Inequivalence of the classes of classical and quantum harmonic potentials: Proof by example

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There are an infinite number of potentials whose classical angular velocity ω_c is independent of energy. There are also an infinite number of potentials whose quantum eigenenergies are equally spaced, $\hbar\omega_q$. If $\omega_c = \omega_q$, then in the WKB approximation these classes are the same. (Certain specific potentials belong to both of these classes exactly.) However, we show by a different specific example that in general these classes are not equivalent.

I. INTRODUCTION

The simple harmonic-oscillator potential,

$$V(x) = \tfrac{1}{2}m\omega^2 x^2 , \tag{1.1}$$

is perhaps the most studied potential, for both the classical and quantum problems. Its special properties are many. For instance, its classical angular velocity $\omega_c(E)$ or, alternatively, its period

$$\tau_c(E) = 2\pi/\omega_c(E) , \tag{1.2}$$

is independent of energy,

$$\omega_c(E) = \text{const} \equiv \omega_c = \omega . \tag{1.3}$$

The quantum analog of Eq. (1.3) is that the quantum eigenenergies are equally spaced:

$$(E_{n+1} - E_n)/\hbar \equiv \omega_q(n) = \text{const} \equiv \omega_q = \omega, \quad n = 0, 1, \ldots . \tag{1.4}$$

Equation (1.4) is the quantum analog of the harmonicity in (1.3) for the following reason: In any potential having equally spaced eigenenergies, a wave packet, no matter what its shape, will return to its original $t = t_0$ shape whenever

$$t = t_0 + 2\pi j/\omega_q, \quad j = \text{integer} . \tag{1.5}$$

This can be seen by decomposing any time-dependent state into eigenstates,

$$\Psi(x,t) = \sum_n a_n \psi_n(x) \exp[-i\omega_q t(n + \tfrac{1}{2})] , \tag{1.6}$$

and observing that the equal spacing of the levels means that

$$\Psi^*(x,t_0)\Psi(x,t_0) = \Psi^*(x, t = 2\pi j/\omega_q + t_0) \times \Psi(x, t = 2\pi j/\omega_q + t_0) . \tag{1.7}$$

Further, for the harmonic oscillator one can construct a particular set of states, called the "coherent states (of the harmonic oscillator)," which follow the motion of a classical particle as well as possible.[1] These states have expectation values $\langle x(t)\rangle$ given by the values $x_c(t)$ that a classical particle with energy

$$E = \langle H\rangle - E_0 \tag{1.8}$$

would have. Also, these states have wave packets that do not change their shapes with time. (They have the shape of the ground-state Gaussian, only displaced.)

There is another exactly solvable system which also has many of these properties. This is the "harmonic oscillator with centripetal barrier" (HOCB) or "isotonic oscillator" potential,[2,3]

$$V(x) = U_0(ax - 1/ax)^2, \quad U_0 \equiv \nu^2 \mathcal{E}_0 \equiv \lambda(\lambda+1)\mathcal{E}_0, \quad \mathcal{E}_0 = \frac{\hbar^2 a^2}{2m} . \tag{1.9}$$

Here the classical angular velocity is also independent of energy,

$$\omega_c = 4\mathcal{E}_0\nu/\hbar , \tag{1.10}$$

and the eigenvalues are equally spaced,

$$E_n = \mathcal{E}_0\nu(4n + 3 + 2\lambda - 2\nu), \quad n = 0, 1, 2, \ldots \tag{1.11}$$

so that

$$\omega_q(n) = (E_{n+1} - E_n)/\hbar = \omega_q = \omega_c . \tag{1.12}$$

The "minimum-uncertainty coherent states"[1,2,4,5] for this system have been studied.[2,3] They have many of the classical properties of the harmonic-oscillator coherent states, even though these coherent states do change their shapes during an oscillation. (They must, since the potential is not symmetric about $ax = 1$.)

