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Rotationally invariant field theory on lattices. II. Internal symmetry

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We construct gauge theories on the Euclidean Poincaré transform related ensemble of lattices. Some examples of their effective propagators are derived. Even though the lattice spacing is kept nonzero (on the order of the Planck length), the effective ensemble propagators are rotationally invariant.

I. INTRODUCTION

In this paper we examine the ensemble-averaged lattice theory[1] for interacting scalar-vector and spinor-vector gauge theories. The notation used follows naturally from the previous paper.[1]

We begin (Sec. II) with the usual lattice derivative for scalar fields coupled to gauge fields. We try to stay as close as possible to conventional lattice formalisms,[2,3] but some differences arise. In particular, we employ the natural map from the group to the algebra (the logarithm). In this formalism the difference between noncompact and compact theories[4,5] corresponds to the difference between the "direct" logarithm and the principal-value logarithm. We do this because our approach generalizes to lattices with nontrivial metrics.[1]

We obtain the Poincaré-averaged action for the decoupled Abelian gauge theory. In an appendix we show the derivation for the two-dimensional version.

Then in Sec. III we calculate the lattice-averaged U(1)-gauge coupled fermion system in momentum space. The answer is so complicated that it appears only useful perturbatively. Of course, in view of the smallness[1] of ϵ, that should not be too bad for most purposes. We note that the Minkowski version has a nondegenerate energy and thus one does not get a multiplicity of fermions.

II. INTERACTING GAUGE FIELDS

In this section we discuss the action for scalars, ϕ (C^n vectors), interacting by a gauge-covariant coupling with gauge fields B_μ^A associated with SU(n) generators. Because the momentum space is compact, there appears to be a difficulty with gauge covariance, if one uses the path-ordered exponential of the gauge field. But if the group elements for each link are constructed using the variables natural in the Weinberg-Salam model,[6] the *physical field variables*, no problem arises. From a geometrical standpoint this means we use path-ordered integrals of the connection; not just the gauge field.[7]

As in the Weinberg-Salam model[6] we define

$$W_\mu^A i\Lambda_A = \Omega^{-1} B_\mu^A i\Lambda_A \Omega + \Omega^{-1}\partial_\mu \Omega\,, \qquad \Phi = \Omega^{-1}\phi\,. \tag{1}$$

The Λ_A are matrices representing SU(n) generators. In the SU(2) case Φ has only one physical degree of freedom; W_μ^A has three more than B_μ^A. W_μ^A and Φ are the gauge-*invariant* quantities. Of course, it is still true that the covariant derivatives of Φ and ϕ are proportional so the theory reduces, as $\epsilon\to 0$, to the usual theory

$$\partial_\mu \Phi + iW_\mu^A \Lambda_A \Phi = \Omega^{-1}(\partial_\mu\phi + iB_\mu^A\Lambda_A\phi)\,. \tag{2}$$

Let I be an n-tuple of integers; $I_{\pm\nu}$ locates its neighbors:

$$I \equiv (I_1, \ldots, I_n)\,, \qquad I_{\pm\nu} \equiv (I_1, \ldots, I_\nu \pm 1, \ldots, I_n) \equiv I \pm \nu\,. \tag{3}$$

The embedding of Z^n in R^n is given by

$$x_I = \epsilon I\,, \qquad x_{I+\nu} = \epsilon I_{+\nu} = \epsilon(I+\nu)\,, \tag{4}$$

with ϵ the lattice spacing and x in R^n. We introduce the arc from x_I to $x_{I+\nu}$ given by $x = x_I(1-\lambda) + x_{I+\nu}\lambda$ with $\lambda \in [0,1]$. The group element for the link x_I to $x_{I+\nu}$ is given by the path-ordered exponential

$$U(I+\nu, I) = P\left[\exp\left(\int_I^{I+\nu} W_\mu^A i\Lambda_A dx^\mu\right)\right]\,, \tag{5}$$

where $\int_I^{I+\nu} dx^\mu$ is shorthand for $\int_0^1 (x_{I+\nu}^\mu - x_I^\mu)d\lambda$. The covariant derivative of Φ is therefore

$$\mathfrak{D}\Phi\,\big|_I = \epsilon^{-1}[U(I+\nu, I)\Phi(I+\nu) - \Phi(I)]\,. \tag{6}$$

Because W_μ^A will be sampled on the ensemble of lattices,[1] not just one, we will not assume W_μ^A constant on links. For if it were constant for one lattice orientation, it would not necessarily be so for another.

