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Dirac theory and its Melosh quantum-mechanical representation

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Some properties of the Melosh quantum-mechanical representation are derived and discussed on the basis of interrelations between Foldy-Wouthuysen, Eriksen, and Melosh transformations.

The existence of a unitary transformation relating the $SU(6)_{W,\,\mathrm{strong}}$ and $SU(6)_{W,\,\mathrm{currents}}$ groups has been suggested by Gell-Mann.[1] Such a transformation has been constructed by Melosh[2] in the context of the free-quark model. Different aspects of the Melosh proposal(s) have already been discussed[3-7] and some of their implications in related fields have also been studied.[8-12] So, recent contributions state and deal with Melosh transformations, but for brevity we refer the reader to the literature[2-12] and to the references therein.

Actually, in relativistic quantum mechanics, a unitary transformation which eliminates the α_1 and α_2 terms in the *free* Dirac Hamiltonian is often called a "Melosh transformation" for evident reasons,[3] although it has, in principle, nothing to do with the original context.[1,2] In fact, such a relativistic quantum-mechanical transformation belongs to a *class* of unitary transformations whose different properties have already been emphasized and discussed. In particular, Weaver[9] compared the so-called Melosh transformation with those of Foldy and Wouthuysen[13] (FW), Cini and Touschek[14] (CT), and Majorana[15] in the free case. He also studied[10] the energy eigenvalues for a charged Dirac particle in a homogeneous magnetic field and recognized the corresponding Melosh transformation as the Tsai proposal.[16]

Each transformation of the above-mentioned class has specific physical interests. For example, the FW and CT transformations are meaningful when nonrelativistic and ultrarelativistic limits of the Dirac theory, respectively, are under study; the Majorana and Melosh transformations show direct relevance to physics when the high-energy limit is taken into account, or when the transverse momentum (of the particle) is large compared to its longitudinal momentum. These physical situations evidently are intimately connected with the null-plane formalism and the study of the infinite-momentum frame. Let us only mention that, in the original Melosh context, connections between constituent quarks and current quarks have been proposed through the use of lightlike (or null-plane) charges which are especially suitable in dealing[7] with the infinite-momentum limit. So the study of "objects" like the Melosh transformation, besides their specific interests in connection with the precise context of the Dirac theory, could, maybe, lead to interesting remarks in connection with characteristic features in strong and weak interactions.

Here, the aim of this note is to give more information and properties on the *quantum-mechanical* Melosh transformation, seen as an element of this interesting class of unitary transformations acting on the Dirac equations and operators. Our new results are, *first*, the use of projection operators giving a deeper insight into the physical meaning of the transformation and, *second*, the explicit contents of the Melosh *representation* by comparison with the original FW results.[13]

Let us, at once, recall that the Melosh transformation takes the explicit form[17,18]

$$U_M=\exp(iS_M),\quad S_M=\tfrac{1}{2}\tan^{-1}\left(-i\,\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{m}\right),\qquad U_M^\dagger=U_M^{-1}\,, \tag{1}$$

where the index $\perp$ refers to the transverse components, m is the nonzero rest mass of the Dirac particle, and $\vec{\gamma}=\beta\vec{\alpha}$ are (anti-Hermitian) Dirac matrices. Applied to the free Dirac Hamiltonian

$$H_0=\vec{\alpha}\cdot\vec{p}+\beta m\,, \tag{2}$$

U_M gives

$$H'_{0,M}=U_M H_0 U_M^{-1}=\beta E_{p,\perp}+\alpha_3 p_3\,, \tag{3}$$

where $E_{p,\perp}=(\vec{p}_\perp^{\,2}+m^2)^{1/2}$ corresponds to the usual "energy" $E_p=(\vec{p}^{\,2}+m^2)^{1/2}$.

Now, let us mention that in a recent work[19] we constructed the "Eriksen-type" form[20] of the Melosh transformation in the quantum-mechanical context. In fact, we noticed that the interesting role of the matrix β in the Foldy-Wouthuysen-Eriksen developments has to be played by the Dirac spin matrix Σ_3 when Melosh-Eriksen interrelations are considered. Then we proposed the following Eriksen-type form of Eq. (1):

$$U_M=\tfrac{1}{2}(1+\Sigma_3\lambda)[1+\tfrac{1}{4}(\Sigma_3\lambda+\lambda\Sigma_3-2)]^{-1/2} \tag{4}$$

with

$$\Sigma_3 = -\frac{i}{2}(\vec{\alpha}\times\vec{\alpha})_3 = i\gamma_1\gamma_2 \tag{5}$$

and

$$\lambda = \Sigma_3(m + \vec{\gamma}_\perp\cdot\vec{p}_\perp)E_{p,\perp}^{-1}. \tag{6}$$

