Rotationally invariant field theory on lattices. III. Quantizing gravity by means of lattices

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Whether one views the lattice as merely regulatory or physically real, lattice gravity is different from continuum gravity because the symmetry group of coordinate transformations is destroyed by the lattice. Since negative-norm kinetic terms can no longer be removed by specifying a gauge, the functional integrals will not converge. We suggest including a quartic potential in the action which obliges the action to converge regardless of the sign of those terms. This potential must depend on a background metric in order that a number of desirable features may hold. For example, consider the conformal factor: One needs a quartic term to cancel the wrong-sign kinetic energy. But one also wants the theory to have asymptotically flat manifolds and to admit the flat manifold as a solution. This requires a canceling quadratic. This in turn requires a background structure. This structure is supplied in a natural way by the lattice itself. The approach to the continuum is examined.

I. INTRODUCTION

If one accepts the idea that functional integrals are (somehow) limits of multiintegrals on lattices,[1,2,3] then it is desirable to have a lattice version of every sector of physical theory currently available including gravity.[4,5] There are three views possible on the importance of lattices in field theory. One, they simply represen, a regulation scheme for continuum theories.[6,7] Two, since one does not yet know how to define precisely the field-theory limit of multiintegrals, other than perturbatively,[1,3,8] it is not merely desirable but absolutely essential to have a lattice version of each theory in order to claim that one has a well-defined theory and in order that one may examine that limit.[9] Three, because gravitation introduces a special scale, the Planck length,[10-15] there is a special lattice picked out by gravitation which thus deserves special study,[5,10,11,13] particularly in light of the renormalization difficulties presented by that scale. The first two view the lattice as being of transitory importance. The third suggests a physical reality for the Planck lattice, the one with 10^{-33} cm spacing. In any case, lattice gravity is worth studying.

What is unique about gravitation is that immediately upon separating the points in the derivative, i.e., regulating by going to a lattice, one *completely destroys the continuous symmetry group of the limit theory*. There are some difficulties even in gauge theories, as discussed in paper II,[10] but nothing as serious as this. Since the convergence of the functional integral depends essentially upon the existence of this symmetry group to reduce the degrees of freedom from ten to two, and since that group is destroyed immediately upon going to the lattice, we are confronted with a situation in which the usual, naive approaches to putting a theory on the lattice will not work.

As indicated in the abstract, consider the conformal factor Ω,

$$g_{\mu\nu}=\Omega^2\bar{g}_{\mu\nu}\,,\qquad \det\bar{g}=1\,,\qquad (\det g)^{1/2}=\Omega^4\,. \tag{1}$$

In order that the kinetic term for Ω converge we introduce a quartic potential (cosmological term) with an arbitrarily small, appropriately signed coefficient. In order to have flat space as one possibility we need a quartic. The two such terms one can construct without a background metric are inadequate. The lattice, with plausible assumptions about regularity and physical limits (small spacing, etc.), introduces such a background already. A short expansion of that argument is the following. The Euclidean functional integral does not converge if all components of the metric are integrated because some of them have the wrong sign.[16,17]

In the weak-field limit set

$$g_{\mu\nu}=\delta_{\mu\nu}+M_P^{-1}H_{\mu\nu}\,. \tag{2}$$

M_P is the Planck mass (~10^{19} GeV). The trace of $H_{\mu\nu}$ has the wrong sign for its kinetic energy since the action is[5,16]

$$-\mathcal{A}=-\int d^4x\Big(\tfrac{1}{4}f^{\mu\nu|\lambda}f_{\mu\nu|\lambda}-\tfrac{3}{32}h^{|\lambda}h_{|\lambda}-\tfrac{1}{2}f_{\nu\mu}{}^{|\mu}f^{\nu\lambda}{}_{|\lambda}+\tfrac{1}{4}f_{\nu\mu}{}^{|\mu}h^{|\nu}\Big)\,, \tag{3}$$

where $h=\delta^{\mu\nu}H_{\mu\nu}=\mathrm{tr}H$ is the trace and $f_{\mu\nu}=H_{\mu\nu}-\frac{1}{4}h\delta_{\mu\nu}$ is trace free. Usually this is not a problem because one can go to the transverse traceless gauge for a pure spin-two particle,

$$f_{\nu\mu}{}^{|\mu}=0\,,\qquad h=0\,. \tag{4}$$

Only the first term is left in Eq. (3).

In the strong-field region the problem is seen differently. The Einstein-Hilbert action splits as follows[11,15,16,18]:

$$-\mathcal{A}=\int d^4x(\Omega^2\bar{R}+6\partial_\mu\Omega\bar{g}^{\mu\nu}\partial_\nu\Omega)\,. \quad (5)$$

$\bar{R}$ is the curvature scalar of $\bar{g}$.

Clearly, the kinetic term for Ω has the wrong sign. But, as given by Feynman and Gupta,[19] this is just a self-consistently coupled power-series expansion (in M_P^{-1}), an interacting version of the fields which appear in the weak limit. It is still convergent if the spin-two fields only are quantized. This has been confirmed in Faddeev and Popov's[20] paper in which they quantize the pure spin-two Arnowitt-Deser-Misner (ADM)[21] structure in a gauge; then use ghosts to get the Einstein-Hilbert action back. Thus the general problem of convergence is cured by the gauge invariance. But on the lattice the gauge invariance is gone. Each component of g must converge.

On the lattice it is possible to assure convergence of the action functional for Ω if one includes an arbitrarily small, negative Ω^4 term in the action integral even if the kinetic term has the wrong sign. This term is a cosmological constant term since[18]

$$\lambda(\det g)^{1/2}=\lambda\Omega^4\,. \quad (6)$$

The reason this works is that on lattices the kinetic energy term is bounded by a constant times Ω^2 because the momentum is bounded.[10] Thus the quartic term will ultimately win out and cause convergence.

Of course, introducing a cosmological constant term wrecks asymptotic flatness and makes a constant flat space (time) an impossibility. It is possible if Ω has no potential as a boundary condition on the differential equations. If one wishes to include flat space (time) as an allowed solution to the equations of motion, one must alter the potential for Ω so that Ω, a constant, is permitted by the equations. Because of the usual degree limits on potentials one is motivated to look for quadratic terms and, in fact, we have a number of conditions we would like this potential to satisfy:

(1) be of degree four in Ω,

(2) the Ω functional integral converges (for bounded momenta),

(3) asymptotic flatness and flat space (time) are allowed,

(4) the other functional integrals ($\bar{g}_{\mu\nu}$) converge (for bounded momenta),

(5) the preferential effects are small enough to be experimentally compatible,

(6) the action is constructed from contracted tensors times the usual metric volume factor,

(7) the domain of the functional variables is not bounded.

If one does not make this assumption the potential can be replaced by a step-function domain of integration. The parameters in the potential induce a similar characteristic *large* (cosmological) distance.[22]

Because of the very nature of lattices' description, we are led to consider them as embeddings of a given original hypercubic lattice in higher-dimensional space. The preferred metric is the natural (flat) metric of the space to be embedded. See Secs. II, III, and IV and below in this introduction. The potential which satisfies all of these conditions is nearly unique. It is

$$[-2\lambda^2+\lambda^2 g^{\mu\nu}G_{\mu\nu}-\lambda'^2(g^{\mu\nu}G_{\mu\nu})^2]\sqrt{g}\,. \quad (7)$$

Here, $G_{\mu\nu}$ is the preferred lattice metric, which is taken in the usual coordinates to be $\delta_{\mu\nu}$. It is possible that for a certain range of λ'' one may also be able to include a term

$$-\lambda''^2 g^{\mu\nu}G_{\nu\rho}g^{\rho\lambda}G_{\lambda\mu}\sqrt{g} \quad (8)$$

and still achieve convergence.

The λ parameters measure the strength of the preferential background structure. Convergence occurs for all nonzero values of λ for all components of $g_{\mu\nu}$ by the same mechanism that worked for Ω even with the wrong sign in the kinetic energy. See Sec. VI.

It is possible to relate our potential to a loosening of the usual traceless metric constraint of the limit theory. Rewrite Eq. (8) as

$$-\lambda'^2\sqrt{g}\left[\left(\mathrm{tr}g^{-1}-\frac{\lambda^2}{2\lambda'^2}\right)^2-\left(\frac{\lambda^2}{2\lambda'^2}\right)\left(\frac{\lambda^2}{2\lambda'^2}-4\right)\right]. \quad (9)$$

Now we choose the ratio of λ's. Thus if

$$\lambda^2/2\lambda'^2=4 \quad (10)$$

we find the potential is

$$-\lambda'^2\sqrt{g}(\mathrm{tr}g^{-1}-4)^2\,. \quad (11)$$

In the expansion

$$g^{\mu\nu}=\delta^{\mu\nu}+\epsilon H^{\mu\nu} \quad (12)$$

we find the potential is

$$-\lambda'^2\sqrt{g}\,\epsilon^2\,\mathrm{tr}^2 H\,. \quad (13)$$

In the limit that λ'^2 goes to infinity the exponential integral becomes a functional δ function which projects out the negative-norm trace field. This seems justified on the basis of experiment[4,23] and because the physical vacuum will be a minimum for flat space.

It should not be surprising that metric theories behave differently on a lattice compared to the continuum. The very nature of their geometries differ. To even describe a lattice one must give the locations of each vertex relative to (enough) others or relative to some coordinate system. Then the (Euclidean) lengths of the separation vectors give rise to *natural* distances (after all, one needs to know not merely the angles but how far to go to locate the next point as being in the lattice). There is no abstract lattice in the sense of an abstract manifold. Indeed, a manifold is described using equivalence classes of coordinate charts; one need not have any concept of distance at all. Instead, one has the *option*[24] of imposing a local distance by introducing an arbitrary symmetric two-tensor for which arc length is defined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu . \tag{14}$$

Riemann, the inventor of the manifold, knew these were intrinsically different. In his *Habilitationsschrifte* he said, "In a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in the continuous one it must come from something outside."[25,26]

Knowing that we must ultimately relate any lattice theory of gravity to conventional theories, we examine the relationship between the metric and the lattice vectors. We are physically interested in only those manifolds which arise as limits of lattices (with metric). (See Sec. II.) In order that one have a number of desirable features such as asymptotic flatness, regularity (the lattice equivalent of isotropy), flat-space rotational invariance, the usual number of degrees of curvature, etc., we are led to considering ensembles of lattices which are hypercubical tessellations embedded in ten-dimensional space. See Fig. 1. The hypercubical tessellation of R^4 is the *unique* tessellation by a regular hypersolid.[27] In other words, we locate vertices in R^{10} by means of a function X which we think of as mapping not merely a single lattice $\mathrm{Z}^4 \subset \mathrm{R}^4$, but the ensemble of Poincaré relatives of Z^4 in R^4. This covers all of R^4; thus we consider maps from R^4 into R^{10} which have Fourier transforms in the ball of radius π/ϵ, with ϵ called the lattice spacing. This is, of course, in the spirit of the preceding papers. The choice of ten dimensions is motivated by the Janet-Burstin-Cartan[26,28] theorem we discuss below and in Sec. II.