Since the cells are in one-to-one correspondence with the vertices, we associate the positive displacement cell (framed by $I, I+\mu,\ I+\nu$, etc.) with the Ith vertex. The loops around this cell's plaquettes have the following group element associated to them:

$$U_{I\mu\nu}=U(I,I+\mu)U(I+\mu,I+\mu+\nu)U(I+\mu+\nu,I+\nu)\times U(I+\nu,I)\,. \tag{7}$$

It is an element of SU(n) for each I and pair $\mu\nu$.

Just as exp maps the algebra into the group, Ln maps the group into the algebra. Diagramatically,

$$\begin{aligned}&\text{algebra}\xrightarrow{\exp}\text{group}\,,\\&\text{group}\xrightarrow{\text{Ln}}\text{algebra}\,.\end{aligned} \tag{8}$$

For matrix representations with iA in the algebra and G in the group,

$$\begin{aligned}iA&=\ln G=\sum_{n=1}^{\infty}\frac{1}{n}(I-G)^n\,,\\G&=\exp(iA)=\sum_{n=1}^{\infty}\frac{1}{n!}(iA)^n\,.\end{aligned} \tag{9}$$

There is an ambiguity in the definition of the logarithm depending on whether or not one chooses the principal part (Ln or ln). Thus

$$\begin{aligned}&\text{Ln}e^{i\theta}=i\hat{\theta}\,,\quad \hat{\theta}=\theta \bmod 2\pi\,,\\&\ln e^{i\theta}=i\theta\,.\end{aligned} \tag{10}$$

This difference is what leads to the compact versus noncompact ambiguity in lattice field theories. Note that our approach is somewhat different from that of Wilson.[2-5] Thus, depending on the theory desired, one can take $F^A_{I\mu\nu}$ to be given by either

$$iF^A_{I\mu\nu}\Lambda_A=\epsilon^{-2}\,\text{Ln}U_{I\mu\nu}$$

or (11)

$$iF^A_{I\mu\nu}\Lambda_A=\epsilon^{-2}\ln u_{I\mu\nu}\,.$$

We will ignore this distinction in the following. The choice is left to the reader. We choose this approach as one which generalizes to lattices with nontrivial metrics. One readily verifies that in the Abelian theory these definitions yield

$$\begin{aligned}F_{I\mu\nu}&\equiv\epsilon^{-2}\oint\!\!\oint_{I_{\mu\nu}\text{plaquette}}F_{\alpha\beta}dx^\alpha\wedge dx^\beta\\&=\epsilon^{-2}\oint_{I_{\mu\nu}\text{bndry}}(A_\alpha+\partial_\alpha\omega)dx^\alpha\\&=\epsilon^{-2}\oint_{I_{\mu\nu}\text{bndry}}A_\alpha dx^\alpha\,.\end{aligned} \tag{12}$$

Thus for the gauged scalar-vector theory we consider the following action:

$$\mathfrak{a}=\epsilon^4\sum_I\left[\epsilon^{-2}\sum_\nu|U(I+\nu,I)\Phi(I+\nu)-\Phi(I)|^2-M^2|\Phi(I)|^2-\tfrac{1}{4}\sum_{\mu\nu}\frac{\text{tr}}{N}(F_{I\mu\nu}F_{I\mu\nu})\right]. \tag{13}$$

The Abelian version is

$$\mathfrak{a}=\epsilon^4\sum_I\left\{\sum_\nu\epsilon^{-2}\left|\exp\left[i\int_I^{I+\nu}(A_\alpha+\omega_{|\alpha})dx^\alpha\right]\Phi(I+\nu)-\Phi(I)\right|^2-M^2|\Phi(I)|^2-\tfrac{1}{4}\sum_{\mu\nu}F_{I\mu\nu}F_{I\mu\nu}\right\}. \tag{14}$$

