

U(1) problem: Current algebra and the θ vacuum

Pran Nath*

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

R. Arnowitt

Department of Physics, Northeastern University, Boston, Massachusetts 02115

(Received 7 January 1980)

The effective Lagrangian which gives a full solution of the U(1) problem is obtained from the $1/N$ expansion of quantum chromodynamics. This Lagrangian satisfies all the anomalous n -point Ward identities for arbitrary q^2 , and includes chiral and SU(3) breaking. The relationship between the η' mass and the θ dependence of the vacuum energy in the absence of quarks and the form of the strong CP violation in the effective-Lagrangian framework are deduced.

The U(1) problem has been a long-standing puzzle in the quantum-chromodynamic (QCD) theory of strong interactions.¹ Recently, Witten² has proposed a resolution of the U(1) problem by analyzing the effect of the axial U(1) anomaly on the QCD Lagrangian within the framework of the $1/N$ expansion.^{3,4} While the anomaly produces effects which are $1/N$ smaller than the leading terms, it contributes a significant nonzero contribution to the mass of the singlet 0^- meson in the chiral limit, thus splitting the singlet-octet masses in that limit. In this note we show that these ideas lead directly to an effective-Lagrangian formulation which (1) satisfies all anomalous n -point Ward identities (WI) and PCAC (partial conservation of axial-vector current) conditions of $U(3) \times U(3)$ current algebra, (2) includes arbitrary amounts of chiral and SU(3) breakdown, and (3) includes the strong-CP-violating θ -dependent effects. The satisfaction of the anomalous WI is accomplished by the introduction of Kogut-Susskind⁵-type poles needed to account for the nonvanishing topological charge. The general WI of course includes the soft-pion limit. Thus the effective Lagrangian automatically realizes all the soft-pion results of Witten: the η' grows an additional mass due to the anomaly (which does not vanish in the chiral limit); this mass can be related to the θ dependence of the vacuum energy $E(\theta)$ in the absence of quarks; all effects of θ are seen to disappear in the chiral limit. In addition, a simple expression for CP-violating effects are obtained. However, since the WI are satisfied for arbitrary q^2 (and not just in the soft-pion limit) and due to the fact that no assumptions are made regarding the amount of chiral or SU(3) breakdown, the formalism offers a realistic approach to the calculation of processes that depend on the U(1) anomaly or the topological charge (such as the $\eta \rightarrow 3\pi$ decay and the neutron

dipole moment). As we will see in some simple examples here, these corrections required by the real world can sometimes be considerable.

1. *Effective Lagrangian.* An effective Lagrangian, in terms of phenomenological fields involving the physically observed mesons and obeying all the current-algebra and PCAC conditions, but *not* including the U(1) anomaly, has, of course, been known for a long time.^{6,7} Recently, in a detailed analysis, Witten⁴ has pointed out that the leading terms of the $1/N$ expansion of QCD automatically imply an effective-Lagrangian description of the physical mesic interactions. Now, the fundamental QCD Lagrangian automatically obeys the current-algebra constraints, and so one may expect that the effective Lagrangian deduced from it in the $1/N$ expansion be similarly constrained. The only way this could not be true would be if the current-algebra constraints did not obey the $1/N$ conditions (i.e., mixed different orders in $1/N$). That this is *not* the case, can be seen from the following consideration. Let F represent the strength of any of the interpolating constants coupling the bilinear quark currents to the mesic fields (e.g., for the 0^- nonet of mesons φ_a one has $\langle 0 | \bar{q} \gamma^\mu \gamma^5 \frac{1}{2} \lambda_a q | q, b \rangle = i q^\mu F_{ab}$). In the $1/N$ expansion one has $F \sim \sqrt{N}$. Now from Refs. 6, and 7 one may see directly that current algebra implies that an arbitrary n -point contribution to the effective Lagrangian, $\mathcal{L}_I^{(n)} = g^{(n)} \varphi_1 \cdots \varphi_n$, obeys $g^{(n)} \sim F^{-(n-2)}$ and so $g^{(n)} \sim N^{1-n/2}$. This is precisely the constraint required by the $1/N$ expansion.³