Thus we shall define the metric as follows. First, the lattice derivation of X is

$$D_\mu X^A \big|_I = \epsilon^{-1}[X^A(I+\mu) - X^A(I)] . \tag{15}$$

Here I runs from 1 to 4; A runs from 1 to 10. The notation is explained further in Sec. III. $I+\mu$ means the location obtained by shifting I by adding 1 to the μth component, i.e., μ is given by

$$\mu = (0, \ldots, \overset{\mu}{1}, \ldots, 0) , \tag{16}$$

where 1 appears in the μth position. Now set

$$g_{\mu\nu} = D_\mu X^A D_\nu X^B \eta_{AB} . \tag{17}$$

η^{AB} is a flat metric in R^{10} (possibly indefinite).

In the limit as the lattice derivative approaches the actual derivative we find

$$g_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B \eta_{AB}. \tag{18}$$

When the number of variables in X^A matches those in $g_{\mu\nu}$, ten in four dimensions, one can solve this equation uniquely in a patch. This is the Janet-Burstin-Cartan (JBC) theorem.[28] The rigidity of the hypercubical lattice plus asymptotic flatness essentially restricts us to one patch (all of R^4). See Sec. III.

In a special coordinate system, perfect coordinates, one can write the embedding as

$$\begin{aligned} &X^A = (x^\alpha, u^a(x)) , \\ &x^\alpha \text{ in } \mathrm{R}^4 , \\ &u^a : \mathrm{R}^4 \to \mathrm{R}^6 . \end{aligned} \tag{19}$$

The last six coordinates are functions of the first four. In these coordinates the curved manifold is like a soap bubble stretching out from the plane in which the bubble-ring lies. See Fig. 2. The problem of defining the volume of the embedded lattice is introduced in Sec. II.

In Sec. IV we calculate curvature, connections, etc. for embedded submanifolds.

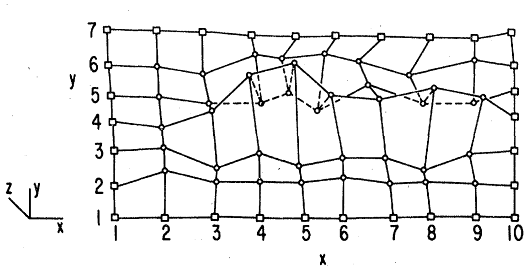

FIG. 1. A two-lattice (net) in three-space.

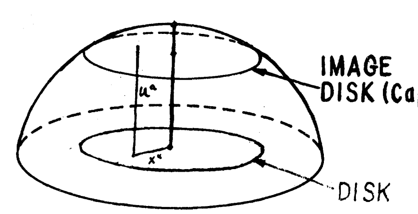

FIG. 2. Hemisphere described in perfect $(x^\alpha, u^a(x^\alpha))$ coordinates.

Next we consider the Einstein-Hilbert action in Sec. V and discuss the effect of using X^A vs $g_{\mu\nu}$ as variables. Because of the uniqueness of the JBC[26,28] theorem, the only effect should be a Jacobian. Then we introduce the conformal factors' convergence problem.

In Sec. VI we go through the details of Ω convergence on the lattice. We introduce the constraint of having a flat solution and examine available quartic potentials with that condition (and the others outlined above). The solution is not quite unique. It does involve the preferred lattice metric, but modifies the Einstein equations as little as possible. An example of a theory in which matter fields are included is written out.

In Sec. VII the structure of the action is formalized and the argument for convergence on $\bar{g}$ is described. We discuss the choice of quantum variables. This leads us into an examination of the Feynman-Gupta[19] limit of the theory and the question of positivity of the action.[17] We show the relation of our potential to the constraints for pure spin-two gravity. These are that the limit spin-two particle state should be symmetric, transverse, and traceless. The approach to that limit can involve the idea of functionally integrating on coordinate transformations. These have quanta which can be admitted on a lattice. They amount to phonons.

There is a brief synoptic conclusion and there are three appendices. The first defines some needed mathematical terminology. The second reviews the Regge[13] theory of lattice gravity and indicates why we have not simply used that theory, no metric, no coordinates. The last appendix discusses the caveats involved in claiming that only a Jacobian arises in changing variables.[29-32] It also contains an amusing speculation on curvature arising as a correction to a simpler theory (somewhat in the manner of Sakharov[33]).

II. THE ASSUMPTIONS ON THE SPATIAL STRUCTURE

Rather than consider the most general lattices possible, we shall make some physically reasonable assumptions which restrict the possibilities substantially. There exists a theory (Regge[13]) of curvature for general lattices. But it is not clear how to write down the action for coupled matter fields since no use of the metric or even coordinates is made (see Appendix B). We shall make assumptions which seem plausible for individual lattices but are also selected so that one can readily proceed to the ensemble. But in the end our technical assumptions can be replaced with one physical assumption: Space is described by one R^4 coordinate patch.

First, we assume that each cell in the four-lattice has a neighborhood which can be (smoothly) mapped onto a neighborhood of the flat four-lattice cell preserving the relations of the vertices, edges, etc. Note that our "lattices" include their vertices, edges, (etc.). [Perhaps "net" or "network" is a more descriptive name than lattice since vertices and links (etc.) are included in the structure.] Because of Nash's[34] isometric embedding theorem which says that any flat, *multi*-patch four-manifold can be embedded in R^{230} (or R^{46} if it is compact), we can continue each neighborhood map and view the totality as embedding a piece of R^4 into R^{230}. (We restrict the dimension shortly.) Second, we assume that space stripped of matter fields is isotropic. Thus we require that the bare lattice has cells which are identical and regular. While you can tile a floor with squares, triangles, or hexes, the unique tessellation of R^n for $n \geq 3$ using regular polytopes is the (hyper) cubic one.[27] We therfore assume that the flat lattice is hypercubic. In a sense this is a pity. If there were a tessellation with regular simplices it would be nicer because the cubic lattice is not as rigid as the simplicial lattice. We shall deal with this problem shortly. Third, in order to define scattering states we shall assume that space is asymptotically flat.

Our three assumptions imply that the entire lattice can be described on a single coordinate patch R^4. (If one wants, one can simply take this as our assumption.) This implication is more easily seen in two dimensions. The precise relationship between the number of vertices, edges emanating from them and the nature of the two-cells (squares) permits only bending. The problem occurs in trying to go from one lattice patch to another. Each overlaping map must be the identity in order that the cells in the overlap be squares also. The fact that the overlap between one cell and the next is the identity does the trick in higher dimension also. Since we assume asymptotic flatness, our lattice can be viewed as a map from a

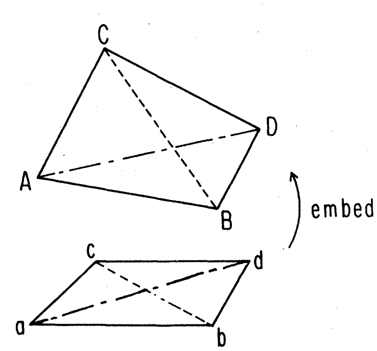

FIG. 3. A two-cell embedded in R^3.

hypercubical lattice which includes all of R^4 (but only one copy) into a space of higher dimension.

The most general metric on a single global coordinate patch can be obtained by embedding in R^{10}. This result is the Janet-Burstin-Cartan theorem. We shall restrict attention to this case unless otherwise indicated.

The regularity assumption presents us with something of a dilemma since the Ith quantity in the action sum is thus weighted by the volume of the Ith "curved" hypercubical cell. But this quantity is not necessarily well defined. For example, in two dimensions we can ask: What is the area of a square which is embedded in three-space while preserving its straight edges (perimeter embedding)? As can be seen in Fig. 3, the image vertices form vertices for a tetrahedron. One is then confronted with an ambiguity. Which pair of triangular faces yields the *correct* embedded area? This ambiguity does not persist if one embeds the innards of the square.

Three approaches are possible here. One, we can subdivide the initial square (hypercube) into triangles (four-simplices). Then we assign the sum of the embedded volumes of the collection of triangles (simplices) to the square (hypercube). Since there is no tessellation in terms of equilateral four-simplices, but instead only the cubic one, we must find a simplicial decomposition of the cube. The easiest to envision is the following. In two dimensions subdivide the square by introducing a point at its center and connect it to the vertices making four triangles. We will call this subdivision "barycentric."[35] In three dimensions, first make the barycentric subdivision of each face of the cube. This yields fourteen vertices, among other things. Now introduce one more vertex at the center; connect it to the fourteen other vertices. One finds $6\times 4=24$ simplices (tetrahedra). We shall call the process of introducing barycentric subdivisions of the faces, then the volume, then the hypervolumes, etc., the sequential barycentric (SB) subdivision. It is evident that in d dimensions a cube will be split into $2d!!/2=2^{d-1}d!$ simplices. Thus in 2, 3, 4 dimensions we find 4, 24, 192 subsimplices. The volume of the Ith cube is the sum of the volumes of the $2^{d-1}d!$ subsimplices. However, to preserve the original cubic structure we have matter fields only at the original 2^d vertices for a given cube and gauge fields only along the $d2^{d-1}$ original links.

Two, instead of being quite so literal about the concept of volume, we make the observation that thus far we have only needed the vertices and links, no areas, volumes, etc. Given only link lengths for the Ith cube, one can define a distorted cube's volume as a suitable product of its link's lengths.

Three, we can introduce a separate conformal field whose magnitude determines what one means by volume.

III. THE METRIC

Lattice separation vectors induce a natural metric. On the lattice a "metric" theory of gravity should be described in terms of the lattice vectors. We are only interested in manifolds which are limits of lattices and therefore inherit their natural metric structure. In this section we will examine the relation between the lattice vectors and the metric.

If we require that the undisturbed lattice reflect the principle of isotropy, we are led to regular lattices in four dimensions. The unique regular tessellation of R^4 is the hypercubical lattice.[27] If each "curved" hypercube has a neighborhood which can be smoothly mapped onto a piece of a hypercubical tessellation of R^4 (as in a manifold), the rigidity of the structure is such that the entire curved lattice can be described in terms of a single global coordinate patch.

If we wish to consider all possible $g_{\mu\nu}$'s in the continuum limit we can obtain them as follows.

