PHYSICAL REVIEW D VOLUME 23, NUMBER 2 15 JANUARY 1981

Brief Reports

Brief Reports are short papers which report on completed research with no expectation of follow-up publication or are addenda to papers previously published in the **Physical Review** *by the same authors. A Brief Report may be no longer than 3½ printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Geometrical origin of gauge theories

R. Gambini and A. Trias

Departamento de Física, Universidad Simón Bolívar, Apartado Postal:80659, Caracas 108, Venezuela

(Received 15 July 1980)

The identification of a basic geometric group structure acting on the manifold of paths $P(M)$ of some manifold M enables us to derive all the kinematical elements of a gauge theory from the single postulate of invariance of the formulation under this group.

The relevance of path dependence to gauge theories was first emphasized by Mandelstam[1] in the context of ordinary electromagnetism. The path-dependent formulation was extended later to the non-Abelian case by Bialynicki-Birula[2] and Mandelstam.[3] As was pointed out by Yang,[4] one of the most interesting features of the formulation consists in its global character in contrast with the ordinary local or differential formulations. Recently, Polyakov[5] has proposed to consider a gauge field as a chiral field defined in the space of all the possible contours, following the ideas of Wilson[6] and Kogut and Susskind[7] on the role of closed strings as elementary excitations in the confining phase of a gauge theory.

In the present paper it is shown how all the kinematical elements of a gauge theory may be derived from the single postulate of invariance of the field under some basic geometric group structure in the manifold of paths. From the mathematical point of view, the postulate provides a natural and unified geometric explanation of most of the properties of a gauge field and identifies the class of path-dependent objects that may be used to represent a gauge field. On the physical side, since the postulate is equivalent to the assumption of invariance of the field under the complete group of motions of its basis space, one gains new insight on the noninteracting character of a classical gauge theory when formulated in the language of path dependence.

Let us then consider some finite-dimensional manifold M and choose some reference point 0 on it. Let us denote by C_0 the set of all closed oriented paths starting at 0. In this set one has the familiar noncommutative operation that associates with C_1 and C_2 the path C_1C_2 obtained by following first C_1 and then C_2. The operation is associative and has a null element. However, there is no inverse. One could consider as the natural candidate for the inverse of C the path $\overline{C}$ obtained by following C in the opposite sense. However, $C\overline{C}$ is not the null path unless some identification is made. In order to construct such an identification we shall introduce the concept of a tree. From a heuristic point of view, a tree is a "null area" closed path which is contractible "within itself" to the null path. This idea may be stated in more formal terms by considering the homotopy group associated with the continuous set of points that define the path C in the manifold. Then C is a tree if its homotopy group is the identity. With this concept in hand, one may now introduce the relation

$$C_1 = C_2, \quad \text{if } C_1\overline{C}_2 \text{ is a tree,}$$

which is immediately seen to be an equivalence relation. Let us then define a loop as a class of this equivalence relation and the set of loops forms a group that one may denote by L_0. One is also immediately convinced that groups corresponding to different reference points are isomorphic and then one may omit any reference to 0 and denote by $\mathrm{L}(M)$ the group structure to which all the L_0 are isomorphic. We shall know this group as the group of loops of M.

This group could also have been introduced in a more algebraic way in terms of polygonals. Since this optional definition is easier to use in some respects and it is more directly related to functional sums over the group, we are going to discuss it in some detail in the case of R^4. A closed polygonal path may be identified with a finite string of four-dimensional displacements:

$$C = (u_1, u_2, \ldots, u_N), \quad u_1 + u_2 + \ldots + u_N = 0. \qquad (1)$$

One may now define the contraction of a string

$$(u_1, u_2, \ldots, u_N)_C$$

as the operation which replaces any two collinear adjacent vectors in the string by its resulting displacement. Then if u_3 and u_4 are the only collinear adjacent vectors in the string $(u_1, u_2, u_3, u_4, u_5)$, one has

$$(u_1, u_2, u_3, u_4, u_5)_C = (u_1, u_2, u_3 + u_4, u_5)\,. \qquad (2)$$

Let us now define a loop as any string which is equal to its contraction. The operation between loops is defined by

$$(u_1, u_2, \ldots, u_N)(v_1, v_2, \ldots, v_M) = (u_1, \ldots, u_N, v_1, \ldots, v_M)_C \qquad (3)$$

and it is immediately seen that the set of loops forms a group under this operation. This is an optional definition of the group $\mathsf{L}(R^4)$. In both definitions it is clear that a loop is defined as a closed path for which any trivial null area part is declared nonexistent.