As we will discuss later, there are an infinite number of confining classical potentials whose

angular velocities ω_c are independent of energy.[6] We call these generalized *classical harmonic potentials*. There are also an infinite number of quantum potentials whose eigenenergies are equally spaced with $\Delta E_n/\hbar = \omega_q$. We call these generalized *quantum harmonic potentials*. The question we ask is, when $\omega_c = \omega_q$ are these two classes of potentials the same?

Each of the two potentials above belongs to both of these classes. Further, as will be shown in Sec. II, the two classes are the same in the WKB approximation. However, by particular example we will demonstrate that they are not the same in general. We will show that there is a potential which exactly has equally spaced quantum eigenvalues, but which does not have a classical angular velocity that is independent of energy. This potential is the one obtained by Abraham and Moses[7] (AM) using the Gel'fand-Levitan formalism.[8] (By itself the Gel'fand-Levitan formalism can be used to generate an infinite number of potentials whose quantum eigenvalues are equally spaced.) In Sec. III we give our demonstration by example, and close with a discussion.

II. GENERALIZED HARMONIC POTENTIALS

The classical period in any potential with a single minimum is given by

$$\tau_c(E) = (2m)^{1/2} \int_{x_L(E)}^{x_R(E)} \frac{dx}{[E - V(x)]^{1/2}} = 2\pi/\omega_c(E) , \tag{2.1}$$

where $x_{L,R}(E)$ are the classical turning points for energy E. Allowing the minimum value of the potential, V_m, to be away from the origin, the analysis of Landau and Lifshitz[6] shows that

$$x_R(V) - x_L(V) = \frac{1}{\pi(2m)^{1/2}} \int_{V_m}^{V} \frac{\tau_c(E)dE}{(V - E)^{1/2}} . \tag{2.2}$$

If $\tau_c(E) = \tau_c = 2\pi/\omega_c$ is independent of energy, then

$$x_R(E) - x_L(E) = \left(\frac{2}{m}\right)^{1/2} \frac{2}{\omega_c} (E - V_m)^{1/2} . \tag{2.3}$$

There are an uncountable number of potentials, V, which satisfy this formula. It is simple to verify that among them are the two exact special cases we have mentioned. For the simple harmonic oscillator,

$$x_{R,L} = \pm\left(\frac{2E}{m\omega_c^2}\right)^{1/2} , \tag{2.4}$$

which yields (2.3) since $V_m = 0$. For the $V_m = 0$ HOCB potential, one has that

$$ax_{R,L} = \left(1 + \frac{1}{4}\frac{E}{U_0}\right)^{1/2} \pm \frac{1}{2}\left(\frac{E}{U_0}\right)^{1/2} , \tag{2.5}$$

so that

$$x_R - x_L = \frac{1}{a}\left(\frac{E}{U_0}\right)^{1/2} = 2\left(\frac{2E}{m\omega_c^2}\right)^{1/2} . \tag{2.6}$$

Lastly, Eq. (2.1) allows us to show that in the WKB approximation the class of generalized classical harmonic potentials is the same as the class of generalized quantum harmonic potentials. This follows since in the WKB approximation

$$(n + \tfrac{1}{2})h = 2(2m)^{1/2} \int_{x_L}^{x_R} [E - V(x)]^{1/2} dx . \tag{2.7}$$

Setting $E = E_n$, taking d/dn of Eq. (2.7), and comparing this result to (2.1) gives[9]

$$\omega_c(E_n) = \frac{1}{\hbar}\frac{dE_n}{dn} . \tag{2.8}$$

Thus, for equally spaced levels, with

$$E_n = \hbar\omega_q(n + \text{const}) \tag{2.9}$$

one has

$$\omega_c(E_n) = \omega_q = \text{const} , \tag{2.10}$$

and the two classes of potentials are the same in the WKB approximation.

III. THE EXAMPLE

Using the Gel'fand-Levitan formalism,[8] one can consider a confining potential, remove any particular bound state corresponding to a particular eigenvalue, and leave all others unchanged. In particular, one can start with the harmonic-oscillator potential and remove the ground state. This is what Abraham and Moses (AM) have done explicitly.[7] (Observe that in this way alone one can generate a countable infinity of generalized quantum harmonic potentials. One first removes the ground state, then the first excited state, and so on.)