As for gauge invariance, let V be in the group

$$\begin{aligned}&\phi\to V^{-1}\phi\,,\\&\Omega\to V^{-1}\Omega\,,\\&B_\mu\to V^{-1}B_\mu V+V^{-1}\partial_\mu V\,.\end{aligned} \tag{15}$$

Thus

$$W_\mu=\Omega^{-1}B_\mu\Omega+\Omega^{-1}\partial_\mu\Omega\to W_\mu\,. \tag{16}$$

Also,

$$\begin{aligned}&U_{I\mu\nu}\to U_{I\mu\nu}\,,\\&F_{I\mu\nu}\to F_{I\mu\nu}\,,\end{aligned} \tag{17}$$

and so the action is manifestly gauge invariant. Note that in the Abelian case one can use

$$\int_I^{I+\nu}\partial_\mu\lambda dx^\mu=\lambda(I+\nu)-\lambda(I) \tag{18}$$

to see that $\int_I^{I+\nu}A_\mu dx^\mu$ goes to $\int_I^{I+\nu}(A_\mu+\partial_\mu\lambda)dx^\mu$ in a gauge transformation.

We will not write out the Fourier transform of the coupled scalar-vector piece because of the complicated convolution terms. We show the example of a fermion coupled to a gauge field in the next section. Here we will compute only the effective propagator for the Abelian gauge field.

The action for the electromagnetic field is

$$\mathfrak{a}=\tfrac{1}{4}\epsilon^4\sum_I F_{I\mu\nu}F_{I\mu\nu}\,, \tag{19}$$

where

$$F_{I\mu\nu}=\epsilon^{-2}\int_0^1\int_0^1 d\tau\,d\sigma(\Delta^a_\mu\Delta^b_\nu-\Delta^a_\nu\Delta^b_\mu)\times\partial_a[A_b(x_I+\Delta_\mu\tau+\Delta_\nu\sigma)] \tag{20}$$

with

$$\Delta^a_\mu=\Lambda^a_\mu\epsilon\,, \tag{21}$$

and

$$x=x_I+\Delta_\mu\tau+\Delta_\nu\sigma\,, \tag{22}$$

and

$$A_\nu(x)=\int_{|k\epsilon|<\pi}\frac{d^4k}{(2\pi)^2}A_\nu(k)e^{ik\cdot x}\,. \tag{23}$$

Thus

$$\mathcal{A}=\tfrac{1}{4}\epsilon^{-4}\int_{|k\epsilon|<\pi} d^4k \sum_{\mu\nu}(ik\cdot\Delta_{[\mu}A\cdot\Delta_{\nu]})^{\dagger}(ik\cdot\Delta_{[\mu}A\cdot\Delta_{\nu]}) \times\left[\frac{\sin(k\cdot\Delta_\mu/2)}{(k\cdot\Delta_\mu/2)}\frac{\sin(k\cdot\Delta_\nu/2)}{(k\cdot\Delta_\nu/2)}\right]^2, \tag{24}$$

where the bracketed indices are antisymmetrized.

To obtain the effective action (the average over the ensemble) it is convenient to use the coset decomposition of an element G of SO(4):

$$G=\exp\begin{bmatrix}0 & a\\ -a^T & 0\end{bmatrix}\exp\begin{bmatrix}0 & & 0 & \\ & & & \\ & & 0 & b\\ 0 & & -b^T & 0\end{bmatrix}\exp\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&c\\0&0&-c^T&0\end{bmatrix} \tag{25}$$

with $a\in \mathrm{R}^3$, $b\in\mathrm{R}^2$, and $c\in\mathrm{R}$. The result of this calculation is (compare to the Appendix)

$$\mathcal{A}^{\text{eff}}=\tfrac{1}{4}\int_{|k\epsilon|<\pi} d^4k\sum_{\mu\nu}(k_{[\mu}A_{\nu]})^{\dagger}(k_{[\mu}A_{\nu]})S(k^2) \tag{26}$$

with

$$S(k^2)=\sum_{N=0}^{\infty}\frac{1}{N!\,(N+2)!}R_N\left(-\frac{\epsilon^2k^2}{4}\right)^N, \tag{27}$$

where

$$R_N=(2N+2)C_N \tag{28}$$

and

$$C_N=\sum_{l,m=0}^{N}\frac{1}{l+1}\frac{1}{2l+1}\frac{1}{m+1}\frac{1}{2m+1}\frac{N!}{l!\,m!}\delta_{N,l+m}. \tag{29}$$