Let us now turn our attention to open paths. Given two points 0 and 0′ in the manifold, one may consider the set of all oriented open lines joining 0 with 0′. Two lines $P_{00'}$ and $Q_{00'}$ are declared equivalent if the closed path $P_{00'}\overline{Q}_{0'0}$ is a tree. Then a path is defined as one of these classes of equivalence and the set of all open paths between 0 and 0′ denoted as $\mathsf{P}_{00'}$. The group of loops L_0 operates in a natural way on this set:

$$L_0 \times \mathsf{P}_{00'} \to \mathsf{P}_{00'}\,,$$

by associating with the pair (L, P) the path P' obtained by following first the loop L and then the path P. Every path in $\mathsf{P}_{00'}$ may be obtained by operation of the group of loops on some fixed reference path. Hence L_0 is a complete group of motions for the set of open paths.

A gauge field is then defined as a chiral field taking values in the set of open paths. It is postulated that the field is invariant under the action of the group of loops. The invariance is stated in the usual way by assuming the existence of a (necessarily unfaithful) unitary representation of the group of loops by matrices W belonging to the gauge group G:

$$W(L_1 L_2) = W(L_1)W(L_2)\,,\; W(L_0) = 1\,, \qquad (4)$$

and the gauge field $\Phi(P)$ transforms under the action of a loop by

$$\Phi\,(LP) = W(L)\Phi(P)\,. \qquad (5)$$

The differential equations of the path-dependent formulation are usually written in terms of the Mandelstam[1] derivatives D_μ which measure the change induced in the gauge field when some path $P(x)$, ending at x, is extended by an infinitesimal vector u_μ to reach the point $x+u$:

$$\Phi\{P_E(x+u)\} = (1 + u^\mu D_\mu)\Phi\{P(x)\}\,. \qquad (6)$$

Let us consider two paths $P(x)$ and $P'(x)$ with the same ends. The path P' may be obtained by operation of the loop $(P'\overline{P})$ on P. Hence, according to (5) one must have

$$\Phi\,(P') = W(P'\overline{P})\Phi(P)\,. \qquad (7)$$

If we now extend both paths with the same infinitesimal vector u_μ one must also have

$$\Phi\,(P'_E) = W(P'_E\,\overline{P}_E)\Phi(P_E)\,. \qquad (8)$$

However, the loops $(P'_E\overline{P}_E)$ and $(P'\overline{P})$ are identical since they differ by a tree. Hence, according to (6) one may write

$$D_\mu\Phi\,(P') = W(P'\overline{P})D_\mu\Phi(P)\,. \qquad (9)$$

Hence, $D_\mu\Phi$ transforms like Φ and the Mandelstam derivative is covariant as a direct consequence of the defining property of the group of loops.

The local structure of the representation $W(L)$ may be obtained as usual by considering the infinitesimal elements of the group of loops. Let us then introduce the infinitesimal loop $\delta L(Puv\overline{uv}\overline{P})$ obtained by following some path P ending at x, then an infinitesimal parallelogram spanned by vectors u_μ and v_μ, and finally going back to the origin along $\overline{P}$. This loop is infinitesimal since, for $u_\mu, v_\mu \to 0$, it reduces $P\overline{P}$, which is identical to the null loop. Hence, for fixed P, one may expand $W(\delta L)$ in powers of u_μ and v_μ to obtain

$$W(\delta L) = 1 + u^\mu B_\mu(P) + v^\mu C_\mu(P) + \tfrac{1}{2}(u^\mu v^\nu + u^\nu v^\mu)S_{\mu\nu}(P) + \tfrac{1}{2}(u^\mu v^\nu - u^\nu v^\mu)A_{\mu\nu}(P)\,, \qquad (10)$$