Using the dimensionless variables

$$z = ax, \quad a = (m\omega_q/\hbar)^{1/2} , \tag{3.1}$$

$$V = \hbar\omega_q v, \quad E_n = \epsilon_n \hbar\omega_q , \tag{3.2}$$

the AM potential is[7]

$$v = v_0 + v_1 , \tag{3.3}$$

$$v_0 = \tfrac{1}{2}z^2, \tag{3.4}$$

$$v_1 = 4\phi(\phi - z), \tag{3.5}$$

$$\phi(z) = \frac{e^{-z^2}}{\pi^{1/2}\,\mathrm{erfc}(z)}, \tag{3.6}$$

where $\mathrm{erfc}(z)$ is the complementary error function

$$\mathrm{erfc}(z) = \frac{2}{\pi^{1/2}} \int_z^\infty e^{-t^2} dt\,. \tag{3.7}$$

As explained in AM, the eigenvalues and normalized eigenstates of the problem can be derived from the Gel'fand-Levitan formalism as

$$\epsilon_n = n + \tfrac{1}{2}, \quad n = 1, 2, \ldots \tag{3.8}$$

$$\chi_n(z) = \psi_n(z) - \left(\frac{2}{n}\right)^{1/2} \phi(z)\psi_{n-1}(z), \tag{3.9}$$

where $\psi_n(z)$ are the ordinary harmonic-oscillator eigenfunctions

$$\psi_n(z) = (\pi^{1/2} 2^n n!)^{-1/2} e^{-z^2/2} H_n(z), \quad n = 0, 1, 2, \ldots\,. \tag{3.10}$$

For our purposes it is enough simply to verify that Eqs. (3.8) and (3.9) satisfy the Schrödinger equation and that the $\chi_n(z)$ are orthonormal.[10]

We plot v_0, v_1, and v in Fig. 1. v_1 satisfies the limits

$$\lim_{z \to \begin{cases} +\infty \\ 0 \\ -\infty \end{cases}} v_1(z) = \begin{cases} 2 \\ 4/\pi = 1.273\,24 \\ 0\,. \end{cases} \tag{3.11}$$

Numerical analysis shows that the minimum of v, v_m, and its location z_m are

$$v_m = 0.982\,68, \quad z_m = -0.826\,98\,. \tag{3.12}$$

Using the fact that (prime denotes d/dz)

$$\phi' = 2\phi(\phi - z), \tag{3.13}$$

a power-series expansion about the minimum gives

$$v(z) = v_m + 0 + \tfrac{1}{2}v''(z_m)(z - z_m)^2 + \cdots, \tag{3.14}$$

where

$$v''(z) = 1 + 16[(\tfrac{3}{2}z - z^3)\phi + (-2 + 7z^2)\phi^2 - 12z\phi^3 + 6\phi^4], \tag{3.15}$$

so that

$$v''(z_m) = 1.161\,73\,. \tag{3.16}$$

This by itself shows that the AM potential is not a harmonic classical potential. For large z the AM potential approaches the harmonic oscillator, which in these units means

$$\lim_{\epsilon \to \infty} \omega_c^2(\epsilon) = v''(\infty) = 1 \equiv \omega_q^2\,. \tag{3.17}$$

However, Eq. (3.15) tells us that

$$\lim_{\epsilon \to v_m} \omega_c^2(\epsilon) = v''(z_m) = 1.161\,73\,. \tag{3.18}$$

Thus, we have shown that $\omega_c(\epsilon)$ is dependent on energy and not equal to the constant $\omega_q = 1$. In terms of periods $\tau_c = 2\pi/\omega_c$, Eq. (3.18) tells us that

$$\lim_{\epsilon \to v_m} \frac{\tau_c(\epsilon)}{\tau_q} = [v''(z_m)]^{-1/2} = 0.927\,78\,. \tag{3.19}$$

