Coherent states for general potentials. VI. Conclusions about the classical motion and the WKB approximation

Michael Martin Nieto and L. M. Simmons, Jr.
Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545

Vincent P. Gutschick
Life Sciences Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545
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The minimum-uncertainty coherent states empirically provide an approximation to the motion of a classical particle. This can be understood conceptually. One can obtain a relationship between the WKB approximation and the definition of the minimum-uncertainty coherent states. This definition is in terms of the "natural" quantum operators, which connect only adjacent energy eigenstates. The classical form of these operators is also related to the WKB approximation. In the Appendix we comment on the origin of "exact" WKB results.

I. INTRODUCTION

In this series of papers[1-6] we have investigated generalizations of the concept of "coherent states" to quantum-mechanical systems characterized by anharmonic potentials.[1-7] We have reviewed[1] the origins and methodology of coherent states for the harmonic oscillator and discussed other generalizations, these generalizations having been applied mainly to systems with equally spaced levels. Our method, the minimum-uncertainty coherent states (MUCS), is motivated by Schrödinger's original emphasis[8] on quantum-mechanical wave packets that follow the motion of a classical particle. From this point of view, a natural name for our states might be "quasi-classical states," but we have chosen to retain the name coherent states to emphasize the connection to other work.

In the MUCS method one first finds those "natural" classical variables X_c and P_c that vary sinusoidally with the classical angular velocity ω_c:

$$X_c = A(E)\sin\omega_c t\,, \tag{1.1}$$

$$P_c = m\omega_c A(E)\cos\omega_c t\,, \tag{1.2}$$

where

$$P_c = m\dot{x}X_c' = pX_c'\,. \tag{1.3}$$

Our notation is $\dot{y} = dy/dt$, $y' = dy/dx$. The variable $X_c(x)$ is the solution (with appropriate boundary conditions) to

$$X_c' = (\tfrac{1}{2}m)^{1/2}\omega_c\left[\frac{A^2 - X_c^2}{E - V(x)}\right]^{1/2}, \tag{1.4}$$

or

$$0 = X_c'' - \frac{V'}{2(E-V)}X_c' + \frac{m\omega_c^2}{2(E-V)}X_c\,. \tag{1.5}$$

Observe that the substitution $X_c = A\sin\theta(x)$ relates X_c to the angle variable

$$\theta(x) = (\tfrac{1}{2}m)^{1/2}\omega_c\int_{x_0}^{x}\frac{dx}{[E-V(x)]^{1/2}}\,. \tag{1.6}$$

For a complete period this integral is proportional to the first-order WKB approximation to the quantum-mechanical density of states.

The next step in the MUCS procedure is to convert the classical variables X_c and P_c into quantum-mechanical operators in the usual way ($x_c \to x$, $p_c \to -i\hbar d/dx$, $E \to H$). In this process it may be necessary to adjust H by a zero-point contribution and to symmetrize the operators appropriately. The resulting (Hermitian) quantum-mechanical operators will obey a commutation relation

$$[X,P] = iG\,, \tag{1.7}$$

where G will generally not be a c number.

Note that, in every example to which this method has been applied, the operators X and P have nonvanishing matrix elements only between adjacent energy eigenstates. That is, these natural quantum operators can be written as the Hermitian sums and differences of raising and lowering operators. For convenience we shall call this the "off-diagonal property" and note that although x and p are off-diagonal for the harmonic oscillator, and X and P as defined above are off-diagonal for anharmonic potentials, x and p are *not* off-diagonal for anharmonic potentials.

The last steps in the MUCS procedure begin by finding the three-parameter set of states (minimum-uncertainty states) that yields an equality for the uncertainty relation defined by Eq. (1.7):

$$(\Delta X)^2(\Delta P)^2 \geq \tfrac{1}{4}\langle G\rangle^2\,. \tag{1.8}$$

It is a consequence of the off-diagonal property that the ground-state wave function is a minimum-uncertainty state (corresponding to special values of $\Delta X/\Delta P$, $\langle X\rangle$, and $\langle P\rangle$). We specialize the minimum-uncertainty states by choosing the ground-state value for $\Delta X/\Delta P$, thereby obtaining a two-parameter set of minimum-uncertainty states that includes the quantum analog of a classical particle at rest. These are the "minimum-uncertainty coherent states."