The above discussion represents then a derivation of the earlier effective Lagrangians^{6,7} from the fundamental QCD theory. These Lagrangians do not include the effects of the U(1) axial anomaly, $(\frac{2}{3})^{1/2} N_1 (g^2/32\pi^2) F \cdot \bar{F}$, since these are $1/N$ smaller than the leading mesic terms. (Here $N_1 = 3$ is the number of light quarks.) This effect may be included

by introducing an additional "anomaly field" $K_\mu(x)$ which couples to Kogut-Susskind-type⁵ poles,^{8,9} such that $Q(x) \equiv \partial_\mu K^\mu$ is the topological charge density.¹⁰ The total effective Lagrangian now has the form

$$\begin{aligned} \mathcal{L} = & -\varphi_a^\mu \partial_\mu \varphi_a + \frac{1}{2} \varphi_a^\mu \varphi_{\mu a} - \frac{1}{2} \varphi_a \mu_{ab} \varphi_b + \frac{1}{2C} (\partial_\mu K_\nu)^2 \\ & - \left(\frac{2}{3}\right)^{1/2} N_i F_i^{-1} \varphi_a \partial_\mu K^\mu + \mathcal{L}_{CA}(\chi_A, \partial_\mu K^\mu) - \theta \partial_\mu K^\mu. \end{aligned} \quad (1)$$

Here $\varphi_a, \varphi_a^\mu, a=1, \dots, 9$ represent the nonet of pseudoscalar meson fields (we are using first-order formalism), the set $\{\chi_A\}$ are all the mesic fields, \mathcal{L}_{CA} is the remaining parts of the current-algebra Lagrangian (excluding the φ_a kinetic energy, but now including interactions involving $\partial_\mu K^\mu$), and μ_{ab} (diagonal in $a, b=1, \dots, 7$) is the pseudoscalar meson mass matrix arising due to the breakdown of chiral symmetry. (Note $\mu_{ab} \rightarrow 0$ in the chiral limit.) We will see below that the parameter C is related to the strength of the topological charge.

We proceed as in standard current-algebra analyses^{6,7} to require that Eq. (1) lead to the PCAC condition for the axial-vector currents A_a^μ with anomaly:

$$\partial_\mu A_a^\mu = F_{ab} \mu_{bc} \varphi_c + \delta_{a9} \left(\frac{2}{3}\right)^{1/2} N_i \partial_\mu K^\mu. \quad (2)$$

[This will be seen below, e.g., Eq. (7) to determine the anomaly couplings of \mathcal{L}_{CA} .] To see the significance of the field $K^\mu(x)$, we first look at the quadratic parts of Eq. (1). This leads to the free field propagators $\Delta_{ab} = i\langle T(\varphi_a \varphi_b) \rangle$ and $\Delta_{\mu\nu} = i\langle T(K_\mu K_\nu) \rangle$:

$$\Delta_{ab} = \delta_{ab} (q^2 + m_a^2)^{-1}, \quad m_a^2 \delta_{ab} = \mu_{ab} + \frac{2}{3} N_i^2 C F_i^{-1} \delta_{a9} F_i^{-1} b9, \quad (3)$$

$$\Delta_{\mu\nu} = -C(\eta_{\mu\nu}/q^2) + (q_\mu q_\nu/q^4) \frac{2}{3} N_i^2 C^2 (F_i^{-1} a9)^2 (q^2 + m_a^2)^{-1}, \quad (4)$$

where F_i^{-1} is the matrix inverse of F_{ab} , and m_a are the physical masses of the pseudoscalar nonet. Equations (3) and (4) include the effects of chiral and SU(3) breakdown in the mass growth of the pseudoscalar mesons. In the chiral and SU(3) limit, however, when $\mu_{ab} \rightarrow 0, F_{a9} \rightarrow (N_i/6)^{1/2} F_i \delta_{a9}$, we immediately deduce all the results of Witten.² Thus, one obtains then $m_8 \equiv m_\eta \rightarrow 0$ while $(m_9)^2 \equiv (m_\eta)^2 \rightarrow 4N_i F_i^{-2} C$, removing the paradox of the light ninth pseudoscalar meson. (Note $m_\eta^2 \sim 1/N$ since $F_i \sim \sqrt{N}$.) Further, from QCD one may establish¹¹