Assume that there exists a map X from the lattice Z^4, viewed as a subset of R^4, into a space of dimension at least ten. Thus let the map X be a smooth embedding of R^4 into R^{10} which contains the lattice image

$$X: R^4 \to R^{10}, \quad \text{smoothly}. \tag{20}$$

The curved lattice vertices are expressed as

$$X(\epsilon I)\in R^{10}, \quad \text{if } I=(I_1,\ldots,I_4)\in Z^4; \tag{21}$$

we shall write

$$X_I \equiv X(I) \equiv X(\epsilon I). \tag{22}$$

The reason that we have chosen the number ten is that if we define for each ϵ

$$g_{\mu\nu}(\epsilon)\big|_I \equiv D_\mu X^A D_\nu X^B \eta_{AB}\big|_I, \tag{23}$$

η_{AB} is the R^{10} metric, with

$$D_\mu X = \epsilon^{-1}\int_{x=\epsilon I}^{x'=\epsilon(I+\mu)} dX \tag{24}$$

$$= \epsilon^{-1}[X_{I+\mu} - X_I], \tag{25}$$

$$I+\mu = I+(0,\ldots,1,\ldots,0),$$

where 1 appears in the μth position. In the limit as $\epsilon\to 0$,

$$g_{\mu\nu} = \lim_{\epsilon\to 0} g_{\mu\nu}(\epsilon)$$

$$= \partial_\mu X\cdot\partial_\nu X. \tag{26}$$

In the lattice definition an orientation is chosen by selecting which sector goes with which vertex. When there is a single global coordinate patch, ten dimensions suffice to obtain all possible $g_{\mu\nu}$ because of the Janet-Burstin-Cartan theorem (Sec. III).

Clearly it is *a priori* possible that X is a map into R^p for p greater than 10 (we overcount g's) or less that ten (if not all components of g are needed physically). But ten suffices to get all of g.

It is worth noting that the continuum limit is quite different from the lattice because of the enormous class of maps sending the plane onto itself invertibly. Even if we include infinite differentiability, these maps form the enormous group of diffeomorphisms. A single lattice is clearly *not* preserved under diffeomorphisms. The limit is special in that it "develops" a large invariance structure which is not there at any finite step. But since our lattice will be so small that differences are approximately the same as lattice derivatives, no classical experimental consequence should be observable. Note that once one has chosen the lattice sites, relabeling by using other coordinates is clearly possible. The action will be trivially invariant under this "alias" transformation.

While it is possible to view links as straight lines in R^{10} for an individual lattice, it makes more sense if they are geodesics (when the ensemble is embedded).

IV. EMBEDDINGS

As we did for other fields,[10] we will take the lattice gravitational action to be a sampling of the embedding functions on a particular lattice (in the spirit of the "net" terminology, this is a "casting"). In this section we will therefore give the essential equations for embedding manifolds.

Thus let $X: R^d \to R^{d+p}$ smoothly. Let X^A denote the components and let η_{AB} be the metric in R^{d+p}. We will actually be interested in the Euclidean case here but will give general formulas. η_{AB} has the following format:

$$\eta_{AB} = \begin{vmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \eta_{ab} \end{vmatrix}, \tag{27}$$

$$1 \le A, B \le d+p, \quad 1 \le \alpha, \beta \le d, \quad d+1 \le a, b \le d+p.$$

$\eta_{\alpha\beta}$ is the flat R^d metric. η_{ab} is the normal-space metric. We lower indices using η_{AB}. By identifying a, b with a pair of antisymmetrized indices $(\alpha\alpha') \to a$, $(\beta\beta') \to b$, we can give a natural metric on the normal space: $\eta_{\alpha\beta}\eta_{\alpha\beta'} - \eta_{\alpha\beta'}\eta_{\alpha\beta}$ for η_{ab}.

Now we can list the interesting geometrical objects,[18]

$$\begin{aligned} &g_{\mu\nu} = X^A{}_{|\mu}\eta_{AB}X^B{}_{|\nu} \quad \text{(induced) metric}, \\ &\Gamma^\lambda_{\mu\nu} = g^{\lambda\alpha}X^A{}_{|\alpha}\eta_{AB}X^B{}_{|\mu\nu} \quad \text{connection}, \\ &P^{AB} = \eta^{AB} - X^A{}_{|\alpha}g^{\alpha\beta}X^B{}_{|\beta} \quad \text{normal projector}, \\ &R_{\mu\alpha\nu\beta} = P_{AB}X^A{}_{|\mu[\nu]}X^B{}_{|\alpha[\beta]} \quad \text{curvature}, \\ &N^A_{\mu\nu} = P^A{}_B X^B{}_{|\mu\nu} \quad \text{normal tensor}, \\ &N^A = N^A_{\mu\nu}g^{\mu\nu} \quad \text{normal}. \end{aligned} \tag{28}$$

The normal projector satisfies $P^A{}_B X^B{}_{|\mu} = 0$, antisymmetrize bracketed indices.

The Ricci tensor is given by

$$R_{\mu\nu} = N_A X^A{}_{|\mu\nu} - N_{A\mu\alpha}X^A{}_{|\nu\beta}g^{\alpha\beta}. \tag{29}$$

The minimal volume or minimal hypersurface equations are gien by varying $\int \sqrt{g}\, d^4x$ with respect to X^A. These are soap-bubble surfaces,

$$\begin{aligned} 0 &= N^A \\ &= P^A{}_B X^B{}_{|\mu\nu}g^{\mu\nu} \\ &= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\, g^{\mu\nu}\partial_\nu X^A). \end{aligned} \tag{30}$$

Note that on a minimal hypersurface,

$$\begin{aligned} R_{\mu\nu} &= -N_{A\mu\alpha}X^A{}_{|\nu\beta}g^{\alpha\beta} \\ &= -P_{AB}X^A{}_{|\mu\alpha}X^B{}_{|\nu\beta}g^{\alpha\beta}. \end{aligned} \tag{31}$$

Introduce "perfect" coordinates which correspond to a perfect splitting of X into space(time) and "normal" coordinates. The formulas look simpler there:

$$\begin{aligned} &X^A = (x^\alpha, u^a), \\ &X^A{}_{|\mu} = (\delta^\alpha{}_\mu, u^a{}_{|\mu}), \\ &X^A{}_{|\mu\nu} = (0, u^a{}_{|\mu\nu}). \end{aligned} \tag{32}$$

Thus we find

$$\begin{aligned} &g_{\mu\nu} = \eta_{\mu\nu} + u^a{}_{|\mu}\eta_{ab}u^b{}_{|\nu}, \\ &\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\alpha}u^a{}_{|\alpha}u^b{}_{|\mu\nu}\eta_{ab}, \\ &P^{ab} = \eta^{ab} - u^a{}_{|\mu}g^{\mu\nu}u^b{}_{|\nu}, \\ &R_{\mu\alpha\nu\beta} = P_{ab}u^a{}_{|\mu[\nu]}u^b{}_{|\alpha[\beta]}. \end{aligned} \tag{33}$$

A hypersurface is minimal if and only if

$$u^a{}_{|\alpha\beta}g^{\alpha\beta} = 0. \tag{34}$$

V. THE EINSTEIN-HILBERT ACTION

Having introduced X one must choose appropriate dynamics. There are a number of questions about this choice which we will begin discussion of in this section.

If one starts from the Euclidean Einstein-Hilbert action with gravitational constant κ,

$$\mathfrak{A} = \kappa \int R\sqrt{g}\, d^4x \tag{35}$$

and varies with respect to X, the equations are

$$G^{\mu\nu}=R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}$$

and (36)

$$0=\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}G^{\mu\nu}\partial_\nu X^A)$$
$$=\Gamma^\alpha_{\mu\nu}G^{\mu\nu}X^A{}_{|\alpha}+G^{\mu\nu}x^A{}_{|\mu\nu}.$$

So far we have only used the existence aspect of the JBC theorem. But given appropriate initial (boundary) conditions, there is a uniqueness proof also. We point out that the Cartan proof of embedding[28] depends upon the Cartan-Kähler theorem[26] which in turn depends on the Cauchy-Kowalewski[26] theorem which means that the solution is uniquely determined by its initial (boundary) conditions. We assume that the limits of the functional integrals appropriately reflect this fact so that only one X^A to a $g_{\mu\nu}$ is permitted. Thus no extraneous solutions to Eq. (34) are allowed, only $G_{\mu\nu}=0$. There is still a question of functional Jacobian, but that should not affect the allowed classical solutions. Some further discussion of these points is found towards the end of the next section and in Appendix C.

To obtain the lattice versions of the usual geometrical objects, we replace the partial derivatives (differences). Thus

$$\begin{aligned}
g_{\mu\nu}&=D_\mu X^A\eta_{AB}D_\nu X^B\,,\\
\Gamma^\lambda_{\mu\nu}&=g^{\lambda\alpha}D_\alpha X^A\eta_{AB}D_\mu D_\nu X^B\,,\\
P^{AB}&=\eta^{AB}-D_\alpha X^A g^{\alpha\beta}D_\beta X^B\,,\\
R_{\mu\alpha\nu\beta}&=P_{AB}D_\mu D_{[\nu]}X^A D_\alpha D_{[\beta]}X^B\,,
\end{aligned}\tag{37}$$

$g^{\lambda\alpha}$ is, as usual, the inverse metrix to $g_{\alpha\mu}$. The most straightforward approach to gravitation is to consider the lattice version of the Einstein-Hilbert action

$$\mathcal{A}=\sum_I\sqrt{g}\,(\epsilon^4\kappa)P_{AB}D_\mu D_\nu X^A D_\alpha D_\beta X^B g^{\mu[\nu]}g^{\alpha[\beta]}\,.\tag{38}$$

Alternatively, since the lattice theory is no longer coordinate-transformation invariant, one might start from a specific coordinate system. Since the metric is nearly flat, $g=I+H$, with H small, and since perfect coordinates give a natural choice for $H=\partial u\otimes\partial u$, we will write out that example:

$$\mathcal{A}=\sum_I\sqrt{g}\,(\epsilon^4\kappa)P_{ab}D_\mu D_\nu u^a D_\alpha D_\beta u^b g^{\mu[\nu]}g^{\alpha[\beta]}\,.\tag{39}$$

Since we are interested in the "ensemble" of lattices, we will view the action as a sampling of the values of $X:\mathrm{R}^4\to\mathrm{R}^{10}$ or $U:\mathrm{R}^4\to\mathrm{R}^6$ on lattices of R^4. As in previous papers[10] all variables will be given support on the intersect of the first Brillouin zones of the lattices. This set is the ball $B_{\pi/\epsilon}(p)$ of radius π/ϵ in momentum space.

There are three immediately apparent difficulties with these choices of action: One, coordinate-relabeling invariance is gone on the lattice. Two, the action is of higher order in the (lattice) derivatives of u^a than is usual. Three, even without the use of embeddings the Einstein-Hilbert action suffers from the conformal factor problem. This problem is that the Einstein-Hilbert action can be arbitrarily negative.[16,17] This is a problem in quantizing $\sqrt{g}R$ as a kinetic term for g. It is a problem even in the conventional theory. If one sets

$$g_{\mu\nu}\equiv\Omega^2\bar{g}_{\mu\nu}\tag{40}$$

with

$$\det\bar{g}=1\,,$$

then (in Euclidean locally R^4 manifolds)

$$\int\sqrt{g}Rd^4x=\int(\bar{R}\Omega^2+6\partial_\mu\Omega\partial_\nu\Omega\bar{g}^{\mu\nu})d^4x\,.\tag{41}$$

The factor Ω^2 is quadratic to preserve the metric signature. We ignore surface terms. R as kinetic energy term for g has the opposite sign to that of whatever scalar fields there may be. Hence the kinetic energy of Ω (the conformal field) can be made arbitrarily large and it has the wrong sign. It therefore appears that the functional integral will diverge.