In papers II (Ref. 2) and III (Ref. 3) of this series we applied our MUCS formalism to a collection of exactly solvable one-dimensional examples. Paper IV (Ref. 4) contained the generalization to multi-dimensional systems and discussed exactly solvable three-dimensional cases. These papers also discussed other coherent-states formalisms that we could generalize to anharmonic potentials.

Paper V (Ref. 5) was devoted to numerical studies of the time evolution of our minimum-uncertainty coherent states and examined their approximation to the classical motion and their coherence properties.[5,9] Our intuitive preference for the MUCS formalism comes from its physical basis in the approximation to the classical motion. But further, the numerical studies[5,9] revealed that the MUCS have coherence properties as good as or better than those obtained with other formalisms, for any given system.

In this concluding paper we discuss some points raised by our work. They amount to asking why the method works in the first place. Specifically, why do the natural quantum operators obtained from the natural classical variables have the off-diagonal property? Why does this property lead to a set of MUCS which approximates the classical motion? Finally, can this classical motion of our MUCS wave packets be understood in terms of the WKB approximation?

II. CLASSICAL AND QUANTUM FREQUENCIES

The WKB method is well known as an approximate quantum-mechanical technique that bridges the gap between classical and quantum treatments; hence the alternative name, semiclassical approximation. (We list a selection of the huge literature dealing with these topics in Refs. 10-18.)

The fact that our MUCS simulate the classical motion very well suggests that there should be some connection between these coherent states and the WKB approximation. Before dealing with this question we remark in this section on the connection between the classical frequency and the WKB approximation to the quantum frequency. This discussion is stimulated by the work of Klein,[15-17] Weldon,[16] and Li.[17]

These authors consider a bound-state system with eigenenergies $E(n)$ and define a frequency by

$$\omega(n)\equiv\frac{1}{\hbar}\frac{dE(n)}{dn}. \tag{2.1}$$

They then prove[19] that "as $n\to$ large it is this quantity $\omega(n)$ which is the classical frequency $[\omega_c(E(n))]$ entering into the Fourier analysis of Newton's equation for a periodic motion." This result is not surprising. Classically, the period $\tau(E)$ is given exactly by

$$\tau(E)=\frac{2\pi}{\omega_c(E)}=(2m)^{1/2}\int_{x_L}^{x_R}\frac{dx}{[E-V(x)]^{1/2}}. \tag{2.2}$$

Also, the WKB approximation states that

$$(n+\tfrac{1}{2})h=2(2m)^{1/2}\int_{x_L}^{x_R}[E-V(x)]^{1/2}dx. \tag{2.3}$$

Setting $E=E(n)$, taking d/dn of Eq. (2.3) and comparing this result to (2.2), yields

$$\omega(n)=\omega_c(E(n)). \tag{2.4}$$

Furthermore, defining the difference approximation to the classical angular velocity by

$$\Omega(n)\equiv\frac{1}{\hbar}\frac{\Delta E(n)}{\Delta n}=\frac{E(n+1)-E(n)}{\hbar}, \tag{2.5}$$

these authors also obtain, for large n,

$$\Omega(n)\simeq\omega_c(E(\bar{n})), \tag{2.6}$$

with

$$\bar{n}\equiv n+\tfrac{1}{2}. \tag{2.7}$$

Equation (2.5) is correct to order n^{-2}.

In our series of papers we have studied six exactly soluble quantum-mechanical systems: (i) the harmonic oscillator, (ii) the "harmonic oscillator with centripetal barrier" or "isotonic oscillator," (iii) the symmetric Pöschl-Teller potential, (iv) the symmetric Rosen-Morse potential, (v) the Morse potential, and (vi) the radial hydrogen atom. For each of these systems we have verified that Eq. (2.4) gives the exact classical frequency

$$\omega(n)=\omega_c(E(n)), \tag{2.8}$$

where $\omega_c(E(n))$ is the angular velocity of a classical particle with energy $E(n)$ in the appropriate potential. Note that our result is that Eq. (2.8) is *exact for all n*, not just for large n (see Table I). Further note that although this result might be expected for the harmonic oscillator, for which the WKB method is exact, it is surprising that it holds for some examples, such as the hydrogen atom for which the WKB series does not terminate. In this sense it is less surprising that, as also

TABLE I. For the six potentials listed, we give the angular velocity $\omega_c(E)$ of a classical particle with energy E, the quantum energy eigenvalues $E(n)$, the quantum angular velocity $\omega(n)$ defined by Eq. (2.5). The last column gives the ground-state value of $\frac{1}{2}\langle G\rangle/(\Delta P)^2$ which fixes this parameter for all members of the set of coherent states. We use the definitions $\mathcal{E}_0 \equiv \hbar^2\alpha^2/2m$ and $\mathcal{E}_R = 2me^4/\hbar^2$ and keep all dimensional constants. For the entry in the last row (column 6) we refer the reader to paper IV (Ref. 4) since a different convention was used for this example.