IV. DISCUSSION

By using the dimensionless form of Eq. (2.1), the quantity

$$P(\epsilon) \equiv \tau_c(\epsilon)/\tau_q \tag{4.1}$$

can be numerically calculated for the AM potential. This is shown in Fig. 2 and Table I. At $\epsilon = v_m$, $P(\epsilon)$ has the correct limiting value 0.927 78 of Eq. (3.19). For $\epsilon > v_m$, $P(\epsilon)$ rises and passes unity just before $\epsilon = 2$. $P(\epsilon)$ continues to rise until it reaches a very small maximum of about 1.0015

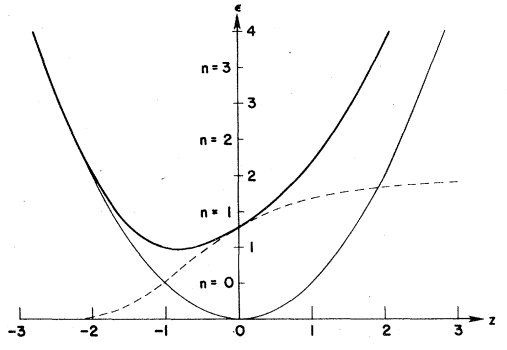

FIG. 1. The harmonic-oscillator potential $v_0(z)$ is a light curve, the contribution $v_1(z)$ is a light dashed curve, and the complete AM potential $v(z)$ is a heavy curve. The number eigenstates are also indicated.

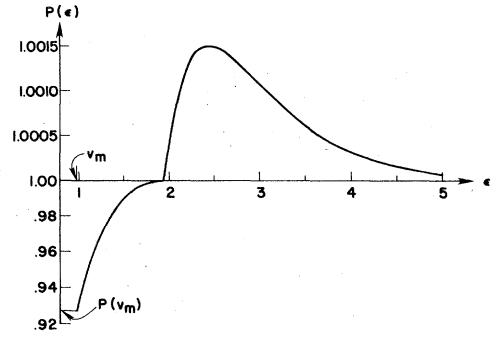

FIG. 2. $P(\epsilon)$ plotted as a function of ϵ. There are two scales for $P(\epsilon)$. The scale for $P(\epsilon) > 1$ is greatly expanded.

TABLE I. z_L, z_R, and $P(\epsilon)$ are shown for increasing values of ϵ. The first row gives the numerical values of the quantities v_m, z_m, z_m, and $[v''(z_m)]^{-1/2}$, respectively.

ϵ	z_L	z_R	$P(\epsilon)$
0.982 68	−0.826 98	−0.826 98	0.927 78
0.985	−0.889 39	−0.762 96	0.928 28
0.99	−0.936 81	−0.712 05	0.929 32
1.00	−0.994 08	−0.647 83	0.931 39
1.25	−1.439 26	−0.038 33	0.970 33
$\frac{4}{\pi}$ = 1.273 24	−1.463 52	0.0	0.972 79
1.50	−1.660 51	0.321 79	0.989 31
1.75	−1.831 42	0.605 67	0.997 34
2.00	−1.977 39	0.843 65	1.000 40
2.25	−2.108 06	1.050 28	1.001 37
2.50	−2.228 19	1.234 11	1.001 49
2.75	−2.340 49	1.400 61	1.001 32
3.00	−2.446 64	1.553 47	1.001 06

near $\epsilon = 2.5$. (Note that Fig. 2 has two scales.) After this $P(\epsilon)$ rapidly approaches its asymptotic value of unity. [At $\epsilon = 8$, $P(\epsilon)$ deviates from unity by less than one part in 10^6.]

Although we have proven that the AM potential has an energy-dependent $\tau_c(\epsilon)$, this dependence is actually very slight. The ground state of this potential is $\epsilon_0 = 1.5$, and $P(\epsilon_0) = 0.989\,32$. This means that it is only for classical energies below the quantum ground-state energy that $P(\epsilon)$ deviates from unity by more than one percent. Therefore, the result of Sec. II that a quantum harmonic potential, in the WKB approximation, is also a classical harmonic potential is not invalidated. The imprecision of this approximation is small and only significant for classical energies below the ground-state energy.

