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Nonlinear σ model in the loop expansion

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The nonlinear σ model in four dimensions is discussed in the context of the loop expansion. Since the model is perturbatively nonrenormalizable, divergences not of the form of the Lagrangian are of course expected; what is perhaps surprising is that there are divergences which appear not to be invariant under the original nonlinear symmetry. We demonstrate, however, that these apparently noninvariant terms do not contribute to on-mass-shell quantities and may be eliminated order by order by a field redefinition involving space-time derivatives. The linear σ model is then examined in detail; it is shown how the nonlinear model, including the apparently noninvariant terms, emerges as the limit of the linear model as the σ mass goes to infinity. Finally, we compare our approach with other treatments of the "noninvariant" terms in the nonlinear model.

I. INTRODUCTION

In a recent paper, we studied the effects of heavy, strongly interacting Higgs particles on the low-energy weak interactions.[1] The natural mass scale for strongly interacting Higgs bosons is of order 1 TeV, well beyond currently accessible energies. The gauged nonlinear σ model, which is the formal limit of the standard model as the mass of the physical Higgs particle goes to infinity, was found to be an extremely useful tool for organizing and cataloging all possible effects. The basic idea of the analysis was rather simple: Because the nonlinear model is perturbatively nonrenormalizable, divergent structures appear in the loop expansion which are not of the form of the original Lagrangian. The power or logarithms of the cutoff which multiply these structures can be interpreted as powers or logarithms of the Higgs mass M_H of the linear model. This is because the linear model with finite M_H is renormalizable; the new divergences only appear as the limit $M_H \to \infty$ is taken (thereby generating the nonlinear theory). Thus, all effects which grow with M_H in the linear model can be summarized by writing down the new divergent structures which appear in the nonlinear model. The dimension and form of these structures is determined simply by power counting and by the explicit nonlinear chiral symmetry.

However, in the implementation of this program, a problem arises which is worth studying in more detail. The nonlinear σ model, even without gauge fields, generates off-mass-shell divergences in the loop expansion which do not appear to be invariant under the nonlinear symmetry of the Lagrangian.

This phenomena must be clearly distinguished from simple nonrenormalizability of the theory. One of course expects divergences in perturbation theory which are not of the form of the original Lagrangian; what is surprising is that some of these divergences do not even appear to have the original symmetry off the mass shell. Although we discovered this phenomenon independently, its existence has, in fact, been noted previously by several authors.[2-6]

In the complete analysis of the renormalization properties of the two-dimensional model by Bardeen *et al.*,[6] it was found that the symmetry could be preserved by a simple redefinition in which the new field at any space-time point is a given function of the old field at that point. However, in four dimensions the situation is more complicated. As we will see, the necessary redefinitions invlove space-time derivatives of the fields.[7] Approaches not involving field redefinition have also been suggested[2-5]; we compare those methods with our own at the end of Sec. II and in the Conclusion.

Another question which needs elucidation is the precise manner in which the linear model acts as a regulated version of the nonlinear model. In particular, how is it possible that apparently noninvariant terms can be generated from the linear model, which is certainly invariant under the original linear symmetry in each order of the loop expansion? Our discussion will make clear the origin of these terms and show explicitly why they do not contribute to on-mass-shell quantities, at least at one loop. An understanding of the relation between the linear and nonlinear models is certainly critical for the type of analysis performed in Ref. 1; it is also necessary in order to discuss power (nonlogarithmic) divergences in the nonlinear model since dimensional regularization sets all such divergences to zero. (The only other chirally invariant regularization scheme we know of is lattice regularization, which is calculationally difficult.) As an example, we use these methods to demonstrate that the quadratic divergences do not violate the Ward identities of the gauged nonlinear theory.

The subtleties of the loop expansion in theories with nonlinear symmetries have been a matter of interest for some time. A decade ago, when the nonlinear σ model was being used extensively as a shortcut to current-algebra results, it was discovered[8] that the careful preservation of the chiral symmetry in the Feynman rules was necessary to prevent violation of the symmetry by one-loop quartic divergences. In contrast, the present work deals with a different kind of apparent violation of chiral symmetry which first appears in the one-loop logarithmic (and, in a certain sense, quadratic—see Sec. V) divergences. As mentioned above, a major goal of this study has been a deeper understanding of the preservation of nonlinear symmetry in order to apply the techniques to Ref. 1. In addition, the nonlinear σ model seems to be a good place to begin the study of more complicated nonlinear symmetries which appear in, for example, some supergravity models. These models may have additional complications which set them apart from the σ model[5]; still, it seems logical to learn as much as possible about the simpler case first. Finally, four-dimensional gauged nonlinear σ models, of the type discussed in Ref. 1 and in Sec. V of this paper, have recently been used in a description of the strong interactions.[9] As a phenomenological model, "massive gauge-invariant quantum chromodynamics (QCD)" (as the theory is called in this context) has some rather attractive features (e.g., confinement by vortices) which appear in a nonperturbative, semiclassical treatment. A thorough understanding of the loop expansion in the nonlinear model may help to shed light on this approach.