See Fig. 1 for a plot of $S(k^2)$. $C_N\approx 1/(2N+1)$ for very small $N(<5)$. However, it starts to grow thereafter. See Fig. 2 and Table I. Therefore, $R_N\approx(2N+2)/(2N+1)$ $(N<5)$ and so

$$S\approx\sum_{0\le N<5}\frac{1}{N!\,(N+2)!}\frac{2N+2}{2N+1}\left(-\frac{k^2\epsilon^2}{4}\right)^N. \tag{30}$$

There is an integral representation for C_N,

$$C_N=4\int_0^1 dy\int_0^y d\bar{y}\int_0^1 dx\int_0^x d\bar{x}(\bar{x}^2+\bar{y}^2)^N. \tag{31}$$

Clearly, $C_N\ll 2^N$, and so the series will converge.

We note that just as averaging the square of the derivative (a vector) of a scalar-generated conventional Bessel's functions, averaging the square of the derivative of the vector field (an antisymmetric tensor) has generated these generalized Bessel functions. Further extensions to generalized Bessel's functions associated with the averaging of the derivatives of higher-rank tensors (especially antisymmetrized tensors of rank p, p-forms) are possible. Since we do not make application of these functions we will not give their series representations here. But the procedure is simple. One averages the appropriate tensor of rank p over a cell of rank p of size $\sim\epsilon$. Take the sum of the squares. Represent it in terms of a plane-wave series and compute the rotational

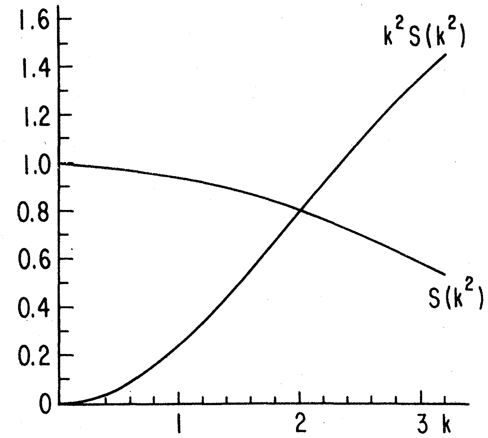

FIG. 1. Euclidean $S(k^2)$ and $k^2S(4^2)$ with $\epsilon=1$ for small k.

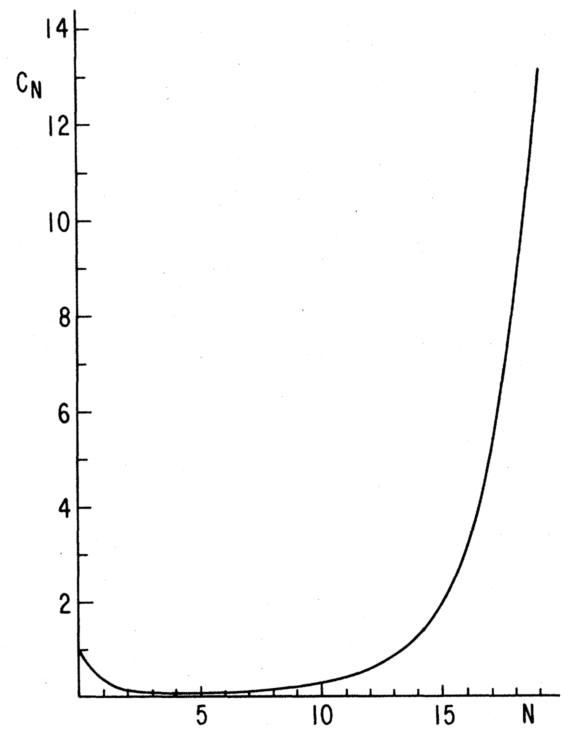

FIG. 2. C_N vs N.

TABLE I. Values for C_N.