The shape of the curve for $P(\epsilon)$ can also be understood on intuitive physical grounds. To do this first observe that a particle in a symmetric confining potential

$$V(x) = K|x|^k \tag{4.2}$$

will have energy eigenvalues which, in the WKB approximation, vary as[11]

$$E_n \propto n^{2k/(k+2)}. \tag{4.3}$$

From Eq. (2.8) this means that

$$2\pi/\tau_c(E_n) = \omega_c(E_n) \propto n^{(k-2)/(k+2)} \propto (E_n)^{(1/2-1/k)}, \tag{4.4}$$

or

$$\frac{d\tau_c(E)}{dE} \propto (1/k - \tfrac{1}{2}) E^{(1/k-3/2)}. \tag{4.5}$$

Equation (4.5) implies that

$$\frac{d\tau_c(E)}{dE} \gtreqless 0 \quad \text{as} \quad k \lesseqgtr 2. \tag{4.6}$$

Now applying the above to the AM potential, for large energies the AM potential is barely contained within the harmonic-oscillator potential. This means that it has an effective $k(E)$ slightly larger than 2 which approaches 2 as ϵ gets large. From (4.6) this implies that, for large E, $d\tau_c/d\epsilon$ should be slightly negative with τ_c approaching un unity as ϵ gets large. Contrariwise, Eq. (3.19) tells us that at low energies $\tau_c(\epsilon)$ is below unity. These two observations mean that $\tau_c(\epsilon)$ must have a maximum above unity.

Another way to see the energy dependence of $\tau_c(\epsilon)$ is to observe that Eq. (2.3) is not satisfied. That is,

$$1 \neq \frac{8(\epsilon - v_m)}{(z_R - z_L)^2}, \tag{4.7}$$

which can be shown from the numerical values given in Table I.[12]

One of the implications of our findings is that the coherent states for this potential will not follow the motion of a classical particle exactly. Because the AM potential is a quantum harmonic potential with period τ_q, the coherent-state wave packets will oscillate with period τ_q for all values of ϵ. They will change their shapes slightly during an oscillation, but will return to their original shapes after every oscillation. However, since the AM potential is not an harmonic classical potential, but has an energy-dependent $\tau_c(\epsilon)$, this means that for very large times, a wave packet will eventually get out of phase with the position of a classical particle with the same energy $E_c = \langle H \rangle - E_0$ and starting position $\langle x(0) \rangle = x_c(0)$.

In conclusion, observe that we have proven that all quantum harmonic potentials are not classical harmonic potentials. Whether the reverse statement holds remains an open question.

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[9]Although Eq. (2.8) is derived only in the WKB approximation, it turns out to be *exact for all n* for a number of potentials which are exactly solvable. See, especially, the table in Ref. 5.
[10]Proving the orthonormality of the eigenfunctions χ_n resolves down to showing that

$$\int_{-\infty}^{\infty} \frac{e^{-3z^2}}{[\mathrm{erfc}(z)]^2} H_n(z)H_m(z)dz$$

$$= \frac{\pi^{1/2}}{2} \int_{-\infty}^{\infty} \frac{e^{-2z^2}}{[\mathrm{erfc}(z)]} [H_n(z)H_{m+1}(z) + H_{n+1}(z)H_m(z)]dz$$

for all n and m. This can be done by writing both Hermite polynomials of the form $H_{j+1}(z)$ as

$$H_{j+1} = 2zH_j - H_j' ,$$

and then integrating *one* of the integrals involving an H_j' by parts.
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[12]A new quantum harmonic potential would be obtained by removing the ground state of the AM potential. Its classical angular velocity might have a greater energy dependence than that of the AM potential. However, for the same reasons as given in this section, the energy dependence should remain modest.