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Comment on Fujikawa's analysis applied to the Schwinger model

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Fujikawa's analysis of the noninvariance of the fermion path-integral measure under axial gauge transformations is used to derive the solution of the Schwinger model. Subtleties in proceeding from infinitesimal to finite axial gauge transformations are presented in the Appendix.

Fujikawa[1] has recently shown that the fermion path-integral measure $[D\psi(x)][D\bar{\psi}(x)]$ is not invariant under γ_5 transformations, and that the Jacobian gives rise to an extra factor which corresponds in the path-integral formalism in four space-time dimensions to the Adler-Bell-Jackiw anomaly. In this note, we use Fujikawa's analysis to give a simple derivation of the solution of the Schwinger model.[2]

In two-dimensional Euclidean space, the generating functional for the Schwinger model is

$$Z[J,\eta,\bar{\eta}]=\int [DA][D\psi][D\bar{\psi}] \times \exp\left\{-S[A,\psi,\bar{\psi}] + \int d^2x[A_\mu(x)J_\mu(x)+\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]\right\}, \quad (1)$$

with

$$S[A,\psi,\bar{\psi}]=\int d^2x[-i\bar{\psi}\gamma_\mu(\partial_\mu - ieA_\mu)\psi + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \text{gauge-fixing terms}] \quad (2)$$

and

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}. \quad (3)$$

In two dimensions, one can always find Φ so that[2]

$$e\gamma_\mu A_\mu = \gamma_\mu \partial_\mu \Phi. \quad (4)$$

In particular, if we pick the gauge

$$\partial_\mu A_\mu = 0, \quad (5)$$

then

$$\Phi=\gamma^5\alpha(x) \quad (\gamma^5 = i\gamma^1\gamma^2), \quad (6)$$

where

$$\alpha(x) = -ieB(x) \quad (7)$$

and B satisfies

$$\partial_\mu\partial_\mu B = F_{12} \equiv \partial_1 A_2 - \partial_2 A_1. \quad (8)$$

We can take

$$A_1 = -\partial_2 B, \quad A_2 = \partial_1 B. \quad (9)$$

Equation (4) would seem to suggest that the field A_μ behaves like a pure gauge field, and the gauge transformation

$$\psi(x)=e^{i\gamma^5\alpha(x)}\psi'(x), \quad \bar{\psi}(x)=\bar{\psi}'(x)e^{i\gamma^5\alpha(x)} \quad (10)$$

reduces the action S [see (2)] to

$$S_1[A,\psi',\bar{\psi}'] = \int d^2x[-i\bar{\psi}'\gamma_\mu\partial_\mu\psi' + \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \text{gauge-fixing terms}]. \quad (11)$$

Thus the Schwinger model would appear to be the theory of a free massless fermion and a massless free "photon" in two dimensions. On the other hand, we know[2] that the photon has mass $m=e/\sqrt{\pi}$.

The resolution of this paradox lies in Fujikawa's observation that the measure $[D\psi][D\bar{\psi}]$ is not invariant under a chiral transformation, i.e., that $[D\psi][D\bar{\psi}] \neq [D\psi'][D\bar{\psi}']$. As shown in the Appendix, the correct transformation is

$$[D\psi][D\bar{\psi}] = [D\psi'][D\bar{\psi}']\exp\left[\frac{-e^2}{2\pi}\int d^2x\, A_\mu(x)A_\mu(x)\right] \quad (12)$$

for $\psi, \bar{\psi}$ transforming according to (10) with α satisfying (7) and (8), and A_μ satisfying (9).

The effective action then becomes

$$S_1[A,\psi',\bar{\psi}'] + \frac{e^2}{2\pi}\int d^2x\, A_\mu(x)A_\mu(x), \quad (13)$$

which describes a free fermion and a photon of mass $m=e/\sqrt{\pi}$. Green's functions can be straightforwardly obtained by differentiation of (1) with respect to J, η, and $\bar{\eta}$ followed by the substitutions (10) and (12). This gives formulas which agree with those of Schwinger.[2] This agreement is another vindication of Fujikawa's analysis.

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APPENDIX

We wish to derive formula (12) of the text, which we do in two steps. First, we repeat Fujikawa's argument[1] in two dimensions for infinitesimal γ^5 transformations. Then we integrate this result to finite axial gauge transformations [so that we can apply the gauge transformation indicated by Eqs. (7)–(10) of the text]. This part of the argument is subtle and we are indebted to S. Coleman for indicating the correct approach to us.

By repeating Fujikawa's argument[1] for two dimensions, we obtain straightforwardly that for infinitesimal transformations $\delta\alpha(x)$ such that

$$\psi(x)=e^{i\gamma^5\delta\alpha(x)}\psi'(x)=[1+i\gamma^5\delta\alpha(x)]\psi'(x)\,, \qquad \text{(A1)}$$

then

$$D\psi\, D\bar{\psi}=D\psi' D\bar{\psi}'\left[1+\frac{ie}{2\pi}\int\epsilon_{\mu\nu}F_{\mu\nu}(x)\delta\alpha(x)d^2x\right]. \qquad \text{(A2)}$$

One can easily check that this gives the correct Ward identities for axial gauge transformations in the Schwinger model.