Another view of this problem persists even in the weak-field limit. Take

$$g_{\mu\nu}=\delta_{\mu\nu}+\epsilon H_{\mu\nu}\tag{42}$$

with $H_{\mu\nu}$ small compared with $\delta_{\mu\nu}$. Let

$$H_{\mu\nu}=f_{\mu\nu}+\tfrac{1}{4}\epsilon h\delta_{\mu\nu}\tag{43}$$

with $f_{\mu\nu}\delta^{\mu\nu}=0$. Note that

$$\Omega\approx1+\tfrac{1}{4}\epsilon h\approx e^{\epsilon h/4}\tag{44}$$

in the weak-field region. That is, the trace and the determinant are related,

$$\det g\approx\mathrm{tr}g-3.\tag{45}$$

For weak fields we can see that $\mathcal{L}=R\sqrt{g}$ becomes[5,16] (up to surface terms)

$$-\tfrac{1}{4}f_{\mu\nu|\lambda}f^{\mu\nu|\lambda}+\tfrac{3}{32}h^{|\lambda}h_{|\lambda}+\tfrac{1}{2}f_{\nu\mu}{}^{|\mu}f^{\nu\lambda}{}_{|\lambda}-\tfrac{1}{4}f_{\nu\mu}{}^{|\mu}h^{|\nu}\,.\tag{46}$$

The trace field h has the wrong sign for the kinetic energy. We see that only one of the modes of $g_{\mu\nu}$ has the negative sign problem. It appears, at least in the weak limit, that quantizing only the pure spin-two (i.e., trace free) piece of $H_{\mu\nu}$ eliminates the problem. The view that gravitation is a nonlinearly coupled theory on a flat background is at least consistent with our single flat-global coordinate patch view. This view is sometimes called the particle or spin-two version of gravity (as opposed to the geometric).[19,21]

To quantize gravity one begins in the spin-two formulation (ADM)[21] and goes to $\sqrt{g}R$ by introducing the Faddeev-Popov[20] gauge determinant, etc. Thus whatever wrong kinetic energy factors might appear to contribute will be cancelled by the ghosts. This cancellation depends essentially upon the freedom of choosing a gauge. One is not free to do that on the lattice.

VI. LATTICES, CONFORMAL CONVERGENCE AND CHOOSING THE ACTION

In the previous section we found that the most straightforward lattice extension of Einstein-Hilbert gravity has a number of difficulties: gauge noninvariance, order of derivatives in the action, and the conformal convergence problem (inherited from the continuum version). These are compounded by the problem of defining the lattice cell volume (see Sec. II).

In this section we shall examine the possibility of quantizing the $\sqrt{g}R$ lattice action. We cannot reduce the number of degrees of freedom from ten to two since we do not have gauge freedom. Thus we are once again confronted with the negative-norm problem.[16,17] We treat the conformal factor separately to isolate the difficulties.

There are two properties that we wish the conformal field action to have. One, we want it to converge. Two, we want asymptotic flatness. In the case of a conformally flat manifold

$$g_{\mu\nu}=\Omega^2\delta_{\mu\nu}\,, \tag{47}$$

and the kinetic term $\kappa\sqrt{g}R$ reduces to $6\kappa\,\partial\Omega\cdot\partial\Omega$. Because the lattice has a maximum momentum $p^2<\pi^2/\epsilon^2$, the kinetic piece of the action is bounded:

$$-\mathcal{A}_{\rm kin}=\int d^4x\,6\kappa\,\partial\Omega\cdot\partial\Omega<6\pi^2K\epsilon^{-2}\int d^4x\,\Omega^2 = 6\pi^2\epsilon^{-4}\int d^4x\,\Omega^2\,. \tag{48}$$

Of course, this is only the leading term of the lattice kinetic energy, but the same idea applies. We use only the first term for simplicity and have written out $-\mathcal{A}$ so that functional convergence is more apparent. Now include a cosmological constant term

$$\mathcal{A}_{CC}=\kappa\Lambda\int d^4x\sqrt{g}=6\pi^2\epsilon^{-4}\lambda^2\int d^4x\,\Omega^4\,, \tag{49}$$

$$\Lambda=6\pi^2\kappa\lambda^2=6\pi^2\epsilon^{-2}\lambda^2\,, \tag{50}$$

λ^2 can be arbitrary but not zero. The other factors $6\pi^2$ are extracted for ease, and will be reabsorbed later in this section:

$$-\mathcal{A}<6\pi^2\epsilon^{-4}\int d^4x(\Omega^2-\lambda^2\Omega^4) = -6\pi^2\epsilon^{-4}\lambda^2\int d^4x[(\Omega^2-\tfrac{1}{2}\lambda^{-2})^2-\tfrac{1}{4}\lambda^{-4}]\,. \tag{51}$$

The functional integral will now converge for every $\lambda\neq 0$.

This is nice except that we have lost asymptotic flatness. Once one has introduced a potential for Ω, which is what the cosmological constant term amounts to, the only way to admit solutions with Ω equal to one everywhere (in order to admit completely flat space as a possibility) is if the potential is chosen appropriately (without a potential one simply imposes a boundary condition on the homogeneous differential operator). Because of the quartic limit on potentials, we are led to considering a quatratic term in the manner of the Higgs potential. Including an Ω^2 term breaks coordinate-relabeling covariance.

Let us look at the ways in which we get terms proportional to Ω. Given the fact that we are looking at embeddings, there is the possibility of using the natural (flat) metric of the embedding space, not just the reduced one. Thus we find four possibilities:

$$\begin{aligned}&(1)\quad \Omega^2=(\det g)^{1/4}\,,\\ &(2)\quad \Omega^2\bar{R}=\sqrt{g}R-\tfrac{3}{8}(\sqrt{g}\,)^{-3/2}\partial\sqrt{g}\cdot\partial\sqrt{g}\,,\\ &(3)\quad \Omega^2\bar{g}^{\mu\nu}\delta_{\mu\nu}=g^{\mu\nu}\delta_{\mu\nu}\sqrt{g}\,,\\ &(4)\quad \Omega^2\bar{g}_{\mu\nu}\delta^{\mu\nu}=g_{\mu\nu}\delta^{\mu\nu}\,.\end{aligned} \tag{52}$$

It is important to note that in a different coordinate system the background metric $G_{\mu\nu}=\delta_{\mu\nu}$ will undergo the standard Jacobian coordinate transformation. The second case (2) will not help achieve a flat Higgs-type solution because $\bar{R}$ vanishes in that limit. The first is an improper degree tensor density.

If one includes gauge fields B_μ [in SU(n), e.g.] coupled to matter fields ϕ (in C^n, e.g.) in the usual way,[12] there is another term

$$\Omega^2\bar{g}^{\mu\nu}(\phi^\dagger_{|\mu}+i\phi^\dagger B_\mu)\cdot(\phi_{|\nu}-iB_\nu\phi) = g^{\mu\nu}(\phi^\dagger_{|\mu}+i\phi^\dagger B_\mu)\cdot(\phi_{|\nu}-iB_\nu\phi)\sqrt{g}\,. \tag{53}$$

But with the usual theories, its vacuum expectation value is zero. This will not help achieve flat space (time). Besides one would not expect to have to add in extra field structure to achieve flatness.

Of course, if in the macroscopic limit of the lattice theory, there is some remnant of the preferred lattice metric (the flat metric, that is) from experiment it must be a very small effect. But even the smallest quartic, negative term will make the full functional integral converge when there is a maximum momentum. Thus we shall consider the three obvious quartic terms for Ω and $\bar{g}_{\mu\nu} = \bar{V}^i_\mu \delta_{ij} \bar{V}^j_\nu$. $\bar{V}^i_\mu$ is the vierbein for $\bar{g}_{\mu\nu}$;

$$\begin{aligned} &\text{(a)}\ \Omega^4 = (\det g)^{1/2}, \\ &\text{(b)}\ (\bar{g}^{\mu\nu}\delta_{\mu\nu})^2 = (g^{\mu\nu}\delta_{\mu\nu})^2(\det g)^{1/2}, \\ &\text{(c)}\ \bar{g}^{\mu\lambda}\delta_{\lambda\rho}\bar{g}^{\rho\sigma}\delta_{\sigma\mu} = g^{\mu\lambda}\delta_{\lambda\rho}g^{\rho\sigma}\delta_{\sigma\mu}(\det g)^{1/2}. \end{aligned} \tag{54}$$

We will require that the potential satisfy the following conditions (assume the usual quartic restriction):

(1) Asymptotic flatness and flat space as an allowed solution;
(2) convergence of the Ω functional integral for the case of bound momenta;
(3) convergence of the functional integral for the other variables;
(4) proper coordinate-relabeling covariance (scalar density);
(5) the preferential effects are minimal;
(6) the domain of the functional variables is not bounded.

If one does not make this assumption the potential can be replaced by a step-function domain of integration. The parameters in the potential induce a similar large (cosmological) distance.[22]

Up to a small quibble, discussed later [Eq. (74)], the potential is unique:

$$\lambda^2\Omega^2(\bar{g}^{\mu\nu}\delta_{\mu\nu}) - 2\lambda^2\Omega^4 - \lambda'^2(\bar{g}^{\mu\nu}\delta_{\mu\nu})^2 = [\lambda^2(g^{\mu\nu}g_{\mu\nu} - 2) - \lambda'^2(g^{\mu\nu}\delta_{\mu\nu})^2]\sqrt{g}. \tag{55}$$

The modification of Einstein's equation is

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -(\lambda^2 - 2\lambda'^2 g^{\lambda\rho}\delta_{\lambda\rho})\delta_{\mu\nu} + \tfrac{1}{2}[\lambda^2(g^{\lambda\rho}\delta_{\lambda\rho} - 2) - \lambda'^2(g^{\lambda\rho}\delta_{\lambda\rho})^2]g_{\mu\nu}. \tag{56}$$

We discuss the physical limit of the λ parameters and their relation to gauge constraints in Sec. VII. For now we merely point out that the functional integral converges for all λ's.

