does not depend on the gauge group. We have verified that the solution at the bifurcation point is absolutely stable under small radial oscillations. Namely, there exists no normalizable solutions to the linearized equations for radial deformations of finite energy. In this respect the solutions we have found share the properties of fluctuations of the Prasad-Sommerfield monopole solutions in flat space-time for SU(2) gauge fields[9,10] and in curved space-time.[11]

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