where one has separated the symmetric and antisymmetric parts of the second-order terms by convenience. By using again the defining property of the group of loops, it is clear that if $u_\mu = \lambda v_\mu$, δL must reduce to the null loop for any v_μ and λ. It is immediately seen that this implies the vanishing of the B, C, and S terms, and then by expanding the remaining contribution in terms of the generators T_a of the gauge group, one finally obtains

$$W\{\delta L(Puv\overline{uv}\overline{P})\} = 1 + i\frac{g}{2}\delta\sigma^{\mu\nu}F^a_{\mu\nu}(P)T_a\,, \qquad (11)$$

which may be considered as the definition of the path-dependent field strength $F^a_{\mu\nu}(P)$.

The path-dependent structure of the generator may also be studied by considering the infinitesimal loop $\delta L(\Pi ab\overline{ab}\,\overline{\Pi}\; Puv\overline{uv}\overline{P}\; \Pi ba\overline{ba}\,\overline{\Pi})$ which contains a perturbation of the path P by the loop $\delta L'(\Pi ab\overline{ab}\,\overline{\Pi})$.

According to (11) we immediately obtain the equations

$$F^{a}_{\mu\nu}(\delta L'\cdot P)=F^{a}_{\mu\nu}(P)+\frac{g}{2}C^{a}_{bc}F^{b}_{\mu\nu}(P)F^{c}_{\alpha\beta}(\Pi)\delta\sigma^{\alpha\beta}\,, \tag{12}$$

where C^{a}_{bc} are the structure constants of G. Equation (12) is in fact a small generalization of the Mandelstam[3] equations to which it reduces when Π is a part of P.

The Bianchi identities may be derived by considering the somewhat elaborated tree

$$L_O=Pabc\overline{bcaP}\;Pac\overline{acP}\;Pcab\overline{abcP}\times Pcb\overline{cbP}\;Pbca\overline{cabP}\;Pba\overline{baP}\,, \tag{13}$$

containing an open path P and three infinitesimal displacements a_μ, b_μ, and c_μ forming a parallelepiped at the end of P. Then by using (11) and the definition of the Mandelstam derivative, it is easy to obtain the equations

$$D_\mu F^{a}_{\nu\lambda}(P)+D_\nu F^{a}_{\lambda\mu}(P)+D_\lambda F^{a}_{\mu\nu}(P)=0\,. \tag{14}$$

One could also have started the study of $W(L)$ with the optional family of infinitesimal loops δL $(Pu\overline{P})$ obtained by following some path P up to x, going then along some vector u_μ, and finally going back to the origin along $\overline{P}$. A similar analysis may be carried out in terms of the associated generator

$$A^{a}_{\mu}(P)=\int_{P(x)}^{x}dy^{\lambda}F^{a}_{\lambda\mu}(\not{P}(y))\,, \tag{15}$$

where $\not{P}(y)$ is that portion of P leading to y. This is the path-dependent potential discussed in a previous paper[8] which allows for the ordinary descomposition of the field strength by introducing the appropiate differential operators.

Hence, all the kinematical elements associated with the degrees of freedom of the path are obtained from the single postulate of invariance of the gauge field under the group of loops. The dynamical equations and the transformation properties of the gauge field under the Poincaré group can then be introduced in the conventional way.

[1]S. Mandelstam, Ann. Phys. (N.Y.) 19, 1 (1962).
[2]I. Bialynicki-Birula, Bull. Acad. Pol. Sci. 11, 135 (1963).
[3]S. Mandelstam, Phys. Rev. 175, 1580 (1968).
[4]C. N. Yang, Phys. Rev. Lett. 33, 445 (1974).
[5]A. M. Polyakov, Phys. Lett. 82B, 247 (1979).
[6]K. Wilson, Phys. Rev. D 10, 2445 (1974).
[7]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
[8]R. Gambini and A. Trias, Phys. Rev. D 22, 1380 (1980).