Potential	$\omega_c(E)$	$E(n)$	$\omega(n) \equiv \frac{1}{\hbar}\frac{dE(n)}{dn}$	$\Omega(\bar{n})$, $\bar{n}=n+\frac{1}{2}$	$\frac{\langle G\rangle}{2(\Delta P)^2}$
$\frac{1}{2}m\omega^2x^2$	ω	$\hbar\omega(n+\frac{1}{2})$	ω	ω	$1/(m\omega)$
$\mathcal{E}_0\nu^2(\alpha x-1/\alpha x)^2$	$4\mathcal{E}_0\nu/\hbar$	$2\mathcal{E}_0\nu[2n+1+(\nu^2+\frac{1}{4})^{1/2}-\nu]$	$4\mathcal{E}_0\nu/\hbar$	$4\mathcal{E}_0\nu/\hbar$	$1/(2\nu\hbar\alpha^2)$
$U_0\tan^2\alpha x$ $U_0\equiv\mathcal{E}_0\lambda(\lambda-1)$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0+E}{\mathcal{E}_0}\right]^{1/2}$	$\mathcal{E}_0[2n\lambda+n^2+\lambda]$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0+E(n)}{\mathcal{E}_0}\right]^{1/2}$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0+E(\bar{n})}{\mathcal{E}_0}\right]^{1/2}$	$1/[(\lambda+\frac{1}{2})\hbar\alpha^2]$
$U_0\tanh^2\alpha x$ $U_0=\mathcal{E}_0 s(s+1)$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E}{\mathcal{E}_0}\right]^{1/2}$	$\mathcal{E}_0[2ns-n^2+s]$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E(n)}{\mathcal{E}_0}\right]^{1/2}$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E(\bar{n})}{\mathcal{E}_0}\right]^{1/2}$	$1/[(s-\frac{1}{2})\hbar\alpha^2]$
$U_0(1-e^{-\alpha x})^2$ $U_0=\mathcal{E}_0\lambda^2$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E}{\mathcal{E}_0}\right]^{1/2}$	$\mathcal{E}_0[2\lambda(n+\frac{1}{2})-(n+\frac{1}{2})^2]$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E(n)}{\mathcal{E}_0}\right]^{1/2}$	$\left(\frac{2\mathcal{E}_0}{\hbar}\right)\left[\frac{U_0-E(\bar{n})}{\mathcal{E}_0}\right]^{1/2}$	$1/[(\lambda-1)\hbar\alpha^2]$
$-\frac{e^2}{r}+\frac{L^2}{2mr^2}$	$\frac{4\lvert E\rvert^{3/2}}{\mathcal{E}_R^{1/2}\hbar}$	$\frac{-E_R}{4n^2}$	$\frac{4\lvert E(n)\rvert^{3/2}}{\mathcal{E}_R^{1/2}\hbar}$	$\frac{4\lvert E(\bar{n})\rvert^{3/2}}{\mathcal{E}_R^{1/2}\hbar}\,\frac{1}{1-\frac{1}{2}(\bar{n})^{-2}+\frac{1}{16}(\bar{n})^{-4}}$	See paper IV, Eq. (4.24)

shown in Table I, Eq. (2.6) is exact (for all n) in every example except the hydrogen atom (for which it is correct to order n^{-2}).

We will find these results useful in the following sections. (In the Appendix we will comment on why first-order WKB calculations can yield exact results for certain systems.)

III. CONNECTION BETWEEN THE CLASSICAL AND QUANTUM MOTION

One of the most striking results to emerge from our study is the off-diagonal property of the natural quantum operators X and P. Why do the natural classical variables defined in Sec. I always yield quantum operators with the off-diagonal property?