$$\left(\frac{d^2 E}{d\theta^2}\right)_{\theta=0} = -[q^\mu q^\nu \Delta_{\mu\nu}(q)]_{q^2=0}, \quad (5)$$

where $E(\theta)$ is the vacuum energy as a function of θ . [The effective Lagrangian, Eq. (1), also produces the identical result.] Thus in the absence of quarks ($N_i=0$) Eqs. (3), (4), and (5) imply $C = (d^2 E/d\theta^2)_{\theta=0}^{\text{no quarks}}$ and so in the chiral limit

$$m_\eta^2 \rightarrow 4N_i F_i^{-2} (d^2 E/d\theta^2)_{\theta=0}^{\text{no quarks}} \quad (6)$$

as obtained by Witten² in the $1/N$ analysis.¹² Further, in the chiral limit *with quarks present*, one can show using Eqs. (3)–(5) that $(d^2 E/d\theta^2)_{\theta=0}$ vanishes as desired since the vacuum energy should be independent of θ in this limit. Thus the simple structure of Eq. (1) contains all the fundamental QCD properties, and will in addition allow one to calculate physical meson processes directly.

As can be seen from Eq. (4), the satisfaction of the two-point anomalous WI implies the existence of two types of ghost poles: a Kogut-Susskind-type $q_\mu q_\nu/q^4$ dipole ghost,⁵ and a $\eta_{\mu\nu}/q^2$ monopole ghost (recall¹² $C > 0$). These ghosts, however, couple only to the anomaly current K^μ and correctly cancel out in physical quantities such as $\tau \equiv i\langle T(Q(x)Q(0)) \rangle$. ($Q \equiv$ topological charge density.) From Eq. (4) and the canonical commutation relations, one finds in the chiral limit that $\tau(q^2=0) \rightarrow C$. Thus in the chiral limit, C represents the quantum fluctuations of the topological charge.¹³ Equation (1) also implies $\langle 0 | \partial_\mu K^\mu | \varphi_a \rangle = (\frac{2}{3})^{1/2} N_i C F_i^{-1} a9$ showing the nonnegligible gluon content¹⁰ of the η' and η .

We now turn to the structure of \mathcal{L}_{CA} of Eq. (1) which is obtained from imposing the current-algebra relations and PCAC on the interaction Lagrangian. In the absence of anomalies, the techniques for doing this are well known.^{6,7} The anomaly introduces additional couplings involving topological charge density $\partial_\mu K^\mu$ only, which, however, are determined in terms of the nonanomaly couplings by the PCAC condition Eq. (2). For example, the nonderivative couplings \mathcal{L}_φ^K involving the spin-0 fields are obtained by iterating the equation

$$\frac{\delta \mathcal{L}_\varphi^K}{\delta \varphi_a} = -F_{ar}^{-1} Z_{rbc} \frac{\delta \mathcal{L}_\varphi^K}{\delta \chi_b} \chi_c - \left(\frac{2}{3}\right)^{1/2} N_i F_i^{-1} a9 \partial_\mu K^\mu, \quad (7)$$

where $\chi_a \equiv (\varphi_a, \sigma_a)$ represent all the pseudoscalar and scalar fields.¹⁴ Thus by inserting the quadratic parts of Eq. (1) that depend on K^μ into the right-hand side and integrating, one obtains the cubic structure

$$\mathcal{L}_\varphi^{K(3)} = \left(\frac{2}{3}\right)^{1/2} N_i [F_{ar}^{-1} Z_{rbc} F_i^{-1} b9] \varphi_a \sigma_c \partial_\mu K^\mu \quad (8)$$

and so forth for the higher-point functions.¹⁵ Equation (7) then precisely guarantees that Eq. (2) holds for these couplings.

