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Stretching a black hole

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The geometry of the event horizon of a Kerr black hole in a magnetic field is examined at the poles and equator. It is shown that there are two distinct domains where the Gaussian curvature is negative at the pole, whereas if the angular momentum parameter exceeds a certain value the equator always exhibits positive curvature. A calculation of the polar and equatorial circumferences indicates that the external field stretches the hole along the axis of symmetry.

We intend to investigate the effect of an external magnetic field that is oriented along the axis of symmetry upon the geometry of the event horizon of a Kerr black hole. This is best achieved by looking at the invariant local measure of the intrinsic deviation of the horizon from a spherical surface. Such a quantity is characterized by the Gaussian curvature, denoted by K, which is independent of the embedding space.[1] This calculation was performed by Smarr[2] for the Kerr metric, where he showed that if the spin of the black hole exceeds a certain limit the event horizon acquires negative Gaussian curvature in the vicinity of the polar caps. The flattening of the polar caps as the rotation increases is analogous to that for a rotating fluid body, although the surface of a material body can nowhere possess negative curvature.

Fortunately an exact solution of the Einstein-Maxwell field equations for a Kerr black hole in an external magnetic field exists in an explicit form.[3] Because we are only concerned with the event horizon, we will give only the two-dimensional line element since the metric is quite complicated. Here B denotes the magnetic field parameter, the field being directed along the symmetry axis of the hole. The line element can be written as

$$ds^2 = E^2 d\theta^2 + G^2 d\phi^2, \tag{1a}$$

where

$$E^2 = E^2(\theta) = \Sigma \Lambda \Lambda^*, \tag{1b}$$

$$G^2 = G^2(\theta) = \frac{(r_+{}^2 + a^2)^2}{E^2} \sin^2\theta, \tag{1c}$$

and

$$\Sigma = r_+{}^2 + a^2\cos^2\theta, \tag{1d}$$

$$\Lambda = 1 + \frac{B^2}{4}\left[(r_+{}^2 + a^2)\sin^2\theta - 2mai\cos\theta(3-\cos^2\theta) + \frac{2ma^2\sin^4\theta}{r_+ + ia\cos\theta}\right], \tag{1e}$$

and where a is the spin parameter (angular momentum per unit mass), m the geometric mass parameter, and

$$r = r_+ = m + (m^2 - a^2)^{1/2} \tag{1f}$$

is the value of the radial coordinate defining the event horizon. For B, the units are determined by the formula[4,5]

$$mB = 8.5 \times 10^{-9}\left(\frac{M}{M_\odot}\right)\left(\frac{B}{10^{12}\text{ gauss}}\right), \tag{2}$$

where M is the hole's mass expressed in solar mass units.

Using the standard procedure to calculate the Gaussian curvature, namely,

$$K = -\frac{1}{2EG}\frac{d}{d\theta}\left(\frac{1}{EG}\frac{d}{d\theta}G^2\right), \tag{3}$$

a very complex expression is derived, which for the pole reduces to

$$K_{\theta=0} = \rho^{-2}m^{-2}(f^2+q^2)^{-2}[\rho(f^2-3q^2) + 2\beta^2(f^2+q^2)], \tag{4a}$$

where

$$\rho = 1 + \beta^4 q^2, \tag{4b}$$

and $\beta = mB$ is defined as a dimensionless distortion parameter.[5] $q = a/m$ and $f = r_+/m$ are both constants with the range of permissible values $0 \leq q \leq 1$ and $1 \leq f \leq 2$. For the equator $\theta = \pi/2$, K becomes

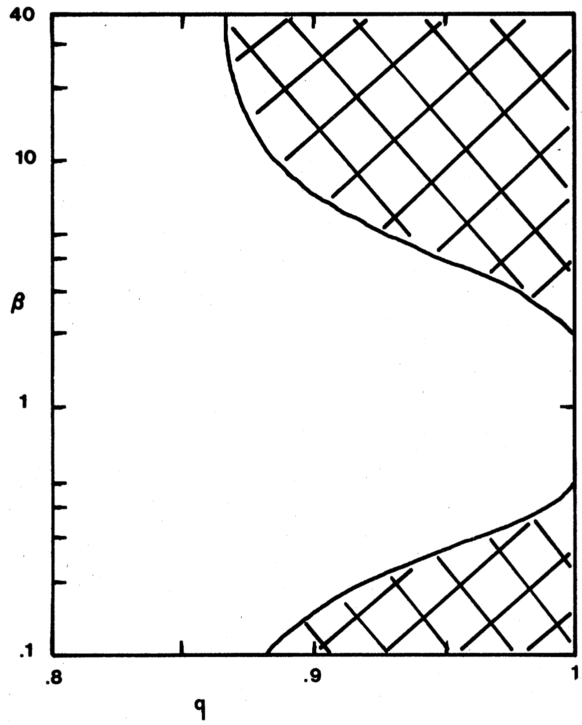

FIG. 1. Plot of β versus q illustrating the zones where K, the Gaussian curvature, is negative, given by the hatched areas. This is for the pole $\theta = 0$.