Thus we have been motivated to consider the action

$$-\mathcal{A} = \int \{\kappa[\Omega^2\bar{R} + 6\partial\Omega\cdot\partial\Omega] - K^2[\lambda^2(2\Omega^4 - \Omega^2\bar{g}^{\mu\nu}\delta_{\mu\nu}) + \lambda'^2(\bar{g}^{\mu\nu}\delta_{\mu\nu})^2]\}d^4x. \tag{57}$$

It is the discrete version of this which is to be quantized on the lattice ensemble. We therefore consider the following lattice version of that action:

$$\begin{aligned} -\mathcal{A} &= \sum_I \{\kappa\epsilon^4[\Omega^2\bar{R} + 6D_\mu\Omega D_\nu\Omega\bar{g}^{\mu\nu}] - \kappa^2\epsilon^4[\lambda^2(2\Omega^4 - \Omega^2\bar{g}^{\mu\nu}\delta_{\mu\nu}) - \lambda'^2(\bar{g}^{\mu\nu}\delta_{\mu\nu})^2]\}\Big|_I \\ &= \sum_I [\Omega^2\epsilon^2\bar{R} + 6(\epsilon D_\mu\Omega)(\epsilon D_\nu\Omega)\bar{g}^{\mu\nu} + \lambda^2\Omega^2\bar{g}^{\mu\nu}\delta_{\mu\nu} - 2\lambda^2\Omega^4 - \lambda'^2(\bar{g}^{\mu\nu}\delta_{\mu\nu})^2]\Big|_I. \end{aligned} \tag{58}$$

On the second line we use the Planck lattice spacing equation

$$\kappa\epsilon^2 = 1. \tag{59}$$

The appearance of ϵ thereafter is for convenience only, one could as well use

$$\Delta_\nu\phi = \epsilon D_\nu\phi = \phi_{I+\nu} - \phi_I. \tag{60}$$

Inclusion of gauge fields coupled with matter fields: One constructs them as terms in addition to the one given above[10]:

$$-\mathcal{A}' = \sum_I \frac{\epsilon^4}{4e^2}\operatorname{tr}(F^\dagger_{\mu\nu}F_{\lambda\rho})\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho} - \frac{\epsilon^4}{2}\Omega^2\bar{g}^{\mu\nu}(\mathfrak{D}_\mu\Phi)^\dagger\mathfrak{D}_\nu\Phi - \Omega^4\epsilon^4(|\Phi|^2 - \mu^2)^2\Big|_I. \tag{61}$$

Here

$$\begin{aligned} U_{I+\nu,I} &= U(I+\nu, I) = P\left[\exp\left(\int_I^{I+\nu} W^A_\mu i\Lambda_A dx^\mu\right)\right], \\ F_{I\mu\nu} &\equiv \epsilon^{-2}\ln(U_{I,I+\mu}U_{I+\mu,I+\mu+\nu}U_{I+\mu+\nu,I+\nu}U_{I+\nu,I}), \end{aligned} \tag{62}$$

and

$$\begin{aligned} W^A_\mu i\Lambda_A &\equiv \Omega^{-1}B^A_\mu i\Lambda_A\Omega + \Omega^{-1}\partial_\mu\Omega, \\ \Phi &\equiv \Omega^{-1}\phi \end{aligned} \tag{63}$$

is a gauge-invariant parametrization of the fields. We refer to these as physical variables because they would give rise to the physical particle states in a Weinberg-Salam model.[22] Thus

$$\mathfrak{D}_\nu\Phi = \epsilon^{-1}[U(I+\nu, I)\Phi(I+\nu) - \Phi(I)]. \tag{64}$$

VII. THE LATTICE GRAVITY ACTION'S STRUCTURE

The total action we have found consists of a sum of terms constructed from the inverse metric and the volume determinant. So it is informative to organize the terms by order of g^{-1}. There are, therefore, three kinds of terms: $\sqrt{g}$, $g^{\mu\nu}\sqrt{g}$, and $g^{\mu\nu}g^{\lambda\rho}\sqrt{g}$ types. In terms of these are of order Ω^4, Ω^2 and $1 = \Omega^0$. If we limit Ω to having a power-series expansion (analytic as $\Omega \to 0$), then there can be no further higher order four-dimensional structures.

The most general appropriately ranked gauge-invariant tensors in a gauge-coupled theory are as follows:

$$\left| \begin{matrix} -\lambda_0{}^2 - \epsilon^4\hat{\lambda}(|\Phi|^2 - \mu^2)^2 \\ -\lambda_2{}^2 g_{\mu\nu} + \beta_2\delta_{\mu\nu} + \epsilon^2\gamma_2 R_{\mu\nu} - \epsilon^4(\mathfrak{D}_\mu\Phi)^\dagger \cdot \mathfrak{D}_\nu\Phi \\ -\lambda_4{}^2 g_{\mu\nu} g_{\lambda\rho} + \beta_4 g_{\mu\nu}\delta_{\lambda\rho} - \lambda'^-\delta_{[\mu\nu}\delta_{\lambda\rho]} - \lambda'^+\delta_{(\mu\nu}\delta_{\lambda\rho)} + \epsilon^2 R_{\mu\lambda\nu\rho} + \frac{\epsilon^4}{4e^2} F^\dagger_{\mu\lambda} \cdot F_{\nu\rho} \end{matrix} \right| . \tag{65}$$

$\delta_{[\mu\nu}\delta_{\lambda\rho]}$ is the $(\nu\rho),(\mu\lambda)$ antisymmetric object, $\delta_{(\mu\nu}\delta_{\lambda\rho)}$ is symmetric. We have suppressed $\lambda_4^\pm$ and $\beta_4^\pm$ since the contraction with

$$|1, g^{\mu\nu}, g^{\mu\nu}g^{\lambda\rho}|\sqrt{g}\,, \tag{66}$$

leads a coalescense of α terms and of β terms (and of γ terms). Thus we define

$$\begin{aligned} \lambda^2 &= \lambda_0{}^2 + \lambda_2{}^2 4 + \lambda_4{}^2 4^2\,, \\ \beta &= \beta_2 + \beta_4 4\,, \\ \gamma &= \gamma_2 + \gamma_4\,. \end{aligned} \tag{67}$$

We relate some of these coefficients to each other,

$$\beta = \lambda^2/2\,. \tag{68}$$

For flatness, as described above,

$$\gamma = 1\,. \tag{69}$$

This is included in the choice of lattice scale. Thus the Ith contribution to the action is obtained by contracting the following tensor structure:

$$-\mathfrak{a}_I = \sqrt{g}\,|1, g^{\mu\nu}, g^{\mu\nu}g^{\lambda\rho}| \left| \begin{matrix} -\lambda^2 - \epsilon^4\hat{\lambda}(|\Phi|^2 - \mu^2)^2 \\ \frac{\lambda^2}{2}\delta_{\mu\nu} + \epsilon^2 R_{\mu\nu} - \epsilon^4(\mathfrak{D}_\mu\Phi)^\dagger \cdot \mathfrak{D}_\nu\Phi \\ -\lambda'^-\delta_{[\mu\nu}\delta_{\lambda\rho]} - \lambda'^+\delta_{(\mu\nu}\delta_{\lambda\rho)} + \frac{\epsilon^4}{4e^2} F^\dagger_{\mu\lambda} \cdot F_{\nu\rho} \end{matrix} \right|_I . \tag{70}$$

The scale, power of Ω, is now entirely in the $g^{-n}(\det g)^{1/2}$ terms as we can see by rewriting with explicit Ω:

$$-\mathfrak{a}_I = |\Omega^4, \Omega^2\bar{g}^{\mu\nu}, \bar{g}^{\mu\nu}\bar{g}^{\lambda\rho}| \left| \begin{matrix} -\lambda^2 - \epsilon^4\hat{\lambda}(|\Phi|^2 - \mu^2)^2 \\ \frac{\lambda^2}{2}\delta_{\mu\nu} + \epsilon^2\bar{R}_{\mu\nu} - \epsilon^2\bar{D}_{\mu\nu} - \epsilon^4(\mathfrak{D}_\mu\Phi)^\dagger \cdot \mathfrak{D}_\nu\Phi \\ -\lambda'^-\delta_{[\mu\nu}\delta_{\lambda\rho]} - \lambda'^+\delta_{(\mu\nu}\delta_{\lambda\rho)} + \frac{\epsilon^4}{4e^2} F^\dagger_{\mu\nu} \cdot F_{\lambda\rho} \end{matrix} \right| . \tag{71}$$

Here we define $\bar{D}_{\mu\nu}$ entirely in terms of the logarithmic derivative of Ω. $d=4$ in space (time):

$$\begin{aligned} \bar{D}_{\mu\nu} &\equiv (d-2)(\ln\Omega)_{\mu\nu} + \bar{g}_{\mu\nu}[\bar{g}^{\lambda\rho}(\ln\Omega)_{\lambda\rho} + (d-2)\bar{g}^{\lambda\rho}(\ln\Omega)_{|\lambda}(\ln\Omega)_{|\rho}]\,, \\ (\ln\Omega)_{\mu\nu} &\equiv (\ln\Omega)_{|\mu\nu} - (\ln\Omega)_{|\lambda}\bar{\Gamma}^\lambda_{\mu\nu}\,, \\ (\ln\Omega)_{|\nu} &\equiv \begin{cases} \Omega^{-1}\partial_\nu\Omega & \text{continuum} \\ \Omega^{-1}D_\nu\Omega & \text{lattice}\,. \end{cases} \end{aligned} \tag{72}$$

From the standpoint of the behavior of Ω, it is more natural to omit $\bar{D}_{\mu\nu}$ and introduce a standard kinetic energy

$$\begin{aligned} -\mathfrak{a}_I = {}& -\Omega^4[\lambda + \hat{\lambda}\epsilon^4(|\Phi|^2 - \mu^2)^2] + \Omega^2\bar{g}^{\mu\nu}[\lambda\tfrac{1}{2}\delta_{\mu\nu} + \epsilon^2\bar{R}_{\mu\nu} - \epsilon^4(\mathfrak{D}_\mu\Phi)^\dagger\mathfrak{D}_\nu\Phi] \\ & - \bar{g}^{\mu\nu}\bar{g}^{\lambda\rho}[\lambda'^-\delta_{[\mu\nu}\delta_{\lambda\rho]} + \lambda'^+\delta_{(\mu\nu}\delta_{\nu\rho)} + \frac{\epsilon^4}{4e^2}F^\dagger_{\mu\lambda} \cdot F_{\nu\rho}] + 6\epsilon^2\bar{g}^{\mu\nu}D_\mu\Omega D_\nu\Omega\,. \end{aligned} \tag{73}$$

Our theory is defined by the functional integral of this on all fields with π/ϵ bounded momentum; $\sum_I \mathfrak{a}_I$ averaged over orientations. One might wish to include a Riemann-tensor-squared term. We shall not do that here.

It is clear that the choice of λ parameters given by

$$\lambda' = 2\lambda' = 2\lambda'^{+} \tag{74}$$

will ensure convergence of

$$-\bar{g}^{\mu\nu}\bar{g}^{\lambda\rho}[\lambda^{-}\delta_{[\mu\nu}\delta_{\lambda\rho]} + \lambda^{+}\delta_{(\mu\nu}\delta_{\lambda\rho)}] = -\lambda'\bar{g}^{\mu\nu}\delta_{\mu\nu}\bar{g}^{\lambda\rho}\delta_{\lambda\rho}. \tag{75}$$

g is itself quadratic in the vierbein[11,12] (or $\partial_\mu X^A$),

$$g_{\mu\nu} = V^i_{\ \mu}\delta_{ij}V^j_{\ \mu}. \tag{76}$$

Therefore this potential term is in fact quartic in vierbein variables which run over the entire real axis. The g variables are positive in order to preserve signature. It seems more normal to use entire variables such as V^i_μ or X^A rather than lower bounded variables. But in any case the potential will lead to convergence of the functional integral for $\bar{V}^\mu_i$, the inverse, determinant-one frame fields (vierbein),[11,12] for the same reason that the Ω integral converged. Even if the kinetic terms have the wrong sign, the fact of lattice momentum cut off plus the degree of the negative potential term ensures it. Since this is true for the inverse determinant-one fields $\bar{V}^\mu_i$, it is true for the straight determinant-one fields.

