Spinor electrodynamics as a dynamics of currents

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Spinor electrodynamics, consisting of the minimally coupled Dirac and Maxwell equations, is shown to be equivalent to sixteen equations for sixteen currents J^0, J^a, $L^{[ab]}$, K^a, K^0, consisting of one scalar bilinear identity, a vector set of four quintic differential equations of third order, a skew tensor set of six cubic identities, an axial-vector set of four bilinear compatibility relations of first order, and one pseudoscalar bilinear identity. The conservation of the vector current J^a follows from the vector set of equations, and the partial conservation of the axial-vector current K^a follows from the remaining twelve equations. There is an additional independent constraining scalar bilinear identity which ensures that these twelve equations are consistent with conservation of the vector current.

I. INTRODUCTION

The results reported in this paper arose out of an effort aimed at confronting the nonlinearity of spinor electrodynamics more directly than is customary in perturbation theory by looking upon Dirac's equation for the four components of the spinor field ψ and their adjoints $\bar{\psi}=\psi^\dagger\gamma^4$

$$i\gamma^a\psi_{,a}+e\gamma^a A_a\psi=m\psi\,,\quad -i\bar{\psi}_{,a}\gamma^a+e\bar{\psi}\gamma^a A_a=m\bar{\psi} \tag{1.1}$$

as equations for the four components of the vector potential A, whose solutions

$$eA^a=(1/2J^0)[2mJ^a-L^{ab}{}_{,b}+i(\bar{\psi}^{,a}\psi-\bar{\psi}\psi^{,a})] \tag{1.2}$$

when substituted into Maxwell's equations

$$F^{ab}{}_{,b}=4\pi eJ^a \tag{1.3}$$

yield a vector set of four differential equations of third order which turn out to be quintic in the sixteen real currents

$$J^0=\bar{\psi}\psi\,,\quad J^a=\bar{\psi}\gamma^a\psi\,,\quad L^{ab}=i\bar{\psi}\gamma^{[a}\gamma^{b]}\psi\,,$$
$$K^a=i\bar{\psi}\gamma^5\gamma^a\psi\,,\quad K^0=\bar{\psi}\gamma^5\psi\,. \tag{1.4}$$

The representations and conventions

$$\vec{\gamma}=\begin{pmatrix}0 & -\vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\quad \gamma^4=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}=-\gamma_5\,, \tag{1.5}$$
$$\delta^{ab}=\delta_{ab}=\mathrm{diag}(-1,-1,-1,+1)\,,\quad \epsilon_{1234}=-\epsilon^{1234}=+1\,,$$
$$\gamma^{[a}\gamma^{b]}=\tfrac{1}{2}(\gamma^a\gamma^b-\gamma^b\gamma^a)$$

are used throughout this paper.

The completion of this system requires an additional 12 independent equations which are derived in Sec. II. Since the gauge-invariant definition (1.4) of the 16 linearly independent real currents in terms of 4 complex spinor components with one arbitrary phase provides only 7 determining real numbers, there must exist 9 independent nonlinear identities between the currents. Such identities are usually written in bilinear form, following the precedent of Pauli.[1] An examination of the complete set of 136 interdependent bilinear identities (given in the Appendix) obtainable by the method of Fierz[2] reveals, however, that the complete set of independent identities consists of a skew tensor set of 6 cubic identities and 3 bilinear (2 scalar, 1 pseudoscalar) identities of the Pauli type. There is also an axial-vector set of 4 compatibility relations, first discovered by Zhelnorovich,[3] which are bilinear differential equations of first order. They arise in the present approach immediately from the observation that instead of solving Dirac's equations for J^0A^a as in (1.2) one can solve alternatively for K^0A^a and the two results must be compatible. Upon substitution of the six cubic identities, the pseudoscalar identity, and one of the scalar identities into the Zhelnorovich equations, one obtains the partial conservation of the axial-vector current, $K^a{}_{,a}=2mK^0$, by contraction with J^a. Contraction with K^a shows that the other scalar identity is a constraint which ensures consistency with the conservation of the vector current, $J^a{}_{,a}=0$, obtained from the vector set of equations.

The elimination of the electromagnetic field is undertaken in Sec. III. The possibility of expressing the field tensor F^{ab} entirely in terms of the currents and their derivatives arises out of a remarkable cubic identity of first order, yielding for $(J^0)^3F^{ab}$ an expression of second order which is irreducibly cubic. Differentiation shows then that $(J^0)^4F^{ab}{}_{,b}$ is irreducibly quartic and of third order, and upon substitution of Maxwell's equations (1.3) one ends up with a vector set of 4 differential equations of third order which are quintic owing to the term $(J^0)^4J^a$.