N	C_N
0	1
1	$\frac{1}{3}$
2	$\frac{17}{90}$
3	$\frac{29}{210}$
4	$\frac{187}{1575}$
5	$\frac{239}{2079}$
6	0.12
7	0.14
8	0.17
9	0.22
10	0.30
11	0.41
12	0.60
13	0.88
14	1.33
15	2.05
16	3.20
17	5.08
18	8.16
19	13.2
0	1
10	0.30
20	21.82
30	4703.0
40	15.5×10^6

average. Evidently the rank of these functions depends on the dimension p of the form and d of the space, and the shape of the cell over which they are averaged. Their arguments are scalars constructed from the p-vectors. The usual Bessel's functions have $p=1$ and $d-2=2\nu$ for J_ν. See comments at the end of the Appendix. We continue S to Minkowski space as described in the previous paper[1]:

$$S_M=\sum_{N=0}^{\infty}\frac{1}{N!\,(N+2)!}R_N\left(\frac{k^2\epsilon^2}{4}\right)^N\left(\frac{1+(-)^N}{2}\right). \qquad (32)$$

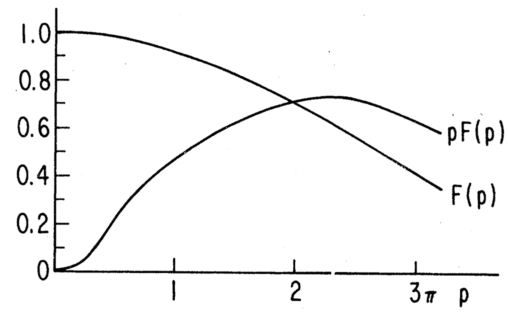

FIG. 3. Euclidean $F(p)$ and $pF(p)$ with $\epsilon=1$ for small p.

III. FERMIONS

In this section we describe the coupling of gauge fields to fermions. We expand the equations in the plane-wave basis for the Abelian case since, in fact, the plane waves are the actual basis functions. The configuration-space functions are viewed as samplings of the momentum-space basis functions (see paper I).[1]

We take the action to be

$$\mathfrak{A}=\epsilon^4\sum_I\left\{\epsilon^{-2}\,\mathrm{Im}\left[\sum_\nu\bar\Psi(x)\gamma\cdot\Delta_\nu U(x+\Delta_\nu,x)\Psi(x+\Delta_\nu)\right]\Bigg|_{x=x_5} -\tfrac14\sum_{\mu\nu}\frac{\mathrm{tr}}{N}\left(F^\dagger_{I\mu\nu}F_{I\mu\nu}\right)\right\}, \qquad (33)$$

$$U(x+\Delta_\nu,x)\big|_{x=x_I}=U(I+\nu,I)\equiv P\left\{\exp\left[i\int_I^{I+\nu}W_\mu^A\Lambda_A\,dx^\mu\right]\right\}. \qquad (34)$$

$F_{I\mu\nu}$ is the same as before. $\Psi=\Omega^{-1}\psi$ as for scalars,

$$\Psi(x)\big|_{I+\nu}=\Psi(I+\nu). \qquad (35)$$

On a lattice there is the well-known problem of energy degeneracy. But from the form of the action we are about to derive we find no energy degeneracy when we take

$$\Psi(x)=\int_{|k\epsilon|<\pi}\frac{d^4k}{(2\pi)^2}e^{-ik\cdot x}\Psi_k \qquad (36)$$

and continue back to Minkowski space. Thus the actual observable fermion energy has no degeneracy.

In the Abelian case we have already analyzed the F^2 piece of the action. We shall analyze the other term which we denote by a prime:

$$\mathfrak{A}'=\sum_{I,\nu}\epsilon^2\,\mathrm{Im}\left\{\bar\Psi\gamma\cdot\Delta_\nu\exp\left[i\int_x^{x+\Delta_\mu}dx^\mu(P_\mu-iW_\mu)\right]\Psi\right\}\Bigg|_{x=x_I}, \qquad (37)$$

$$P_\mu=-i\partial_\mu=-i\frac{\partial}{\partial x^\mu};\quad W_\mu=A_\mu+\partial_\mu\lambda. \qquad (38)$$