In four Euclidean dimensions, any matrix M's inverse is given as

$$\begin{aligned} &M^\mu_i = (\det M)^{-1}\frac{1}{4!}\,\epsilon^\mu{}_{\nu\lambda\rho}\epsilon_i{}^{jkl}M^\nu_j M^\lambda_k M^\rho_l,\\ &\epsilon_{1234} = 1,\\ &\mu,\nu,\lambda,\rho,i,j,k,l \in \{1,2,3,4\}, \end{aligned} \tag{77}$$

as one can readily verify by multiplication. In the case of $\bar{V}$, $\det\bar{V}$ is one. Thus the potential is in fact of order twelve in $\bar{V}^i_\mu$ and six in $\bar{g}_{\mu\nu}$. It certainly converges. While this is somewhat shocking in its degree, one must recall that even a conventional quartic potential for a matter field Φ leads automatically to a term $\Omega^4|\Phi|^4$ because of the determinant-volume factor. Any potential for Φ is of degree greater than four by this reckoning. This is not a disaster on the lattice because the Feynman-Dyson integrals are all finite without subtractions. However, an examination of the approach to the usual continuum theory leads us to an understanding of the origin[22] of this potential term.

If we simply rewrite the potential in the following suggestive form

$$-\lambda'^2\sqrt{g}\left[\left(g^{\mu\nu}\delta_{\mu\nu} - \frac{\lambda^2}{2\lambda'^2}\right)^2 - \left(\frac{\lambda^2}{2\lambda'^2}\right)\left(\frac{\lambda^2}{2\lambda'^2} - 4\right)\right], \tag{78}$$

the choice

$$\frac{\lambda^2}{2\lambda'^2} = 4 \tag{79}$$

leaps out at us. It corresponds to choosing the stationary point and zero of this potential to be the same:

$$-\lambda'^2\sqrt{g}\,(g^{\mu\nu}\delta_{\mu\nu} - 4)^2. \tag{80}$$

When we express

$$g^{\mu\nu} = \delta^{\mu\nu} + \epsilon H^{\mu\nu} \tag{81}$$

we see that the functional integrand becomes a functional δ function for tr$H$ in the limit as λ' goes to infinity,[3,22]

$$\delta(\mathrm{tr}H) = \lim_{\substack{\lambda'\to\infty\\ \text{continue}}} \exp\left(-\lambda'^2\int d^4x\sqrt{g}\,\epsilon^2\,\mathrm{tr}^2H\right). \tag{82}$$

Note there is a mass $M_T{}^2 = \lambda'^2\epsilon^2$. Because tr$H$ is related to the sum of squares of derivatives of the variable (positive definiteness in Euclidean space), it will not vanish in the Euclidean sector unless the derivatives of all the fields vanish. Therefore we cannot impose the limit $\lambda' = \infty$ in Euclidean space. After contours have been rotated to Minkowski space, one can take that limit. The problem is that only the zero vector in Euclidean space is null whereas the entire light cone is null for Minkowski space. For the embedding we have given in the small-field limit

$$\begin{aligned} &H^{\mu\nu} \approx \delta^{\mu\nu}\delta^{\nu\beta}u_{|\alpha}\cdot u_{|\beta},\\ &\mathrm{tr}H \approx \delta^{\alpha\beta}u_{|\alpha}\cdot u_{|\beta} \geq 0. \end{aligned} \tag{83}$$

Note that in general one can give a natural internal metric for the normal space by replacing the index a by an antisymmetric pair $(\alpha\alpha')$. Then give η_{ab} as $\eta_{\alpha\alpha'}\eta_{\beta\beta'} - \eta_{\alpha\beta'}\eta_{\alpha\beta}$. (One might wish to consider embedding $g^{\mu\nu}$ instead.)

This suggests a way in which we can recover the Feynman-Gupta[19] spin-two self-coupled theory on a flat background in the limit. In that theory, the Einstein-Hilbert action for gravitation is consistently built up from a pure spin-two field. (See the Introduction.) In order to be pure spin-two the metric is both traceless

$$g^{\mu\nu}\delta_{\mu\nu} - 4 = H^{\mu\nu}\delta_{\mu\nu} = 0 \tag{84}$$

and transverse

$$\partial_\mu g^{\mu\nu} = g^{\nu\mu}(g_{\mu\alpha|\beta}g^{\alpha\beta}) = 0. \tag{85}$$

We have already used up the four coordinate transformations by going to perfect coordinates (they parametrize the metric in a perfect gauge). So in order to achieve the transverse gauge we would have to include four new variables. Note that there must still be another three constraints to reduce the theory to masslessness (these are obtained by restricted transverse gauge transformations in the limit). We will not write out those constraints, but the combination of tracelessness plus the three constraints needed to project out $(\pm 1, 0)$ from the spin-two are there as long as the field is massless (the functional integral does this automatically in the limit).

Let the transverse coordinates be $\bar{x}^\alpha$. So we write the gauge condition in barred (transverse) coordinates as

$$\frac{\partial}{\partial \bar{x}^\alpha} \bar{g}^{\alpha\beta} = 0 . \tag{86}$$

Introduce the coordinate-transformation matrix V^α_μ and its inverse V^μ_α,

$$V^\alpha_\mu = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} , \qquad V^\mu_\alpha = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} = (V^{-1})^\mu_\alpha . \tag{87}$$

Now express the transverse gauge condition as a set of four differential equations for the four fields

$$\begin{aligned} 0 &= (\bar{\partial}_\alpha \bar{g}^{\alpha\beta}) V^\mu_\beta \\ &= V^\mu_\beta V^\nu_\alpha \partial_\nu (V^\alpha_\lambda V^\beta_\rho g^{\lambda\rho}) \\ &= \partial_\nu g^{\nu\mu} + V^\nu_\alpha V^\alpha_{\lambda|\nu} g^{\lambda\mu} + V^\mu_\alpha V^\alpha_{\lambda|\nu} g^{\lambda\nu} \\ &= \partial_\nu g^{\nu\mu} + V^\nu_\alpha \bar{x}^\alpha_{|\mu\nu|} g^{\lambda\mu} + V^\mu_\alpha \bar{x}^\alpha_{|\lambda\mu} g^{\lambda\nu} . \end{aligned} \tag{88}$$

The difficulty with this relation is that the inverse of the derivatives V^ν_α of the fields $\bar{x}^\alpha$ must be used. This requires that

$$\det \frac{\partial \bar{x}^\alpha}{\partial x^\nu} \neq 0 \quad \text{everywhere} . \tag{89}$$

This means the map is invertible and differentiable everywhere. $\bar{x}^\alpha$ is in fact a diffeomorphism. If we could functionally integrate over these fields $\bar{x}^\alpha$ while imposing this condition we could simply insert the appropriate lattice version of the weakened[3,22] δ function into the functional

$$\lim_{\substack{\eta\to\infty \\ \Sigma\to 0 \\ \text{continue}}} \int \delta[d^4\bar{x}^\alpha] \exp\left[-\eta^2 \sum_I \sqrt{g}\,(\bar{D}_\sigma \bar{g}^{\sigma\nu})(\bar{D}_\lambda \bar{g}^{\lambda\rho})\delta_{\nu\rho}\Big|_I\right] = \delta(\bar{\partial}_\sigma \bar{g}^{\sigma\nu}) . \tag{90}$$

Note that $\bar{x}^\alpha$ is a field of x^ν. By weakened we mean that η^2 is not infinite. This gauge constraint in the infinite $-\eta$ limit requires transversality of g. Now we calculate our field theory with λ'^2 and η^2 finite and then rotate to Minkowski space and finally take the limit. One must be careful to ensure that the right number of independent massless fields persist in that point. One may wish to include an independent mass for the pure spin-two vector which goes away in that limit. In that limit the flat vacuum will be the physical vacuum. Of course before the limit is taken the vacuum may appear (in lowest order) to be turbulent due to both the trace field and longitudinal tensor components' negative-norm contributions. Imposing the constraints will have left us with a Feynman-Gupta[19] action for gravity. That can be rewritten in conventional form by using Faddeev-Popov[20] ghost fields.

The trouble is that there is no easy way to parametrize the diffeomorphisms $\bar{x}^\alpha$ or the determinant condition. There are two ways out of this that may not be too bad. One, just let the vacuum be what it is[17] and use the perfect gauge in which all integrals are straightforward. Two, on the lattice one can replace the differential with a difference constraint. The diffeomorphisms play the role of phonons.

As long as we are dealing with lattice derivatives we can do the following: Introduce u^α by

$$\bar{x}^\alpha = \delta^\alpha_\nu x^\nu + u^\alpha(x) . \tag{91}$$

u^α is called the lattice deformation vector or phonon field. The lattice equivalent of our determinant condition is

$$\det D_\nu \bar{x}^\alpha = \det(\delta^\alpha_\nu + D_\nu u^\nu) \neq 0 . \tag{92}$$

Since u^α can be zero the determinant condition can be interpreted as a bound on the magnitude u^α by noting that $D_\nu u^\alpha$ can never cancel out the identity matrix if it is sufficiently small. Recall that

$$D_\nu u^\alpha|_I = \epsilon^{-1}(u^\alpha_{I+\nu} - u^\alpha_I) . \tag{93}$$

Our restraint must hold everywhere, for all u^α. We can only insure this totally if the components of u^α_I never exceed $\epsilon/2$ in magnitude, for then their difference will not exceed ϵ. We impose this as a constraint on the functional integrals of the u^α. But of course if they are inserted in the obvious way into the lattice potential there will be an effective damping of the higher magnitude u^α's along with the u^a's. This would motivate dropping the potential and dropping assumption σ by imposing cutoffs on the functional variables much as one does for gauge field variables in the compact version of lattice field theory. Or we could more or less get the desired effect by requiring that λ' be enough greater than ϵ^{-2} that the $1/e$ damping from the potential squelches the probability of such a field. The cutoff on u^α is not necessarily independent of the effective (or real) one for u^a. Indeed the power series in ϵ expansion of the action which underlies the Feynman-Gupta viewpoint may not converge if the factor

$$\epsilon H_{\mu\nu} \leftrightarrow D_\mu X \cdot D_\nu X - \eta_{\mu\nu} \tag{94}$$

contains components of order 1. This would tend to support the cutoff domain of functional integrals (no on 6) idea for consistency.

Now the gravitational functional integral we are interested in has (in the limit) severe restraints. It is not the most general asymptotically flat manifold. One R^4 patch suffices. This view coalesces with the Feynman-Gupta[19] view when the series converges but should yield an extension when the ϵH series does not converge. Note that going to Feynman gauge coordinates by means of the u^α when they are constrained can be arranged if the u^a are sufficiently small.