Section IV contains remarks about the possibly general significance of the special results reported in this paper.

II. IDENTITIES AND COMPATIBILITY RELATIONS

The definitions (1.4) imply the sixteen independent linear identities

$$J^A = \tfrac{1}{4}\operatorname{Tr}(\omega\Gamma^A) \quad (A=1,\ldots,16), \tag{2.1}$$

where $\omega = J^B\Gamma_B$ or, in components,

$$\omega_{\beta\alpha} = (J^0 I + J^a\gamma_a + \tfrac{1}{2}iL^{ab}\gamma_{[a}\gamma_{b]} + iK^a\gamma_5\gamma_a + K^0\gamma_5)_{\beta\alpha} = 4\bar\psi_\alpha\psi_\beta . \tag{2.2}$$

The permissible relabeling of spinor indices $\bar\psi_\alpha\psi_\beta\bar\psi_\gamma\psi_\lambda = \bar\psi_\alpha\psi_\lambda\bar\psi_\gamma\psi_\beta$ yields upon multiplication with $\Gamma^A_{\alpha\beta}\Gamma^B_{\gamma\lambda}$ the 136 interdependent bilinear identities[4]

$$J^AJ^B = \tfrac{1}{16}\operatorname{Tr}(\omega\Gamma^A\omega\Gamma^B). \tag{2.3}$$

They enable one to eliminate the spin currents L^{ab} through the skew tensor set of six cubic identities (see Appendix)

$$L^{ab} = \frac{J^0\epsilon^{abcd}J_cK_d + K^0(J^aK^b - J^bK^a)}{J^0J^0 + K^0K^0} \tag{2.4}$$

and to establish that the remaining currents J^0, K^0, J^a, K^a are subject to only three more independent bilinear Pauli identities

$$J^0J^0 + K^0K^0 = J^aJ_a , \tag{2.5}$$

$$J^aJ_a = -K^aK_a , \tag{2.6}$$

$$J^aK_a = 0. \tag{2.7}$$

Since Dirac's equations (1.1), treated as algebraic equations for the components of the vector potential A, allow the alternative solutions

$$eA^a = (1/2K^0)[-\tfrac{1}{2}\epsilon^{abcd}L_{cd,b} + i(\bar\psi^{,a}\gamma^5\psi - \bar\psi\gamma^5\psi^{,a})], \tag{2.8}$$

one has at once the compatibility relations of Zhelnorovich [Ref. 3, Eq. (2.19)]

$$mK^0J^a - \tfrac{1}{2}K^0L^{ab}{}_{,b} + \tfrac{1}{4}(J^0\epsilon^{abcd}L_{cd,b} + J^{b,a}K_b - J_bK^{b,a}) = 0 \tag{2.9}$$

if use is made of the bilinear identities of first order

$$(\bar\psi\psi^{,a})(\bar\psi\gamma^5\psi) + (\bar\psi\psi)(\bar\psi^{,a}\gamma^5\psi) - (\bar\psi^{,a}\psi)(\bar\psi\gamma^5\psi) - (\bar\psi\psi)(\bar\psi\gamma^5\psi^{,a}) = \tfrac{1}{8}\operatorname{Tr}(\omega^{,a}\gamma^5\omega) - (J^0K^0)^{,a} = \tfrac{1}{2}i(J^{b,a}K_b - J_bK^{b,a}) \tag{2.10}$$

obtainable by judicious use of permissible relabeling of spinor indices.

Upon substituting the six identities (2.4) into the Zhelnorovich equations (2.9), taking care to not as yet inadvertently invoke the three Pauli identities, one obtains by contraction with J_a

$$mK^0J^aJ_a - \tfrac{1}{2}(J^0J^0 + K^0K^0)K^a{}_{,a} + \tfrac{1}{4}(J^aJ_a - J^0J^0 - K^0K^0)_{,b}K^b - \tfrac{1}{4}(J^aK_a)_{,b}J^b = 0 , \tag{2.11}$$

and contraction with K_a yields

$$mK^0J^aK_a - \tfrac{1}{2}(J^0J^0 + K^0K^0)J^a{}_{,a} - \tfrac{1}{4}(K^aK_a + J^0J^0 + K^0K^0)_{,b}J^b + \tfrac{1}{4}(J^aK_a)_{,b}K^b = 0 . \tag{2.12}$$