So

$$\mathcal{A}'=\sum_{I,\nu}\epsilon^2\left\{\bar{\Psi}(x)\gamma\;\Delta_\nu\sin\left[\int_x^{x+\Delta_\nu}dx^\mu(P_\mu-iW_\mu)\right]\Psi(x)\right\}\Big|_{x=x_I}, \tag{39}$$

or

$$\mathcal{A}'=\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)!}\int d^4p'\int d^4p\int d^4k_1\cdots\int d^\alpha k_{2n+1}B_\nu(k_1)\cdots B_\nu(k_{2n+1})\bar{\Psi}_{p'}\gamma\cdot\Delta_\nu\Psi_p\delta^4\Big(p'-p-\sum_{j=1}^{2n+1}k_j\Big)\epsilon^{-2}, \tag{40}$$

where

$$B_\nu(k_j)\equiv\Delta_\nu\cdot[(2\pi)^{-2}W(k_j)+\delta^4(k_j)P]\sum_{l_j=0}^{\infty}\frac{(ik_j\cdot\Delta_\nu)^{l_j}}{(l_j+1)!}. \tag{41}$$

The reader may wish to ignore the technical details involved in computing the convolutions when $A_\mu\neq0$. If so, skip to Eq. (49).

Upon averaging over SO(4), only even products of vectors survive. To this end we define the product $[V_1,\ldots,V_M]$ recursively as follows:

$$[0]=1 \tag{42a}$$

and

$$[V_1,\ldots,V_M]=\begin{cases}(M-1)^{-1}\sum_{j=2}^{M}V_1\cdot V_j[V_2,\ldots,\hat{V}_j,\ldots,V_M] & \text{if } M \text{ even} \geq 2\\ 0 \quad \text{if } M \text{ odd}.\end{cases} \tag{42b}$$

The notation $\hat{V}_j$ means that V_j is to be deleted from the expression. Note that there are $(M-1)!!$ terms in the full expansion. Thus

$$\begin{aligned}&[0]=1,\\&[V_1]=0,\\&[V_1,V_2]=V_1\cdot V_2,\\&[V_1,V_2,V_3]=0,\\&[V_1,V_2,V_3,V_4]=\tfrac{1}{3}(V_1\cdot V_2V_3\cdot V_4+V_1\cdot V_3V_2\cdot V_4+V_1\cdot V_4V_2\cdot V_3),\\&\text{etc.}\end{aligned} \tag{43}$$

Since averaged products of the B_ν's will involve terms like $(k_j)^{l_j}$, we define

$$[V_1,(k)^l,V_{l+2}]\equiv[V_1,k,\ldots,k,V_{l+2}], \tag{44}$$

where k appears l times on the right-hand side. Since the $l_j=0$ term in $B_\nu(k_j)$ is unlike the others, we define

$$a_j\equiv a(k_j,m_j)=(2\pi)^{-2}W(k_j)+\delta_{0,m_j}\delta^4(k_j)P \tag{45}$$

and

$$J_\nu=J_\nu(p',p)=\bar{\Psi}_{p'}\gamma_\nu\Psi_p, \tag{46}$$

and

$$2N=2n+2+\sum_{j=1}^{2n+1}l_j. \tag{47}$$

N will be an integer in the effective action

$$\begin{aligned}\mathcal{A}'^{\text{eff}}=\int d^4p'\int d^4p\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1!}\int d^4k_1\cdots\int d^4k_{2n+1}\sum_{l_1\cdots l_{2n+1}}\left(\frac{\epsilon^2}{4}\right)^{N-1}\frac{2N!}{N!(N+1)!}\frac{1}{(l_1+1)!}\cdots\frac{1}{(l_{2n+1}+1)!}\\ \times[a_1,\ldots,a_{2n+1},(ik_1)^{l_1},\ldots,(ik_{2n+1})^{l_{2n+1}},J]\\ \times\delta^4\Big(p'-p-\sum_{l=1}^{2n+1}k_j\Big),\end{aligned} \tag{48}$$

when $A_\mu = 0$ all the l_j sums are trivial; $N=n+1$ and so we find

$$\mathcal{Q}'^{\text{eff}} = \int d^4p \bar{\Psi}_p \not{p} \Psi_p \left[2 \sum_{n=0}^{\infty} \frac{(-)^n}{n!\,(n+2)!} \left(\frac{\epsilon^2 p^2}{4}\right)^n \right]. \quad (49)$$