It may not be necessary to go to the Feynman gauge since the advantages to it are as follows. One, the particle has the usual spin-two properties: It is transverse, traceless, massless. Two, the action is manifestly convergent to leading order in ϵ. These advantages may be outweighed by the facts that one, for the perfect gauge the particles have a natural parametrization in terms of the scalar u^a which avoid some of the difficulties of functionally integrating the coordinate-transformation fields, u^α. Three, if the positive-energy and action conjectures[17] are right the action is bounded from below and flat space is the ground state. Viewed from the perturbative standpoint of propagator plus interactions, proofs of these conjectures seem unusual. They rely on rates of growth and asymptotic flatness.[17] The results are clearly nonperturbative. While the next term may not work, the summed series does. If they are true, no constraints need be imposed except that the conformal problem must be dealt with.[17] We introduce the soft δ functions to study the perturbation limit. They are probably necessary as long as one is on the Euclidean lattice but become less significant in the small-lattice-spacing limit and after rotation to Minkowski space.

As everyone knows there are only two true degrees of freedom for the massless spin-two field. Where do the six degrees of freedom in u^a go? The answer is that in the Minkowski continuum limit there are four restricted gauge conditions (also called dynamical gauge conditions) which reduce the number of degrees of freedom from six to two. Another way of looking at these extra constraints is that they arise from integrating out the nondynamical degrees of freedom, as one does for A_0 in electrodynamics, in which integrating A_0 leads to a δ function.[3] The extra degrees of freedom that are removed are those corresponding to the trace and $(\pm 1, 0)$ parts of the spin-two field in the perturbative theory. This is only easy to see in the small-field limit. We have included the soft δ functions because it appears that these spurious degrees of freedom are present if one is on the Euclidean lattice. (Indeed the trace had better not vanish in Euclidean space.) We will omit writing an explicit δ function to remove the other spin components.

Finally, we comment that in perfect coordinates the natural relationship between the first four coordinates of X and the coordinates for the initial R^4 allows us to readily construct the Poincaré average as in previous papers.

VIII. CONCLUSION

In this paper we have begun an analysis of lattice gravity.[35] Whether one views the lattice as merely regulatory[6,7] or physically real,[5,10,13] one is obliged to analyze quantum field theories there, if one uses functional integrals.[1-3] One important difference between lattice gravity and continuum gravity is the complete destruction of the continuous symmetry group found in the continuum limit. This poses some unique problems for lattice gravity functional convergence which we have answered by proposing a "quartic" potential dependent on the preferred lattice metric $\delta_{\mu\nu}$, which makes the lattice functional integrals converge even if the kinetic terms have the wrong sign. (It is quartic only in Ω.)

The regular flat hypercubic lattice is preferred above the others for reasons analyzed in Sec. II (e.g., it is the *unique* tessellation of R^4 by a regular hypersolid).[27] Because of the very nature of lattices (to describe the vertices requires describing their locations and thus giving a distance)[25,26] we are led to view physical space as an embedding of the hypercubic lattice in R^{10}, but we still have the possibility of using the unembedded lattice's natural, flat structure as a way of constructing the potential.

We went on to discuss the relationship of our potential to the transverse-traceless-symmetric spin-two field approach. One can achieve precisely that theory by including limits of soft functional δ functions related to those constraints.

Performing a transformation to the transverse gauge involves using coordinate-transformation fields which satisfy unusual constraints. [See Eq. (90).] We found that they could be included on a lattice if they did not correspond to large deviations of lattice position.

If the positive-action and energy conjectures are true, then the fields converge for global, nonperturbative reasons and one does not need potentials except to take care of the conformal problem.[17] Still there are advantages for weak-field perturbation theory in going to a gauge.

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APPENDIX A: MATHEMATICAL TERMINOLOGY

General lattices can be thought of as being a connected subset of links and vertices contained in a simplicial complex. See Fig. 4. A simplicial complex is a set built by gluing together building blocks which are n-dimensional triangles, simplices. An n-simplex is any convex set in R^n whose vertices V_i, form n linearly independent vectors $(V_{i+\infty} - V_0)$ for any V_0 in $\{V_i\}$. Such a set of $n+1$ vectors V_i are called convex independent.[36] To be precise, if S is a (closed) simplex with vertices V_i we write

$$[S] = [V_0, \ldots, V_n] = \left\{ V \,\middle|\, V = \sum_{i=0}^{n} a_i V_i, a_i \geq 0 \text{ and } \sum_{i=0}^{n} a_i = 1 \right\}, \qquad \text{(A1)}$$

where (S) will denote the open simplex. A line segment is a 1-simplex; a triangle is a 2-simplex, etc.

We can define a simplicial complex K, as a set of simplices joined together in such a way that the following conditions hold:

$(K1)$ If $(S) \in K$, then all open faces of $[S]$ are in K,

$(K2)$ If (S_1) and $(S_2) \in K$ and $(S_1) \cap (S_2) \neq \phi$, then $(S_1) = (S_2)$. (A2)

$(K2)$ means that the faces of adjoining simplices are identified (are the same). Note that a triangle with a line attached at a vertex is a complex. See Fig. 4. The building blocks need not all be of the same dimension. Such complexes therefore represent a generalization of what is available in the theory of manifolds. It is intuitively obvious (and there is a theorem[6]) that any compact manifold can be smoothly approximated by a simplicial complex. For example, the sphere S^2, can be approximated by the tetrahadron. The topology of these manifolds and their simplicial approximant, in the guise of cohomology theory, is identical by de Rham's theorem.[36,37]

APPENDIX B: REGGE GRAVITY

A theory of curvature on complexes built from simplices of fixed dimension was proposed by Regge.[13] It exploits the proportionality of the Gauss and Riemann curvatures in two dimensions. One views the complex as being built up from n-simplices which are joined along $(n-2)$-dimensional hinges,[11,13] e.g., triangles attached at vertices, tetrahdera at faces, etc. The sum of the angles will not yield in general. This defect of angle multiplied by the area of the $(n-2)$-dimensional "hinge" summed over all hinges is the

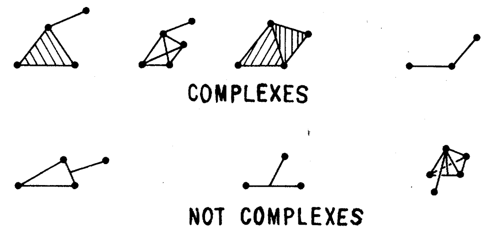

FIG. 4. Some simplicial complexes. Dots denote vertices.

integral of the curvature over the manifold approximated by the complex. This is most readily seen in an example of a two-surface. Take the icosahedron. The triangles are equilateral and five are joined at hinges (vertices in this case). The angle of defect δ, is $360° - (5 \times 60°) = 60° = \pi/3$. As shown in the references, one finds

$$\sum_I \delta_I \Rightarrow \int R\sqrt{g}\, d^2x \quad \text{two dimensions},$$
$$\sum_i \delta_i A_i \Rightarrow \frac{c^3}{16\pi\kappa G} \int R\sqrt{g}\, d^4x \quad \text{four dimensions.} \qquad \text{(B1)}$$

The factor of $\frac{1}{2}$ arises because the Gauss (sectional) curvature[24] is half of Riemann's. Such a relationship exists only in two dimensions[24] since many two-dimensional sections are available in higher dimensions. A_i denotes the "area" of the ith hinge in four dimensions (a two-dimensional object). Unfortunately, there is no way to obtain the action for matter fields without the introduction of a metric. So we pursue this technique no further.

APPENDIX C: HIGHER-ORDER CORRECTIONS AND THE POSSIBILITY THAT GRAVITY MIGHT ARISE FROM THEM

In this appendix we discuss the well-known curvature scalar corrections to the action when there is internal curvature and examine the possibility that this may be an origin of space-time curvature. This idea has a dissimilar formalism, yet kindred spirit to Sakharov's[11,33] proposal for gravitation. While we find it unlikely, for reasons which become clear below, it is still an intriguing concept. We mention it because we need to introduce the general framework of such corrections anyway in order to discuss the change of functional-integral-variables problem. The idea comes along as a sort fo corollary possibility.

Numerous physicists[29,32] have pointed out that the action

$$-\mathfrak{a} = \int \tfrac{1}{2} \bar{g}_{\alpha\beta} \phi^\alpha{}_{|\mu} \phi^\beta{}_{|\nu} \delta^{\mu\nu} d^4x \qquad \text{(C1)}$$

has curvature scalar corrections if $\bar{g}$ is a function of ϕ. Here α, ρ are some internal indices and μ, ν are (flat) space (time). These corrections can be viewed in several ways: as due to operator ordering ambiguities, as due to modifications needed to make Feynman's transition amplitudes obey the expected Schrödinger operator, or as due to modifications required to make a consistent, regularized theory when not using dimensional regularization.[29-32]

The coefficient of the correction is currently under debate depending on the choice of path used in calculating the transition amplitude. If the paths are geodesics the factor is $\frac{1}{6}$; if they are general it is $\frac{1}{8}$.[29-32,38] We will use the latter, more recently derived, number.[38]

Since the corrections are "quantum" we will keep track of $\hbar$. Thus the action is

$$-\mathcal{A} = \int d^4x \{\tfrac{1}{2}\bar{g}_{\alpha\beta}(\phi)\phi^{\alpha}{}_{|\mu}\phi^{\beta}{}_{|\nu}\delta^{\mu\nu} + \tfrac{1}{8}\hbar^2\bar{R}(\phi)[\delta^3(0)]^2\}. \tag{C2}$$

$\bar{R}$ is the curvature scalar computed from $\bar{g}(\phi)$ with ϕ as variable. The other term,

$$\tfrac{1}{2}\hbar \int d^4x \ln \det\bar{g}\,\delta^4(0), \tag{C3}$$

will be subsumed in the functional measure in the usual way.

The object $[\delta^3(0)]^2$ has a heuristic derivation as $(1/d^3x)^2$, one over the spatial lattice volume squared. To derive it one applies the Abers and Lee[3] approach to field theory (as an infinite appropriately coupled quantum mechanics functional) to the one-dimensional correction found by Cheng, DeWitt, and others.[29-32]

The factor $[\delta^3(0)]^2$ is zero only in dimensional regularization. But note that in the only widely considered nonconstant internal $\bar{g}_{\alpha\rho}$ case, the nonlinear sigma model, $\bar{R}$ is a constant inversely proportional to the radius of the constraint sphere squared. It drops out without bothering about regularization.