Let us also recall that one of the well-known interests of an Eriksen-type transformation is its "closed" character allowing one to consider simultaneously the free as well as the interacting cases. Then if we consider the interaction of a charged Dirac particle in a constant homogeneous magnetic external field, we directly recover the Weaver[10] or Tsai[16] considerations through the substitution

$$\vec{p}_\perp \to \vec{\Pi}_\perp = \vec{p}_\perp - e\vec{A}_\perp \tag{7}$$

in Eqs. (4) and (6), e being the charge of the particle. So Eq. (4) becomes the closed form of the Melosh transformation with

$$\lambda^I = \Sigma_3(m + \vec{\gamma}_\perp\cdot\vec{\Pi}_\perp)E_{\Pi,\perp}^{-1}, \tag{8}$$

where the superscript I refers to the interacting case. Let us finally notice that other closed forms of unitary transformations of the Dirac equation have already been discussed. In a recent work,[21] Moss and Okninski developed a method for obtaining a class of unitary transformations which bring the Dirac Hamiltonian for the free electron into a diagonal form. This class includes all the transformations described by Weaver[9] but not the Melosh one because, in this case, the transformed Hamiltonian is not diagonal (only its square is).

Let us now come to the original part of this note. *First*, we construct projection operators associated with the Melosh transformation. Let us write the transformation (1) or (4) in the form

$$U_M = \frac{m + E_{p,\perp} + \vec{\gamma}_\perp\cdot\vec{p}_\perp}{2E_{p,\perp}(E_{p,\perp}+m)^{1/2}}. \tag{9}$$

Then it is easy to show that

$$\Sigma_3 = U_M \lambda U_M^\dagger. \tag{10}$$

Moreover, if we notice that the matrix λ in Eq. (6) can be written as

$$\lambda = \Sigma_3\beta(\vec{\alpha}_\perp\cdot\vec{p}_\perp + \beta m)E_{p,\perp}^{-1} = \Sigma_3\beta H_\perp(H_\perp{}^2)^{-1/2} \tag{11}$$

$$= \Sigma_3\beta\lambda_\perp, \tag{12}$$

and that

$$[\Sigma_3\beta, U_M] = 0, \tag{13}$$

we immediately get

$$\beta = U_M\lambda_\perp U_M^\dagger. \tag{14}$$

Then, from Eqs. (10) and (14), we obtain

$$P_\pm^{\Sigma_3} = U_M P_\pm^\lambda U_M^\dagger, \quad P_\pm^\beta = U_M P_\pm^{\lambda_\perp} U_M^\dagger, \tag{15}$$

where $P_\pm^{\Sigma_3}$, $P_\pm^\lambda$, $P_\pm^\beta$, and $P_\pm^{\lambda_\perp}$ are *projection* operators with an evident physical meaning in the Dirac theory when

$$P_\pm^X = \tfrac{1}{2}(1 \pm X), \quad X \in \{\Sigma_3, \lambda, \beta, \lambda_\perp\}. \tag{16}$$

As an example, the projectors $P_\pm^\beta$ refer, as usual, to the subspaces of large (+) and small (−) components of the Dirac wave function and $P_\pm^{\lambda_\perp}$ to the subspaces of definite positive (+) and negative (−) energies *when* $p_3 \equiv 0$. Thus, through Eq. (15), the Melosh transformation connects these respective subspaces. Here, let us recall that from the FW transformation[13,20] it can be shown that

$$P_\pm^\beta = U_{FW} P_\pm^\lambda U_{\text{FW}}^\dagger, \tag{17}$$

giving correspondences between the subspaces of large (and small) components *and* the subspaces of definite positive (and negative) energies, respectively, these correspondences being expected when *nonrelativistic* considerations have to be investigated. So the physical meaning of the Melosh transformation appears very clearly through Eqs. (15) and by comparison with Eq. (17) for example.

Such considerations on projection operators are also useful when the transverse momentum of the particle is large compared to its longitudinal momentum, i.e., when the physical situation is such that $p_3/(m^2+\vec{p}_\perp{}^2)^{1/2} \ll 1$. Indeed, it is then straightforward to connect our developments with those of Weaver[22]; his K operator is nothing else but our $\beta\Sigma_3$ matrix connecting λ and $\lambda_\perp$ [cf. Eq. (12)]. So, projection operators defined by Eqs. (15) and (16) can be simply interpreted in this modified Majorana representation[22] and the associated two-component forms of the Dirac theory can directly be related to "good" and "bad" component descriptions.[8]

Finally, let us mention that the expression λ in Eq. (12) does not require the explicit specification of the Hamiltonian and, consequently, is well adapted for an extension to interacting cases. For example, in the case of an electron in a magnetic field, the corresponding relation becomes, with Eq. (8),

$$\lambda^I = \Sigma_3\beta\lambda_\perp^I, \quad \lambda_\perp^I = H_\perp^I(H_\perp^{I\,2})^{-1/2}, \tag{18}$$

and the associated projection operators are easily constructed.