Let $F(p^2)$ be the function in the brackets. Note $F(p^2) = (8/p^2\epsilon^2) J_2((p^2\epsilon^2)^{1/2})$. The Minkowski-space version replaces $(-)^n$ by $[(-)^n + 3]/4$ and $p_E \to p_M$. That is,

$$F(p^2) = \frac{2}{p^2\epsilon^2} [J_2((p^2\epsilon^2)^{1/2}) + 3 I_2((p^2\epsilon^2)^{1/2})]. \quad (50)$$

It is evident that the energy will be nondegenerate.

IV. CONCLUSION

We have presented the most straightforward adaptation of conventional lattice gauge theory to the ensemble of lattices which is possible. It was, however, necessary to introduce a different kinetic term for the gauge fields in order to have an approach which works for lattices with nontrivial metrics. The coupled fermion action is, unfortunately, rather unpleasant looking, but should still be useful perturbatively. The fermions do not have the usual lattice fermion energy degeneracy.

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APPENDIX

In this appendix we show how to calculate the effective action for a U(1) gauge boson in two dimensions. As in the text [Eq. (24)] we take the effective two-dimensional action to be

$$\mathcal{Q} = \tfrac{1}{4} \int d^2k \sum_{\mu\neq\nu} |k_a A_b - k_b A_c|^2 \Delta^a_\mu \Delta^b_\nu |\epsilon^{-4} \left[\frac{\sin(k\cdot\Delta_\mu/2)}{(k\cdot\Delta_\mu/2)}\right]^2 \times \left[\frac{\sin k\cdot\Delta_\nu/2}{(k\cdot\Delta_\nu/2)}\right]^2, \quad (A1)$$

where $\Delta^a_\mu = \epsilon\Lambda^a_\mu$ with Λ^a_μ an SO(2) matrix. Set

$$\Lambda^a_\mu = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}^a_\mu . \quad (A2)$$

Let

$$f_{ab} = k_a A_b - k_b A_a \quad (A3a)$$

and

$$P_\mu{}^2 = \left[\frac{\sin(k\cdot\Delta_\mu/2)}{(k\cdot\Delta_\mu/2)}\right]^2, \quad (A3b)$$

and find

$$\begin{aligned} \tfrac{1}{4} \sum_{\mu\neq\nu} |k_a A_b - k_b A_a|^2 \Delta^a_\mu \Delta^b_\nu \epsilon^{-4} P_\mu{}^2 P_\nu{}^2 &= |f_{01}|^2 \sum_{\mu\neq\nu} (\Lambda^0_\mu \Lambda^1_\nu - \Lambda^1_\mu \Lambda^0_\nu)^2 P_\mu{}^2 P_\nu{}^2 \\ &= |f_{01}|^2 P_0{}^2 P_1{}^2 . \end{aligned} \quad (A4)$$

Now expand $P_\mu{}^2$,

$$\begin{aligned} P_\mu{}^2 &= 2\frac{1-\cos(k\cdot\Delta_\mu)}{(k\cdot\Delta_\mu)^2} \\ &= 2\sum_{l=0}^{\infty} \frac{(-)^l}{(2l+2)!} (k\cdot\Delta_\mu)^{2l} . \end{aligned} \quad (A5)$$

Since

$$k\cdot\Delta_0 = \epsilon k_0 \cos\theta + \epsilon k_1 \sin\theta \quad (A6a)$$

and

$$k\cdot\Delta_1 = -\epsilon k_0 \sin\theta + \epsilon k_1 \cos\theta , \quad (A6b)$$

it is convenient to define

$$z = \rho e^{i\alpha} = \tfrac{1}{2}\epsilon(k_0 - ik_1) . \quad (A7)$$

Thus

$$\begin{aligned} k\cdot\Delta_0 &= z e^{i\theta} + \bar{z} e^{-i\theta} \\ &= \rho[e^{i(\alpha+\theta)} + e^{-i(\alpha+\theta)}] \end{aligned} \quad (A8)$$

and

$$\begin{aligned} k\cdot\Delta_1 &= iz e^{i\theta} - i\bar{z} e^{-i\theta} \\ &= \rho i[e^{i(\alpha+\theta)} - e^{-i(\alpha+\theta)}] . \end{aligned} \quad (A9)$$