Although we do not have a formal proof of this result, it is true for our solvable examples. Its physical basis is made clear by the following discussion, which yields some insight into the connection between classical and quantum mechanics and the often used semiclassical bridge, the WKB approximation.

Because the natural classical variables are sinusoidal with the classical angular velocity ω_c, one has [Eqs. (1.1) and (1.2)]

$$\dot{X}_c = \dot{x}X' = P_c/m\,, \tag{3.1}$$

$$\dot{P}_c = -m\omega_c{}^2X_c\,. \tag{3.2}$$

The corresponding natural quantum operators are

$$X = X_c(x)\,, \tag{3.3}$$

$$P = -\frac{i\hbar}{2}\left(X'\frac{d}{dx}+\frac{d}{dx}X'\right). \tag{3.4}$$

They satisfy (by definition) the first quantum equation of motion

$$\dot{X} = -\frac{i}{\hbar}[X,H] = P/m\,. \tag{3.5}$$

Now let us digress to consider a pair of (Hermitian) operators $\hat{X}$ and $\hat{P}$ that have the off-diagonal property:

$$\hat{X}\lvert n\rangle = x_n^-\lvert n-1\rangle + x_n^+\lvert n+1\rangle\,, \tag{3.6}$$

$$\hat{P}\lvert n\rangle = -i(p_n^-\lvert n-1\rangle - p_n^+\lvert n+1\rangle)\,, \tag{3.7}$$

where $x_n^\pm$ and $p_n^\pm$ are real and assumed known. If $\hat{X}$ and $\hat{P}$ are related by Eq. (3.5) then it follows that

$$p_n^\pm = \pm\frac{m}{\hbar}(E_{n\pm1}-E_n)x_n^\pm. \tag{3.8}$$

Using this and the approximate WKB result from Sec. II one can examine the second quantum equa-

tion of motion:

$$\dot{\hat{P}}|n\rangle = -\frac{i}{\hbar}[\hat{P},H]|n\rangle$$

$$= -\frac{m}{\hbar^2}[[\hat{X},H],H]|n\rangle$$

$$= -\frac{m}{\hbar^2}[(E_n - E_{n-1})^2 x_n^- |n-1\rangle + (E_{n+1} - E_n)^2 x_n^+ |n+1\rangle] \quad (3.9a)$$

$$\equiv -m[\Omega^2(n-1)x_n^-|n-1\rangle + \Omega^2(n)x_n^+|n+1\rangle] \quad (3.9b)$$

$$\simeq -m\omega_c^2(E(\bar{n}))\hat{X}|n\rangle . \quad (3.9c)$$

Therefore, in the WKB approximation, Hermitian operators with the off-diagonal property and related by Eq. (3.5) will obey the classical equations of motion.

Observe that this means that one can, in principle, obtain the natural classical variables (and hence a solution to the classical problem) from a complete solution to the quantum problem: One constructs the off-diagonal operators related by Eq. (3.5) and finds their classical analogs. This is related to the empirical observation, especially familiar from one-dimensional examples, that often those problems solvable in closed form classically are also solvable in closed form quantum mechanically (and vice versa).

The above off-diagonal property of the natural quantum operators is a map from q-number objects (the natural quantum operators themselves) to nondiagonal c-number objects (the $x_n^\pm$ and the $p_n^\pm$). The construction of the "classical" MUCS depends on the existence of these maps. This mapping is reminiscent of the quantum methods of Klauder,[20] and Schweber,[21] which were applied to specific systems by Hammer, Shrauner, and De Facio.[22] There the idea was to use Feynman's path-integral formalism in the space of raising and lowering operators with Gaussian measure.

Finally, recall that the MUCS minimize the uncertainty relation defined by the natural quantum operators X and P. But in the WKB approximation X and P follow the same equations of motion as do the natural classical variables X_c and P_c. Therefore it is not surprising that the (initially highly localized) MUCS themselves follow the classical motion.

IV. THE GROUND STATE AS A COHERENT STATE

Given the off-diagonal property of the natural quantum operators, the fact that the ground state belongs to the set of MUCS follows. This is an important fact because it means that the MUCS set contains a quantum representation of a classical particle at rest.