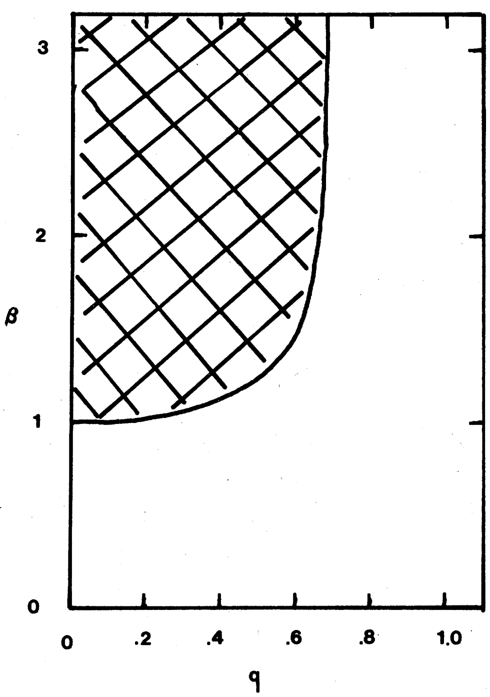

FIG. 2. Plot of β versus q at the equator $\theta = \pi/2$. The hatched area indicates where K is negative. Note that for $q \gtrsim 0.682$ K is always positive.

$$K_{\theta=\pi/2} = \Psi^{-4} f^{-2} m^{-2}\{\Psi^2(1+q^2f^{-2}) - \Psi\beta^2[\tfrac{1}{2}(f^2+q^2)+2q^2f^{-1}+q^4f^{-3}] + \tfrac{1}{4}\beta^4 q^2(3+q^2f^{-2})^2\}, \tag{5a}$$

where

$$\Psi = 1 + \tfrac{1}{2}\beta^2[\tfrac{1}{2}(f^2+q^2)+q^2f^{-1}]. \tag{5b}$$

An analysis of the character of K at the poles and equator should allow us to infer the behavior of K in the neighborhood of these points. Also, the behavior in these regions will give us an idea of the peculiarities of the geometry of a spinning black hole embedded in very strong magnetic fields. As discussed below, we can compute both the polar and equatorial circumferences to get a picture of how the event horizon is distorted in space.

Figure 1 shows the domain of values for q and β for which K is negative at the poles. Interestingly, there are two disjoint regions (indicated by the hatched areas) where K is negative. If $\beta = 0$, Smarr[2] showed that the surface is everywhere positive for $q \leq 0.866$, whereas if $q = 0$, the pole possesses positive K for all β. For the equator, Fig. 2 illustrates the situation as determined from Eqs. (5a) and (5b). For $q \gtrsim 0.682$, K is a positive for all β, and if $q = 0$, $K < 0$ when $\beta > 1$. However, it appears that in the interval $0.682 < q < 0.866$ these two regions exhibit positive K for all β. Is this true for the entire surface? Such an analysis will involve studying the general expression for K.

The magnetic field stretches the black hole along the axis of symmetry. This effect is observed by examining the equatorial circumference C_e and the polar circumference C_p. These quantities are obtained by performing the integrations

$$C_e = \int_0^{2\pi} G(\pi/2)\,d\phi, \tag{6a}$$

$$C_p = \int_0^{2\pi} E(\theta)\,d\theta. \tag{6b}$$

For C_e,

$$\frac{C_e}{m} = \frac{2\pi(f^2+q^2)}{f[1+\frac{1}{4}\beta^2(f^2+q^2+2q^2f^{-1})]}, \tag{7}$$

so that as β increases, C_e/m will decrease for any value of q. C_p/m must be numerically evaluated due to the complexity of E. The results

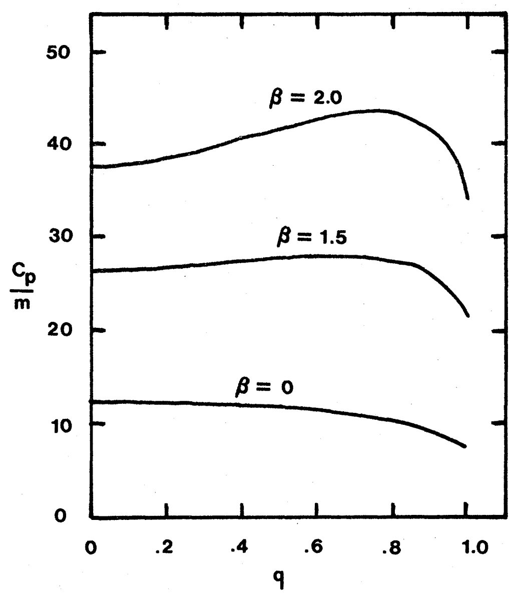

FIG. 3. C_p/m versus q for several values of β. Note how the polar circumference increases for increasing β though the rotation still has an effect.

of this integration for several choices of β are shown in Fig. 3. For $q=0$ and $\beta=0$, C_p/m is exactly 4π (since $f=2$) and increasing q causes the circumference to reduce. As β increases, the hole's horizon becomes elongated along the axis even though the rotation still influences the extent of this elongation.

If, indeed, external fields cause the hole to deform, it is conceivable that the distribution and infall of matter in the vicinity of the hole will be affected. Further, analyses of the vibration frequencies have often assumed a spherical horizon, though this may not be an accurate physical representation under the unusual conditions that may exist at the centers of some galaxies and quasars (if black-hole models are to be considered for such powerful radio sources). It remains to be seen if the effects outlined here can ever occur in nature, since for a black hole with a mass of about 10^9 suns, B must have a value near 10^{11} gauss if we are to have $\beta \sim 1$. Current black-hole dynamo models consider external fields to play an important role in the energy production mechanisms, but to the authors' knowledge, the field strengths are still much less than 10^{11} gauss. Perhaps under certain circumstances our effects may play a role.

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