The anomalous two-point WI originating from the effective Lagrangian of Eq. (1) yield in the 8, 9 channels three equations for the F_{ab} (Ref. 16):

$$(m_\eta F_{88})^2 + (m_\eta F_{89})^2 = \frac{4}{3}(c_K + c_\kappa) - \frac{1}{3}c_\tau, \quad (9)$$

$$F_{88}(F_{88} + \sqrt{2}F_{98})m_\eta^2 + F_{89}(F_{89} + \sqrt{2}F_{99})m_\eta^2 = c_\tau, \quad (10)$$

$$(F_{88} + \sqrt{2}F_{98})^2 m_\eta^2 + (F_{89} + \sqrt{2}F_{99})^2 m_\eta^2 = 3c_\tau + \frac{4}{3}N_1^2 C, \quad (11)$$

where $c_\tau \equiv F_\tau^2 m_\tau^2$, $c_K \equiv F_K^2 m_K^2$. In the SU(3) and chiral limit one would have $F_{88} = F_\tau^2 F_{99} = \sqrt{6}C/m_\eta$, and $F_{89} = 0 = F_{98}$. One may, however, solve these equations rigorously for F_{89} , F_{98} , and F_{99} in terms of $F_\tau \equiv F_{88}$. Thus one finds

$$|F_{89}| = (m_\eta)^{-1} \left[\frac{1}{3}(4c_K + 4c_\kappa - c_\tau) - m_\eta^2 F_\tau^2 \right]^{1/2}.$$

To obtain an estimate of F_{89} we set $F_\eta \approx F_K \approx 115$ MeV and find with chiral and SU(3) breaking that $\alpha \equiv F_{89}m_\eta/F_{88}m_\eta \approx 0.6$, which is not small. The corresponding formula for F_{98} involves the anomaly parameter C . From Eq. (3) one might estimate C by $C \approx (F_\eta)^2(m_\eta^2 - m_\pi^2)/4N_1^2 \approx 5c_\tau$ and then $\sqrt{2}F_{98} + F_{88} \approx \pm \alpha [8N_1^2 C/m_\eta^2(1 + \alpha^2)]^{1/2}$.

These results illustrate that sizable corrections arise in the real world from SU(3) and chiral breaking.

2. Strong-CP-violating effects. The effects of θ on physical processes arise due to the breakdown of chiral symmetry. As pointed out by Baluni,¹⁷ these aspects are more easily seen when a U(1) transformation is made, which eliminates the θ dependence of the fundamental QCD Lagrangian from the gluon sector transforming it into a CP-violating mass matrix $\delta \mathcal{L}_{CP}$ in the quark sector:

$$\delta \mathcal{L}_{CP} = \theta m_u m_d m_s (m_u m_d + m_u m_s + m_d m_s)^{-1} q \bar{t} \gamma^5 q.$$

One may make the identical U(1) transformation on the effective Lagrangian of Eq. (1). The axial-vector current \tilde{A}_a^μ , associated with the U(1) symmetry transformation, satisfies the PCAC condition without anomaly, i.e., $\partial_\mu \tilde{A}_a^\mu = F_{ab} \mu_{bc} \varphi_c$, and is related to the gauge-invariant current A_a^μ by $\tilde{A}_a^\mu = A_a^\mu - (\frac{2}{3})^{1/2} N_1 \delta_{a9} K^\mu$. [Note that in the chiral limit this implies \tilde{Q}_9^5 is conserved and indeed one may explicitly verify that \tilde{Q}_9^5 commutes with the *Hamiltonian* of Eq. (1) in this limit.] The transformation that eliminates the topological charge term $-\theta \partial_\mu K^\mu$ of Eq. (1) is then¹⁷ $U(\theta) = \exp\{i\theta[(\frac{2}{3})^{1/2} N_1]^{-1} \tilde{Q}_9^5\}$, where $\tilde{Q}_9^5 \equiv \int d^3x \tilde{A}_9^0$. In addition one must make a chiral SU(3) \times SU(3) transformation so that the perturbing Lagrangian $\delta \mathcal{L}_{CP}$ is correctly an SU(3) singlet, in conformity with the theorems of Dashen and Nuyts.¹⁸ This can be achieved by determining the coefficients β_i which minimize the quantity $F(\beta_i) = \langle 0 | V^{-1} (\varphi'_a) V | 0 \rangle$, where $V(\beta_i) \equiv \exp(i\beta_i Q_i^5)$, Q_i^5 are the axial charges of SU(3) \times SU(3), and φ'_a are the transformed pseudo-

scalar fields¹⁹:

$$U \varphi_a U^{-1} = \varphi'_a - \theta [(\frac{2}{3})^{1/2} N_1]^{-1} F_{9a}. \quad (12)$$