It should come as no surprise that even if one starts with a Lagrangian, with flat internal metric, but goes to a nonconstant metric through a change of variables, one would get a similar term. That is, let the action initially be

$$-\mathcal{A} = \int \tfrac{1}{2}\delta_{ij}\Phi^i{}_{|\mu}\Phi^j{}_{|\nu}\delta^{\mu\nu}d^4x, \tag{C4}$$

which has no $\hbar^2 R$ term. But change variables to ϕ^α from Φ^i,

$$d\Phi^i = \Phi^i{}_{|\alpha}d\phi^\alpha = \frac{\partial\Phi^i}{\partial\phi^\alpha}d\phi^\alpha. \tag{C5}$$

There will be a functional Jacobian

$$\det\frac{\partial\Phi}{\partial\phi} = (\det\bar{g})^{1/2}, \tag{C6}$$

where

$$\bar{g}_{\alpha\beta} \equiv \Phi^i{}_{|\alpha}\delta_{ij}\Phi^j{}_{|\beta} \tag{C7}$$

is the new effective metric, making the action naively

$$-\mathcal{A}_{\text{naive}} = \int \tfrac{1}{2}\bar{g}_{\alpha\beta}\phi^{\alpha}_{|\mu}\phi^{\beta}_{|\nu}\delta^{\mu\nu}d^4x. \tag{C8}$$

But once again there are corrections, and so the actual functional is

$$-\mathcal{A} = \int \{\tfrac{1}{2}\bar{g}_{\alpha\beta}\phi^{\alpha}_{|\mu}\phi^{\beta}_{|\nu}\delta^{\mu\nu} - \tfrac{1}{8}\hbar^2\bar{\Gamma}^{\alpha}_{\beta\gamma}\bar{\Gamma}^{\beta}_{\alpha\delta}\bar{g}^{\gamma\delta}[\delta^3(0)]^2\}d^4x. \tag{C9}$$

On the lattice the action is a sum

$$-\mathcal{A} = \sum_I \tfrac{1}{2}\bar{g}_{\alpha\beta}D_\mu\phi^\alpha D_\nu\phi^\beta\epsilon^4\Big|_I. \tag{C10}$$

In the one-dimensional case one can quickly see that using the difference of fields is different from using the Feynman procedure of evaluating the integral of the action along the classical path[1,29]

$$\begin{aligned} -\mathcal{A}_{\text{lat}} &= \sum_{i=-\infty}^{\infty} \tfrac{1}{2}\bar{g}_{\alpha\beta}(\phi)[\phi^\alpha(i+1) - \phi^\alpha(i)] \\ &\quad\times[\phi^\beta(i+1) - \phi^\beta(i)]\epsilon^{-1}, \\ -\mathcal{A}_{\text{Feyn}} &= \sum_{i=-\infty}^{\infty} \tfrac{1}{2}\int_{x_i}^{x_i+\epsilon} dt\bar{g}_{\alpha\beta}(\phi)\dot{\phi}^\alpha\dot{\phi}^\beta. \end{aligned} \tag{C11}$$

It is the second one of these which leads to the $\hbar^2$ corrections. The first has no such correction. However, once one goes to a field theory it is necessary to introduce a regulation scheme. The dimensionally regulated zero-ϵ limit of the second theory has no $\hbar^2$ term either. It is therefore the regulated limit of the theories which should be compared for physical consequences. They concur. Since we are using the lattice formalism, we will not encounter any $\hbar^2$ terms which must be removed. The ambiguity in choice of lattice variables results only in a choice of Jacobian.

It is worth noting that the Jacobian for the Einstein-Hilbert functional was calculated by Faddeev and Popov[20] by starting from the two spin-two components in the ADM formalism.[21] No one has published the $\hbar^2$ correction. Because dimensional regularization is so widely used, there is hardly much urgency in doing such a difficult job.

But if we wanted to compare the unregulated version of these theories it seems plausible that one should add a term to A_{Lat}:

$$-\mathcal{A}' = \sum_I \tfrac{1}{8}\hbar^2(\epsilon^{-3})^2\bar{R}_I\epsilon^4. \tag{C12}$$

Is it possible that this can be interpreted as a space (time) curvature? We shall examine this question. First, in order that $\bar{R}$ have the usual dimensions, the fields ϕ^α must be replaced with fields having the dimensions of length. Thus let

$$\bar{X}^\alpha = \hbar^{-1/2}\lambda^2\phi^\alpha , \tag{C13}$$

$\bar{X}^\alpha$ scales as a length if λ is a length.

Define L_I by

$$\sum_I L_I = -\sum_I \epsilon^4 \mathcal{L}_I/\hbar = -\mathcal{A}/\hbar. \tag{C14}$$

Therefore the L_I for the rescaled $-(\mathcal{A}_L + \mathcal{A}')$ is

$$L_I = \left[\tfrac{1}{2}\left(\frac{\epsilon}{\lambda}\right)^4 \bar{g}_{\alpha\beta} D_\mu \bar{X}^\alpha D_\nu \bar{X}^\beta \delta^{\mu\nu} + \tfrac{1}{8}\left(\frac{\lambda}{\epsilon}\right)^4 \epsilon^2 \bar{R}\right]\Bigg|_I . \tag{C15}$$

If we wish ultimately to interpret $\bar{R}$ not as an internal but as a space (time) curvature, there must arise an implicit curved metric structure. This can be achieved by requiring an intimate relationship between the fields $\bar{X}^\alpha$ and the coordinates x^μ. $\bar{X}^\alpha(x^\mu)$ must be a diffeomorphism. That is, it and its inverse $x^\mu(\bar{X}^\alpha)$ must be (infinitely) differentiable functions of each other. We shall temporarily gloss over the difference between lattice and actual derivatives. Furthermore, we define $\bar{g}_{\alpha\beta}$ implicitly as follows:

$$\begin{aligned}\bar{g}_{\mu\nu} &= D_\mu \bar{X}^\alpha \bar{g}_{\alpha\beta} D_\nu X^\beta \\ &= D_\mu \bar{X}^\alpha [\delta_{\alpha\beta} + (D\bar{X}^{-1})^\mu_\alpha (D_\mu \bar{X}^a \eta_{ab} D_\nu \bar{X}^b)(D\bar{X}^{-1})^\nu_\beta] D_\nu \bar{X}^\beta \\ &= D_\mu \bar{X}^\alpha \delta_{\alpha\beta} D_\nu \bar{X}^\beta + D_\mu \bar{X}^a \eta_{ab} D_\nu \bar{X}^b \\ &= D_\mu \bar{X}^A \eta_{AB} D_\nu \bar{X}^B .\end{aligned} \tag{C16}$$

In the continuum limit we see that $\bar{g}_{\alpha\beta}$ is the perfect coordinates metric

$$\bar{g}_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha \bar{X}^a \eta_{ab} \partial_\beta \bar{X}^b , \tag{C17}$$

where we have used the diffeomorphism properties of $\bar{X}^\alpha$. Because of the close relationship between $\bar{X}^\alpha$ and the coordinates x^μ, it seems reasonable to set $\epsilon = \lambda$. Thus we have been led to the lattice action

$$\mathcal{A} = -\tfrac{1}{2}\sum_I \left[D_\mu \bar{X}^A \eta_{AB} D_\nu \bar{X}^B \delta^{\mu\nu} + \left(\frac{\epsilon}{2}\right)^2 \bar{R}\right]\Bigg|_I . \tag{C18}$$

We can approximately interpret $\bar{R}$ as a space (time) curvature, because it is unaffected by diffeomorphisms, if the metric $D_\mu \bar{X}\cdot D_\nu \bar{X}$, couples to matter fields in the appropriate way. This can be done by hand along the usual lines (see Sec. VI). In addition one must introduce a conformal field Ω. One can treat it as an independent field which couples in the appropriate way. Thus consider the action

$$\begin{aligned}\mathcal{A}_{\text{CONF}} = \sum_I \Bigg\{ &-\tfrac{1}{2}\left[D_\mu \bar{X}\cdot D_\nu \bar{X}\delta^{\mu\nu} + \left(\frac{\epsilon}{2}\right)^2 \bar{R}\right]\Omega^2 \\ &+ \tfrac{3}{4}\epsilon^2 D_\mu \Omega D_\nu \Omega \bar{g}^{\mu\nu} \\ &+ \Omega^2 \bar{g}^{\mu\nu}(\mathfrak{D}_\mu \Phi)^\dagger \cdot \mathfrak{D}_\nu \Phi + \Omega^4 U(\Phi) \\ &- \frac{1}{4e^2}\operatorname{tr}(F^\dagger_{\mu\lambda} F_{\nu\rho})\bar{g}^{\mu\nu}\bar{g}^{\lambda\rho}\Bigg\}\Bigg|_I ,\end{aligned} \tag{C19}$$

with internal fields in standard notation.

If one is at large distances on the Planck lattice, one expects this action to be approximated by the integral (after slightly rescaling ϵ)

$$-\mathcal{A} = \int \sqrt{g}\, d^4x \left[-\tfrac{1}{2} g_{\mu\nu}\delta^{\mu\nu}\left(\frac{\delta^4(0)}{\sqrt{g}}\right) - \kappa R + g^{\mu\nu}(\mathfrak{D}_\mu \Phi)^\dagger \mathfrak{D}_\nu \Phi + U(\Phi) - \frac{1}{4e^2} g^{\mu\nu} g^{\lambda\rho} \operatorname{tr}(F^\dagger_{\mu\lambda} F_{\nu\rho})\right] . \tag{C20}$$

Here we have used

$$g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu} \tag{C21}$$

and the following correspondence

$$\int \sqrt{g}\, d^4x\, g_{\mu\nu}\delta^{\mu\nu}\frac{\delta^4(0)}{\sqrt{g}} \iff \sum_I \delta^4 \epsilon^4 g_{\mu\nu}\delta^{\mu\nu}\frac{\epsilon^{-4}}{\Omega^4} . \tag{C22}$$

We see that upon regulation it is the first term which vanishes. One is left with the Einstein-Hilbert-matter action. This strange transformation has arisen because of the unusual scaling properties of $\bar{X}^\alpha$ as a field.

Two more serious questions beset this idea. One, why should the extra degrees of freedom in $g_{\alpha\beta}$ beyond X^α (those corresponding to the six X^a) be quantized? A somewhat plausible answer is that this is precisely how the fixed curvature internal space should be generalized to the arbitrary curvature internal space, and summed over the arbitrary variations. Two, what about the fact that this theory does not converge on the lattice for the same reasons given in the text? This is a fatal flaw to the theory as it stands. Clearly, some modifications such as those described in the text are called for.

We point out that Sakharov's[11,33] proposal, to consider the modifications of zero point for the Lagrangian on a curved space as the source of gravitation, is not entirely dissimilar. In the case he considered, one finds the modifications expressible as a power series in the curvature. The leading term (after renormalization) is

$$\Delta \mathcal{L} = B\hbar R \int k\, dk\, .$$

B is an as yet unspecified constant. If one cuts off k at $\hbar\pi/\epsilon$, one finds

$$\Delta \mathcal{L} = B\hbar^3 R \pi^2 \epsilon^{-2}\, ,$$

where

$$\kappa\epsilon^2 = 1\, .$$

Sakharov suggests setting

$$B\hbar^3\pi^2 = -1$$

and interpreting this correction factor as the Einstein-Hilbert action, though why one should then quantize the metric degrees of freedom is not clear here either. In the path-integral series we just discussed, the relation terminates at $\hbar^2 R$. In his case the power series goes on forever. Still the idea of using the correction factor as the source of gravitation is common.

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