One might conclude that the finite-α version of this is

$$\psi(x)=e^{i\gamma^5\alpha(x)}\psi'(x)\,,$$
$$D\psi\, D\bar{\psi}=D\psi' D\bar{\psi}'\exp\left[\frac{ie}{2\pi}\int\epsilon_{\mu\nu}F_{\mu\nu}(x)\alpha(x)d^2x\right], \qquad \text{(A3)}$$

but this is incorrect. [It will yield the wrong mass for the photon if we choose $\alpha(x)$ according to (7) in the text.] The point is that (A2) is correct as long as the Lagrangian has no axial coupling of the form

$$\bar{\psi}\gamma^\mu\gamma^5\psi A^5_\mu \qquad \text{(A4)}$$

in addition to the usual $\bar{\psi}\gamma^\mu\psi A_\mu$ coupling.

If there had been such a term, the correct infinitesimal version of (A2) would have been

$$D\psi\, D\bar{\psi}=D\psi' D\bar{\psi}'\left[1+\frac{ie}{2\pi}\int\epsilon_{\mu\nu}(F_{\mu\nu}+F^5_{\mu\nu})\delta\alpha(x)d^2x\right], \qquad \text{(A5)}$$

where

$$F^5_{\mu\nu}=-i(\partial_\mu\epsilon_{\nu\rho}-\partial_\nu\epsilon_{\mu\rho})A^5_\rho\,. \qquad \text{(A6)}$$

(A5) and (A6) are trivially obtained from (A2) by recalling that in two dimensions

$$j^5_\mu\equiv\bar{\psi}\gamma^\mu\gamma^5\psi=i\epsilon_{\mu\nu}j_\nu\,, \qquad \text{(A7)}$$

so that (A4) can be rewritten as an ordinary vector coupling.

Now suppose we wish to iterate a sequence of infinitesimal γ_5 transformations. After the first one $[\delta\alpha_1(x)]$, the effective action S contains a term

$$\int d^2x\,\bar{\psi}\gamma^\mu\gamma^5\psi\;\partial_\mu(\delta\alpha_1) \qquad \text{(A8)}$$

after an integration by parts. Thus, the first infinitesimal gauge transformation induces an effective axial-vector coupling of the form (A4) with

$$A^5_\mu=-\frac{1}{e}\partial_\mu\delta\alpha_1\,. \qquad \text{(A9)}$$

Thus $F^5_{\mu\nu}$ depends on $\delta\alpha_1$ and the integration of (A5) for finite α requires a little more care.

The correct equations are

$$\delta(D\psi\, D\bar{\psi})\equiv D\psi' D\bar{\psi}'-D\psi\, D\bar{\psi}$$
$$=D\psi\, D\bar{\psi}\left(\frac{-ie}{2\pi}\int\epsilon_{\mu\nu}[F_{\mu\nu}(x)+F^5_{\mu\nu}(x)]\delta\alpha(x)d^2x\right) \qquad \text{(A10)}$$

and

$$\delta F^5_{\mu\nu}(x)=\frac{i}{e}(\partial_\mu\epsilon_{\nu\rho}-\partial_\nu\epsilon_{\mu\rho})\partial_\rho\delta\alpha(x)\,, \qquad \text{(A11)}$$

subject to

$$F^5_{\mu\nu}=0\,, \qquad \text{(A12)}$$

when $\alpha=0$. The solution of (A11) and (A12) is

$$F^5_{\mu\nu}(x)=\frac{i}{e}(\partial_\mu\epsilon_{\nu\rho}-\partial_\nu\epsilon_{\mu\rho})\partial_\rho\alpha(x)\,, \qquad \text{(A13)}$$

so that

$$\epsilon_{\mu\nu}(F_{\mu\nu}+F^5_{\mu\nu})=\epsilon_{\mu\nu}F_{\mu\nu}-\frac{2i}{e}\partial^2\alpha(x)\,. \qquad \text{(A14)}$$

The solution of (A10) is then

$$D\psi' D\bar{\psi}'=D\psi\, D\bar{\psi}\exp\left\{\frac{-ie}{2\pi}\int\left[\epsilon_{\mu\nu}F_{\mu\nu}(x)\alpha(x)-\frac{i}{e}\alpha(x)\partial^2\alpha(x)\right]d^2x\right\}, \qquad \text{(A15)}$$

where ψ and ψ' are related by (A3).[3] Now choose $\alpha(x)$ according to (7) of the text, i.e.,

$$\alpha(x)=-ieB(x) \qquad \text{(A16)}$$

with

$$\partial^2 B = F_{12} = \tfrac{1}{2}\epsilon_{\mu\nu} F_{\mu\nu}\,, \tag{A17}$$

then substituting into (A15) gives

$$D\psi' D\bar{\psi}' = D\psi D\bar{\psi}\exp\left(\frac{-e^2}{2\pi}\int B\partial^2 B\, d^2x\right) \tag{A18}$$

$$= D\psi D\bar{\psi}\exp\left[\frac{e^2}{2\pi}\int(\partial_\mu B)(\partial_\mu B) d^2x\right] \tag{A19}$$

$$= D\psi D\bar{\psi}\exp\left(\frac{e^2}{2\pi}\int A_\mu A_\mu\, d^2x\right) \tag{A20}$$

from (9). This establishes (12).

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[1]K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D 21, 2848 (1980).

[2]J. Schwinger, Phys. Rev. 128, 2425 (1962).

[3]The quadratic term in α in (A15) appears to violate the group property associated with the composition of two axial gauge transformations. But (A15) is correct only if $F^5_{\mu\nu}$ vanishes. If we perform the axial gauge transformation with parameter $\alpha(x)$ followed by one with $\beta(x)$, then after the α transformation the action S picks up a $F^5_{\mu\nu}$ piece given by (A13). Then in the β transformation, $F_{\mu\nu}$ in (A15) must be replaced by $F_{\mu\nu}+F^5_{\mu\nu}$. This restores the group property.