These transformations, of course, leave the kinetic energy terms invariant. In precise analogy then with the fundamental QCD Lagrangian,¹⁷ the existence of the chiral-breaking mass terms of Eq. (1) give rise to a CP-violating interaction. Thus to linear order in the fields, Eq. (12) produces a $\delta \mathcal{L}_{CP}$ of the form²⁰

$$\delta \mathcal{L}_{CP} = \theta [(\frac{2}{3})^{1/2} N_1 D^{-1}_{99}]^{-1} F_{9a} \sqrt{Z}_{9a} v_0(x), \quad v_0 \equiv (\sqrt{Z} \varphi')_0, \quad (13)$$

where $D_{ab} \equiv \sqrt{Z}^{-1} {}_{ac} \mu_{cd} (\sqrt{Z})^{-1} {}_{db}$ and $(\sqrt{Z})_{ab}$ is the wave-function renormalization matrix of Glashow and Weinberg.²¹

The factor $1/D^{-1}_{99}$ is proportional to the bare mass matrix μ_{ab} and hence vanishes in the limit of perfect chiral symmetry. Thus the effective Lagrangian contains the corresponding properties of the fundamental Lagrangian. Further, by equating the vacuum matrix elements $\langle 0 | [\tilde{Q}_9^5, \delta \mathcal{L}_{CP}] | 0 \rangle$ of the two formalisms, one obtains the relation (for $N_1 = 3$)

$$(D^{-1}_{99})^{-1} (\sqrt{Z}_{9a} F_{9a})^2 = \frac{1}{3} m_u m_d m_s (m_u m_d + m_u m_s + m_d m_s)^{-1} \langle 0 | \bar{q} q | 0 \rangle, \quad (14)$$

which represents a sum rule for the η and η' chiral mass matrix.

There are of course additional CP-violating terms arising from other chiral-symmetry-breaking effects required by current algebra and appearing in the interaction parts of \mathcal{L}_{CA} of Eq. (1). The simplest decay to consider that proceeds via strong CP violation is that of $\eta \rightarrow 2\pi$. These additional effects actually cancel for this process in the soft-pion and $m_\sigma \rightarrow \infty$ limit (corresponding to nonlinear representations of the 0^+ states¹⁴). One obtains then the identical result for this decay as Crewther *et al.*²² calculate from the fundamental QCD Lagrangian. The formalism presented here allows, however, for the inclusion of arbitrary breakdown of chiral and SU(3) symmetry, and for hard-pion corrections. Perhaps the most interesting application of the techniques developed here is to determine the strong-CP-violating effects on the neutron electric dipole moment, which can be calculated with the above formalism without soft-pion²² or bag approximations.¹⁷ This calculation will be considered elsewhere.

Note added. After completing this work we became aware of the paper by C. Rosenzweig, J. Schechter, and G. Trahern [Phys. Rev. D **21**,

3388 (1980)]. This work also gives an effective-Lagrangian description of the U(1) problem, but considers only the SU(3) symmetric, σ -model coupling of spin-zero mesons. In this approximation, our Eq. (7) reduces to the Rosenzweig *et al.* results. We should like to thank Dr. Veneziano

for bringing this work to our attention.

Research is supported in part by the National Science Foundation under Grants Nos. PHY78-09613 and PHY77-22864.

*On sabbatical leave from Northeastern University, Boston, Massachusetts 02115.

¹For a recent review see R. J. Crewther, Riv. Nuovo Cimento 2, 63 (1979) and Phys. Lett. 70B, 349 (1977).

²E. Witten, Nucl. Phys. B156, 269 (1979).

³G. 't Hooft, Nucl. Phys. B72, 461 (1974).

⁴E. Witten, Nucl. Phys. B160, 57 (1979).

⁵J. Kogut and L. Susskind, Phys. Rev. D 11, 3594 (1975).

⁶J. Wess and B. Zumino, Phys. Rev. 163, 1727 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Lett. 19, 1085 (1967); C. G. Callan, S. Coleman, and B. Zumino, Phys. Rev. 177, 2239 (1968); 177, 2247 (1968).