The states that minimize the uncertainty relation (1.7) are given[1] by the solutions to the eigenvalue equation

$$X + \left[\frac{i\langle G\rangle}{2(\Delta P)^2}P\right]\psi = \left[\langle X\rangle + \frac{i\langle G\rangle\langle P\rangle}{2(\Delta P)^2}\right]\psi . \quad (4.1)$$

We want to show that the ground-state wave function satisfies this equation. X and P have the off-diagonal property given by Eqs. (3.6) and (3.7), from which it follows that their matrix elements are related as in Eq. (3.8). In particular,

$$\langle 0|X|0\rangle = \langle 0|P|0\rangle = 0 . \quad (4.2)$$

Then

$$(\Delta X)_0^2 = \langle X\rangle_0^2 = |x_0^+|^2 , \quad (4.3)$$

$$(\Delta P)_0^2 = \frac{m^2}{\hbar^2}(E_1 - E_0)^2|x_0^+|^2 , \quad (4.4)$$

$$\langle G\rangle_0 = -i\langle 0|[X,P]|0\rangle = \frac{2m}{\hbar}(E_1 - E_0)|x_0^+|^2 . \quad (4.5)$$

Thus Eq. (4.1) is satisfied by the ground-state wave function, and the three-parameter set of minimum-uncertainty states that satisfy Eq. (4.1) can be restricted to the two-parameter minimum-uncertainty coherent states by choosing the value of the parameter $\frac{1}{2}\langle G\rangle/(\Delta P)^2$ to be the ground-state value

$$\frac{\langle G\rangle}{2(\Delta P)^2} = \frac{\Delta X}{\Delta P} = \frac{\hbar}{m(E_1 - E_0)} . \quad (4.6)$$

The discussion above has been in the context of one-dimensional problems, for which Eq. (3.5) holds. These results agree with the calculations of papers II and III as listed in Table I. For the three-dimensional problems discussed in paper IV the operators X and P are not related as simply as in (Eq. 3.5); that is, P is not simply $m\dot{X}$. Instead, there is an additional x dependence (r dependence) in the relation between X_c and P_c, namely,[4]

$$\dot{X}_c = -\frac{g(E_c,L_c)}{m}\frac{P_c}{r^2} . \quad (4.7)$$

In the quantum version one has

$$\dot{X} = \frac{-i}{\hbar}[X,H] = \frac{-g}{2m}\{1/r^2,P\} . \quad (4.8)$$

This means that $p_n^\pm$ is not related to $x_n^\pm$ by a simple energy-dependent factor as in Eq. (3.8). This alters Eq. (4.4), but the factor $|x_0^+|^2$ remains and the result still holds that the ground state can be made into a coherent state by taking $\Delta X/\Delta P$ to be $(\Delta X)_0/(\Delta P)_0$. The explicit factors are calculated in paper IV.

V. UNDERSTANDING THE CLASSICAL MOTION

Ideally, one wants coherent-state wave packets that (i) are well localized in position and momentum, with mean positions $\langle x(t)\rangle$ that move (ii) with the classical angular velocity $\omega_c(E)$ and (iii) with the classical amplitude $A(E)$, and (iv) that change shape as little as possible.

As we have discussed, especially in papers I (Ref. 1) and V,[5] it is only for the harmonic-oscillator coherent states that all these properties are perfectly realized. Even for the harmonic oscillator with centripetal barrier (isotonic oscillator),[2] a system with equally spaced levels, these properties are only partly achieved. For example, the shape of the packet changes with time but is restored at the end of every classical period because the eigenfrequencies are multiples of each other.

For more general systems with unequally spaced levels the component frequencies of the wave packet are incommensurate so the packet must spread and ultimately lose coherence. As a result, the coherent states cannot follow the classical motion perfectly for all time. One seeks states that, by some measure, maximally approach all of the desirable properties (i)–(iv). Obviously an arbitrarily chosen wave packet (however well localized) cannot achieve this because not only the intial shape of the packet but also the phase information (velocity) must be correctly chosen.

The MUCS are one way of approaching the stated goals. In some situations, certain other coherent state prescriptions have been observed to approximate very closely the classicality of our MUCS.[5] There may be a broad range of such approximate classical states. In certain situations, however, the MUCS succeed when alternative approaches fail[5] and we know of no instance in which other definitions of coherent states produce better results than the MUCS. For the above reasons and the fact that the MUCS method is physically based on the classical motion, we have preferred it.