If one invokes now the scalar identity (2.5) and the pseudoscalar identity (2.7), one obtains from (2.11) the partial conservation of the axial-vector current,

$$K^a{}_{,a} = 2mK^0 , \tag{2.13}$$

and (2.12) reduces to

$$J^bJ_bJ^a{}_{,a} + \tfrac{1}{2}(K^aK_a + J^aJ_a)_{,b}J^b = 0 , \tag{2.14}$$

showing that the other scalar identity (2.6) is a constraint ensuring consistency with the conservation of the vector current,

$$J^a{}_{,a} = 0 , \tag{2.15}$$

which is a consequence of Maxwell's equations.

III. ELIMINATION OF THE ELECTROMAGNETIC FIELD

The solutions (1.2) enable one to write the gauge-invariant electromagnetic field tensor $F^{ab} = 2A^{[a,b]}$ in the form

$$eF^{ab} = (1/J^0)(2mJ^{[a,b]} + L^{c[a,b]}{}_{,c}) + (1/J^0J^0)(2mJ^{0,[a}J^{b]} - J^{0,[a}L^{b]c}{}_{,c}) + 2G^{ab} , \tag{3.1}$$

where

$$G^{ab} = (i/J^0J^0)\,[\bar\psi\psi)(\bar\psi^{,[a}\psi^{,b]}) + (\bar\psi\psi^{,[a})(\bar\psi^{,b]}\psi)] \tag{3.2}$$

can be expressed entirely in terms of the currents and their first derivatives on account of the remarkable cubic identity

$$G^{ab} = (i/64)(J^0)^{-3}\operatorname{Tr}(\omega^{,a}\omega^{,b]}\omega) \tag{3.3}$$

obtained, using the definition (2.2), by permissible relabeling of spinor indices.

The evaluation of G^{ab} requires consideration of 125 sets of traces. Of these, 84 vanish identically, 28 contribute terms which cancel by antisymmetry and the contributions of the remaining 13 sets of traces add up to

$$G^{ab} = \tfrac{1}{16}(J^0)^{-3}[2K^0J^{c,[a}K_c{}^{,b]} - 8J_cK^{c,[a}K^{0,b]} + 3L_{cd}(J^{c,a}J^{d,b} - K^{c,a}K^{d,b}) - L_{cd}L^{ce,a}L_e{}^{d,b}] . \tag{3.4}$$

Since it is impossible to extract from the traces a common factor J^0, $(J^0)^3 F^{ab}$ is irreducibly cubic and of second order in the currents. Upon formation of the divergence $F^{ab}{}_{,b}$ and its substitution into Maxwell's equations (1.3) one ends up with the vector set of four quintic equations of third order,

$$4\pi e^2 (J^0)^4 J^a = \text{(irreducibly quartic and of third order in the currents)}. \quad (3.5)$$

Writing out the right-hand side of (3.5) is straightforward, but lengthy, and is not needed for the purpose of this paper. This completes the task of reformulating spinor electrodynamics entirely as a dynamics of currents.

IV. REMARKS

Although the power and usefulness of Faraday's field concept as an aid to calculations and as a mnemonic device of enormous propagandistic persuasiveness is beyond question, seen from the viewpoint adopted in this paper, the concepts of the electromagnetic potential, the electromagnetic field, and the spinor field amplitude are artifices which ultimately should be eliminated. The physical reality of electrodynamics resides in the currents, and the formalism ought to reflect that.

The method practiced here can be applied to other branches of field theory, provided the number of components of the gauge field equals the number of components of the source field amplitude. For example, the totally antisymmetric contortion tensor coupled minimally to the spinor field in the Einstein-Cartan theory[5] meets that criterion.

The elimination of the fields demonstrated above shows that the minimal coupling of linear field equations via bilinear interaction terms is deceptive because it hides the high degree of nonlinearity of the system of equations resulting from the elimination of the "vertex part," accomplished here at the classical level.

This system of equations contains not only the salient features of classical electron theory, such as the radiation damping indicated by the appearance of third derivatives, but also all quantum-mechanical features of one-electron theory, such as the diffusion of the wave packet, and the *Zitterbewegung*. It does not incorporate the exclusion principle, and the so-called radiation corrections, which can show up only after quantization of the theory. Therefore, exact solutions of the system (2.4), (2.5), (2.7), and (3.5) with the constraint (2.6), if such can be found, will describe solitons[6] only.