Second, by comparison with the original FW results, we can calculate the different operators—position, velocity, momentum, etc.—belonging to the *Melosh representation* in the *free* case. These straightforward but rather lengthy calculations are summarized in Table I, where we also pointed out the corresponding *mean* operators in the Dirac

TABLE I. The Dirac and Melosh operators in their respective representations.

Operator	Dirac representation	Melosh representation
Position	$\vec{x}$	$\vec{x}'_M=\vec{x}-\frac{(\vec{\Sigma}\times\vec{p}_\perp)_\perp}{2E_{p,\perp}(m+E_{p,\perp})}-\frac{i\beta\vec{\alpha}_\perp}{2E_{p,\perp}}+\frac{i\beta(\vec{\alpha}\cdot\vec{p}_\perp)\vec{p}_\perp}{2E_{p,\perp}{}^2(E_{p\perp}+m)}$
Velocity	$\dot{\vec{x}}=\vec{\alpha}$	$\dot{\vec{x}}'_M=\vec{\alpha}+\frac{\beta\vec{p}_\perp}{E_{p,\perp}}-\frac{(\vec{\alpha}\cdot\vec{p}_\perp)\vec{p}_\perp}{E_{p,\perp}(E_{p,\perp}+m)}$
Momentum	$\vec{p}$	$\vec{p}'_M=\vec{p}$
Hamiltonian	$H_0=\vec{\alpha}\cdot\vec{p}+\beta m$	$H'_{0,M}=\alpha_3p_3+\beta E_{p,\perp}$
Orbital angular momentum	$\vec{L}=\vec{x}\times\vec{p}$	$\vec{L}'_M=\vec{x}'_M\times\vec{p}$
Spin	$\vec{S}=\frac{1}{2}\vec{\Sigma}\equiv\frac{1}{2}\frac{\vec{\alpha}\times\vec{\alpha}}{2i}$	$\vec{S}'_M=\frac{1}{2}\left[\frac{m}{E_{p,\perp}}\vec{\Sigma}+\frac{\vec{p}_\perp(\vec{p}_\perp\cdot\vec{\Sigma})}{E_{p,\perp}(E_{p,\perp}+m)}+\frac{i\beta(\vec{\alpha}\times\vec{p}_\perp)}{E_{p,\perp}}\right]$
Mean position	$\vec{X}=\vec{x}-\frac{(\vec{\Sigma}\times\vec{p}_\perp)_\perp}{2E_{p,\perp}(E_{p,\perp}+m)}+i\frac{\beta\vec{\alpha}_\perp}{2E_{p,\perp}}-\frac{i}{2}\frac{\beta(\vec{\alpha}\cdot\vec{p}_\perp)\vec{p}_\perp}{E_{p,\perp}{}^2(E_{p,\perp}+m)}$	$\vec{X}'_M=\vec{x}$
Mean velocity	$\dot{\vec{X}}=\frac{\vec{p}_\perp}{E_{p,\perp}}\frac{\vec{\alpha}\cdot\vec{p}_\perp+\beta m}{E_{p,\perp}}$	$\dot{\vec{X}}'_M=\frac{\beta\vec{p}_\perp}{E_{p,\perp}}+(\vec{\alpha}\cdot\vec{e}_z)\vec{e}_z$
Mean orbital angular momentum	$\vec{\mathfrak{L}}=\vec{X}\times\vec{p}$	$\vec{\mathfrak{L}}'_M=\vec{X}'_M\times\vec{p}'_M=\vec{x}\times\vec{p}$
Mean spin	$\vec{\mathfrak{S}}=\frac{1}{2}\left[\frac{m}{E_{p,\perp}}\vec{\Sigma}+\frac{\vec{p}_\perp(\vec{p}_\perp\cdot\vec{\Sigma})}{E_{p,\perp}(E_{p,\perp}+m)}-\frac{i\beta(\vec{\alpha}\times\vec{p}_\perp)}{E_{p,\perp}}\right]$	$\vec{\mathfrak{S}}'_M=\vec{S}=\frac{1}{2}\vec{\Sigma}$

representation. These formulas give us a complete insight into the quantum-mechanical context of the Melosh representation. As an example we note that the transverse components of the new mean velocity operator commute with each other but not with the longitudinal component. Such considerations are directly connected with the "good" use of the Melosh transformation in the context of the infinite-momentum limit from where it was originated in hadronic physics.[2,3] As another example, it is easy to point out that the longitudinal components of the new mean orbital and intrinsic angular momentum operators are *separately* constants of motion:

$$[(\vec{\mathfrak{L}}'_M)_3,H'_0]=[(\vec{\mathfrak{S}}'_M)_3,H'_0]=0\,. \tag{19}$$

This was expected in what concerns the third component of the spin operator through its connection with the helicity operator at the infinite-momentum limit.[23] Let us also mention that the results contained in Table I can immediately be compared with those of the FW representation as summarized in particular by De Vries.[24]

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