⁷R. Arnowitt, M. Friedman, P. Nath, and R. Sutor, Phys. Rev. D 3, 594 (1971).

⁸G. Veneziano, Nucl. Phys. B159, 213 (1979).

⁹P. Di Vecchia, Phys. Lett. 85B, 357 (1979). However, we do not find that the propagators stated in this paper follow from the Lagrangian there.

¹⁰Thus $K^\mu(x)$ is the phenomenological field representing the QCD quantity $(g^2/32\pi^2)\epsilon^{\mu\alpha\beta\gamma}A_\alpha^a(F_{\beta\gamma}^a - \frac{1}{3}GC^{abc}A_\beta^b A_\gamma^c)$ and $\partial_\mu K^\mu = (g^2/32\pi^2)\tilde{F}^{\mu\nu}F_{\mu\nu}$. That the η and η' have significant gluon content was previously pointed out by H. Goldberg, Phys. Rev. Lett. 44, 363 (1980).

¹¹As pointed out in Ref. 2, $(d^2E/d\theta^2)_0$ consists of two parts, the usual structure $-i \int d^4x |T(\partial_\mu K^\mu(x)\partial_\mu K^\mu(0))|$ and an additional piece due to the dependence of the gluon canonical momentum on θ . By direct calculation in the temporal gauge, the sum combines to Eq. (5).

¹²Note that in QCD in the temporal gauge we find $C = (g^2/8\pi^2)^2 \langle B_a^i(0)B_a^i(0) \rangle$, where $B_a^i \equiv \frac{1}{2}\epsilon^{ijk}F_{jk}^a$. Thus $C > 0$, as required also in the effective Lagrangian since $m_{\eta'}^2 > 0$.

¹³The quantity $\tau(q^2=0)$ is called $\langle\mu^2\rangle$ by Crewther (Ref. 1).

¹⁴In Eq. (7) we are assuming for simplicity a linear representation for the 0^+ mesons. The nonlinear representations can be obtained from this by taking the limit $m_\sigma \rightarrow \infty$ for the 0^+ masses. The quantity Z_{Iabc} is the constant that enters in the axial-vector current A_a^0 as $A_a^0(x) = Z_{Iabc}\lambda_b^0\chi_c$ and is explicitly calculated in Appendices A and B of Ref. 7.

¹⁵The general n -point contribution for \mathcal{L}_{CA} can be obtained by replacing $\mu_a^2 s_a$ in Eq. (6.4) of Ref. 7 with $\mu_{ab}\phi_b + \frac{2}{3}N_i F^{-1} a^i g^0_\mu K^\mu$.

¹⁶Equations (9)-(12) were first obtained without the anomaly in Ref. 7 [see Eqs. (C3), (C16)-(C18) there]. The modification due to the anomaly was considered in Ref. 1. See also, H. Goldberg, Ref. 10. C_κ is a parameter which accounts for SU(3) breakdown and equals $F_\kappa^2\mu_\kappa^2$ if one assumes a linearly realized κ -meson in the strangeness-changing vector channel.

¹⁷V. Baluni, Phys. Rev. D 19, 2227 (1979).

¹⁸R. Dashen, Phys. Rev. D 3, 1879 (1971); J. Nuyts, Phys. Rev. Lett. 26, 1604 (1971).

¹⁹To linear order in θ , φ'_a is given by (Ref. 14) $\varphi'_a = \varphi_a - \theta[(\frac{2}{3})^{1/2}N_i]^{-1}Z_{13ab}\sigma_b$.

²⁰One finds $\beta_i M_{ia} D_{ak} = \theta[(\frac{2}{3})^{1/2}N_i]^{-1}M_{9a} D_{ak}$, where (Ref. 14) $M_{ab} \equiv F_{ac}\sqrt{Z}_{bc}$.

²¹S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968). Note that it is $(\sqrt{Z}\varphi)_9$ that transforms as an SU(3) singlet (not the physical meson field φ_9). See Appendix A of Ref. 7.

²²R. J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten, Phys. Lett. 88B, 123 (1979); 91B, 487(E) (1980).