Having obtained these states which act so classically, one can ask how they can be prepared and used in real physical situations. In paper I (Ref. 1) we have already mentioned possible applications to quantum-field theory,[22-29] to which we have added the references in Sec. III.[20,21] Our original motivation for this study (see the first article in Ref. 7) was the fact that some molecular physicists find that classical calculations of the interaction of radiation with molecules do much better than they had reason to believe *a priori*. Finally, nuclear physicists may find coherent states useful to describe collective motions.[29,30] Thus, these states and their properties are of potential use in studying coherent physics in real physical situations, not to mention the understanding they shed on how quantum mechanics can be made to mimic classical mechanics.

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APPENDIX: COMMENTS ON EXACT WKB CALCULATIONS

In this paper we observed that an intuitive physical understanding of our results can be had through the WKB formalism. As discussed in Sec. II and listed in Table I,

$$\frac{1}{\hbar}\frac{dE(n)}{dn}\equiv\omega(n)=\omega_c\,(E(n)) \tag{A1}$$

for the potentials we have considered. Further, for these same potentials, Krieger[12-14] and Rosenzweig[12,13] have observed that if one writes

$$\hat{V}(x)=V(x)+V_\rho(x)\,, \tag{A2}$$

where $V_\rho(x)$ is a well-defined analytic function, then WKB energies obtained from the formula

$$h(n+\tfrac{1}{2})=2(2m)^{1/2}\int_{x_L}^{x_R}[E-\hat{V}(x)]^{1/2}dx \tag{A3}$$

are exact. Also, the second- and third-order WKB corrections are zero.

Finally, Dashen, Hasslacher, and Neveu[31,32] have developed a semiclassical WKB approximation for field theory based on the formalisms of Gutzwiller[33] and Maslov.[34] In particular, they[35] and Luther[36] demonstrated that this formalism yields exact spectra for the quantum sine-Gordon theory.

Can one understand why all these particular problems can be made to yield exact answers in the WKB approximation? We observe that, if so, the pertinent phrase seems to be[35-37] "exact integrability." For the sine-Gordon system,[35] "In order to display the canonical structure of the sine-Gordon equation, Faddeev and Takhtajan[38] and McLaughlin[39] have shown that if one writes the nonlinear problem as a Hamiltonian system, one can show that it is integrable. That is, the linear eigenvalue problem of the inverse scat-

tering method is interpreted as a canonical transformation which takes the original Hamiltonian system to an 'action-angle' form."

For the classical quantum-mechanical potential problems discussed by us[1-5] and Krieger[12-14] and Rosenzweig,[12,13] exact integrability can be expressed as the simple statement that $x_c(t)$ can be given exactly in terms of transcendental functions, not just as an implicit integral.

A further justification for this intuitive speculation is to look at two systems where WKB does not give exact answers. In Ref. 13, Rosenzweig and Krieger thought that they could obtain exact WKB eigenvalues for the potential

$$V(x) = A e^{2ax} + B e^{-2ax}, \tag{A4}$$

which is a system they knew is not analytically solvable. However, later[14] they discovered that this is not the case.

Recently, Abraham and Moses[40] have discussed the potential

$$V = \tfrac{1}{2} m\omega^2 x^2 + 4\hbar\omega\phi(\phi - z), \tag{A5}$$

$$z = (m\omega/\hbar)^{1/2} x, \tag{A6}$$

$$\phi = \frac{e^{-z^2}}{\pi^{1/2}\,\mathrm{erfc}(z)}. \tag{A7}$$

This potential has the same eigenenergies as the harmonic oscillator, but with the ground state removed:

$$E_n = \hbar\omega(n + \tfrac{1}{2}), \quad n = 1, 2, \ldots. \tag{A8}$$

However, the classical problem has not been solved analytically. Even so, it has been shown[41] by both analytic approximation and numerical techniques that the classical angular velocity is *dependent* on energy. Therefore,

$$\frac{1}{\hbar}\frac{dE(n)}{dn} = \omega \neq \omega_c(E). \tag{A9}$$

Because $\omega_c(E)$ is dependent on energy, this also means that the Kreiger-Rosenzweig technique cannot yield the correct eigenvalues, except by the trivial artifice of removing the second term in (A5).

The above arguments are intuitive, and may not be exact. However, they do serve as a reason to see if other systems can yield exact WKB results only if they are exactly integrable.

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