It is not obvious how to effect quantization of the currents directly[7,8] without relapsing into the quantization of fields which is inextricably tied to the interaction picture and perturbation expansions in powers of e^2. In full awareness of the remaining problem of quantization, the results reported in this paper are put forward as a classical starting point for an alternative formulation of quantum electrodynamics which aims at bypassing perturbation theory and the perplexities[9] associated with the "true" vertex part in the standard approach.

APPENDIX

Since the bilinear identities appear to have been given only partly in the literature, a complete list of all 15 sets of 136 such identities is recorded below. It required the evaluation, by the standard method, of 375 sets of traces, 85 of which yielded nonvanishing contributions:

$$6J^0J^0 = -2K^0K^0 + 2J^aJ_a - 2K^aK_a + L^{ab}L_{ab}, \quad (A1)$$

$$2J^0J^a = \epsilon^{abcd}L_{bc}K_d, \quad (A2)$$

$$2J^0L^{ab} = 2\epsilon^{abcd}J_cK_d - \epsilon^{abcd}K^0L_{cd}, \quad (A3)$$

$$2J^0K^a = \epsilon^{abcd}L_{bc}J_d, \quad (A4)$$

$$8J^0K^0 = \epsilon^{abcd}L_{ab}L_{cd}, \quad (A5)$$

$$4J^aJ^b = 2\delta^{ab}(J^0J^0 + K^0K^0 - J^cJ_c - K^cK_c) + 4K^aK^b + 4L^{ac}L_c{}^b + \delta^{ab}L^{cd}L_{cd}, \quad (A6)$$

$$J^aL^{bc} = \epsilon^{abcd}J^0K_d + \delta^{ab}(K^0K^c - J_dL^{dc}) - \delta^{ac}(K^0K^b - J_dL^{db}) + J^bL^{ac} - J^cL^{ab}, \quad (A7)$$

$$2J^aK^b = 2J^bK^a - 2\delta^{ab}J^cK_c + 2K^0L^{ab} - \epsilon^{abcd}J^0L_{cd}, \quad (A8)$$

$$J^aK^0 = K_bL^{ba}, \quad (A9)$$

$$4L^{ab}L^{cd} = 4L^{ac}L^{bd} - 4L^{ad}L^{bc} + 4(L^{ae}L_e{}^c\delta^{bd} - L^{ae}L_e{}^d\delta^{bc} + L^{be}L_e{}^d\delta^{ac} - L^{be}L_e{}^c\delta^{ad})$$
$$+ (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})(L^{ef}L_{ef} + 2J^0J^0 - 2K^0K^0 + 2J^eJ_e - 2K^eK_e) + 4(J^aJ^d - K^aK^d)\delta^{bc}$$
$$- 4(J^aJ^c - K^aK^c)\delta^{bd} + 4(J^bJ^c - K^bK^c)\delta^{ad} - 4(J^bJ^d - K^bK^d)\delta^{ac} - 4\epsilon^{abcd}J^0K^0 , \tag{A10}$$

$$L^{ab}K^c = \epsilon^{abcd}J^0J_d + \delta^{ac}(J^bK^0 + L^{bd}K_d) - \delta^{bc}(J^aK^0 + L^{ad}K_d) + L^{ac}K^b - L^{bc}K^a , \tag{A11}$$

$$2L^{ab}K^0 = 2(J^aK^b - J^bK^a) + \epsilon^{abcd}J^0L_{cd} , \tag{A12}$$

$$4K^aK^b = 4J^aJ^b - 2\delta^{ab}(J^0J^0 + K^0K^0 + J^cJ_c + K^cK_c) - 4L^{ac}L_c{}^b - \delta^{ab}L^{cd}L_{cd} , \tag{A13}$$

$$K^aK^0 = J_bL^{ba} , \tag{A14}$$

$$6K^0K^0 = -2J^0J^0 + 2J^aJ_a - 2K^aK_a - L^{ab}L_{ab} . \tag{A15}$$

To obtain the six cubic identities (2.4) multiply (A3) with J^0, (A12) with K^0, and add them. The three bilinear Pauli identities (2.5), (2.6), and (2.7) follow from contractions of (A6), (A13), and (A8). These nine independent identities are now sufficient to satisfy the entire set.

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