Now

$$\begin{aligned} P_0{}^2 P_1{}^2 &= 4 \sum_{l,m=0} \frac{(-)^{l+m}}{(2l+2)!\,(2m+2)!} (k\cdot\Delta_0)^{2l} (k\cdot\Delta_1)^{2m} \\ &= 4 \sum_{l,m=0} \frac{(-)^{l+m}}{(2l+2)!\,(2m+2)!} (-)^m \rho^{2l+2m} \sum_{k=0}^{2l} \sum_{p=0}^{2m} \frac{2l!}{k!\,(2l-k)!} \frac{2m!}{p!\,(2m-p)!} [(-)^p e^{2i(l-k+m-p)(\alpha+\theta)}] . \end{aligned} \quad (A10)$$

When we integrate this over $d\theta$ and divide by 2π, we get a Kronecker delta $\delta_{l+m,k+p}$. Unfortunately, the sum still appears to be quite complicated until one notices that the vanishing of $(l-k+m-p)$ means there is no α dependence. Since there is no α dependence, we may as well choose it to be zero and use the first line of Eq. (A10).

$$\int \frac{d\theta}{2\pi}(k\cdot\Delta_0)^{2l}(k\cdot\Delta_1)^{2m}=(2\rho)^{2l+2m}\int\frac{d\theta}{2\pi}\cos^{2l}\theta\sin^{2m}\theta$$
$$=(4\rho^2)^{l+m}\frac{2l!\,2m!}{4^{l+m}(l+m)!\,l!\,m!}\,. \qquad (A11)$$

Now use the fact that

$$4\rho^2=\epsilon^2k^2=\epsilon^2(k_0{}^2+k_1{}^2)\,. \qquad (A12)$$

Therefore, we can write the effective action as

$$\mathcal{Q}^{\rm eff}=\int d^2k|f_{01}|^2\,4\sum_{l,m=0}\frac{(-)^{l+m}}{(2l+2)!(2m+2)!}\left(\frac{\epsilon^2k^2}{4}\right)^{l+m}\times\frac{2l!\,2m!}{(l+m)!\,l!\,m!}$$
$$\equiv\int d^2k\,\tfrac{1}{4}\,|k_aA_b-k_bA_a|^2S(k^2)\,, \qquad (A13)$$

$$S(k^2)=\sum_{N=0}^{\infty}\frac{1}{N!^2}\left(-\frac{\epsilon^2k^2}{4}\right)^N C_N \qquad (A14)$$

with

$$C_N=\sum_{l,m=0}^{N}\frac{1}{l+1}\,\frac{1}{2l+1}\,\frac{1}{m+1}\,\frac{1}{2m+1}\,\frac{N!}{l!\,m!}\delta_{N,l+m}\,. \qquad (A15)$$

Note that the answer is positive definite, since it is the average of positive quantities. When one computes the effective action in R^4, one can similarly select the direction of k_μ to make life easier.

If we use small disks instead of squares for the averaging region, we find that S is changed to S',

$$S'(k^2)=\frac{4}{\epsilon^2k^2}[J_1((k^2\epsilon^2)^{1/2})]^2\,. \qquad (A16)$$

In terms of coefficients of ϵ^2k^2,

$$S'(k^2)=\sum_{l,m}\frac{(-)^{l+m}}{l!\,(l+1)!\,m!\,(m+1)!}\left(\frac{\epsilon^2k^2}{4}\right)^{l+m}. \qquad (A17)$$

If we compare to this series, we find

$$S(k^2)=\sum_{l,m=0}\frac{(-)^{l+m}}{l!\,(l+1)!\,m!\,(m+1)!}\left(\frac{\epsilon^2k^2}{4}\right)^{l+m}\times\left[\frac{l!\,m!}{(l+m!}\,\frac{1}{2l+1}\,\frac{1}{2m+1}\right]. \qquad (A18)$$

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