The organization of the paper is as follows: Sec. II establishes our notation, discusses some of the general features of the nonlinear σ model, and exhibits the one-loop invariant counterterms. In Sec. III, we calculate the one-loop π self-energy and 4π scattering graphs in a particular parametrization of the theory, and note that the logarithmic divergences are not of the form of the invariant counterterms. It is then shown that the apparently noninvariant terms can be absorbed into a field reparametrization. In Sec. IV, we generalize this calculation for arbitrary initial parametrization and for graphs with arbitrary numbers of external lines by using the background field method. Section V treats the linear model, showing how it can generate the apparently non-invariant terms as $M_H \to \infty$ and why these terms can be absorbed into field redefinition. We also discuss the gauged model and describe how the linear theory acts as regulator, preserving the Ward identities in the case of the quadratic divergences. A summary and some comparisons with other work appear in the final section.

II. GENERAL PROPERTIES OF THE FOUR-DIMENSIONAL NONLINEAR σ MODEL

We begin by writing down the $SU(2)_L \times SU(2)_R$ nonlinear model and describing some of its properties. The fields are represented by the two-by-two matrix

$$M(x) \equiv \sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x), \qquad (2.1)$$

which transforms from the left and right according to the $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)_L \times SU(2)_R$. The nonlinear constraint is given by

$$MM^\dagger = M^\dagger M = f^2, \qquad (2.2)$$

which is enforced by requiring

$$\sigma(x) = [f^2 - \vec{\pi}^2(x)]^{1/2}. \qquad (2.3)$$

The Lagrangian is then simply

$$\mathcal{L}_{NL} = \tfrac{1}{4}\,\mathrm{tr}\,(\partial^\mu M)^\dagger \partial_\mu M. \qquad (2.4)$$

It is useful to make explicit the invariance under the isomorphic group $O(4) \simeq SU(2)_L \times SU(2)_R$. The set $\phi(x) = (\sigma(x), \vec{\pi}(x))$ transforms as an $O(4)$ vector, with $\vec{\pi}$ transforming linearly under the "rotation" or "isospin" subgroup $SU(2)_L + SU(2)_R$. Under the remaining transformations (the "boosts"), $\vec{\pi}$ transforms nonlinearly:

$$\begin{aligned}&\vec{\pi} \to \vec{\pi} + \vec{\epsilon}(f^2 - \vec{\pi}^2)^{1/2},\\ &[(f^2 - \vec{\pi}^2)^{1/2} \to (f^2 - \vec{\pi}^2)^{1/2} - \vec{\epsilon}\cdot\vec{\pi}],\end{aligned} \qquad (2.5)$$

where $\vec{\epsilon}$ is a set of three infinitesimal parameters. The nonlinear Lagrangian in its $O(4)$ form is

$$\mathcal{L}_{NL} = \tfrac{1}{2}(\partial_\mu \phi)^2 = \tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2}\,\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{f^2 - \vec{\pi}^2}. \qquad (2.6)$$

The model (2.4) or (2.6) can be viewed as a limit of a spontaneously broken linear theory

$$\mathcal{L}_{LIN} = \tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + \tfrac{1}{2}(\partial_\mu \sigma)^2 - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - f^2)^2. \qquad (2.7)$$

If $\lambda \to \infty$ with the vacuum expectation value f fixed, the potential term in (2.7) becomes a δ function in a functional integral and the nonlinear model is generated, at least formally. Equivalently one can take the Higgs mass,

$$M_H{}^2 = 2\lambda f^2, \qquad (2.8)$$

large, still holding f constant. Of course, the limit $M_H \to \infty$ does not exist in perturbation theory, which is simply a reflection of the perturbative nonrenormalizability of the nonlinear model. It is, in fact, an old idea[10] that the linear theory with finite M_H can be thought of as a regulated nonlinear theory. In Sec. V, we will make more precise this connection and use it to discuss some of the more subtle properties of the nonlinear model.

The parametrization of the Lagrangian in terms of the $\vec{\pi}$ fields in (2.4) or (2.6) is, of course, not unique. (In fact, we will discover, in Sec. III, that it is necessary to redefine the fields in order to keep manifest the nonlinear chiral symmetry.) The freedom of reparametrization can most easily be expressed in geometric language.[11] If the π^i are thought of as coordinates on a three-sphere of radius f, the Lagrangian (2.6) takes the form

$$\mathcal{L}_{\mathrm{NL}} = \tfrac{1}{2} g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j , \tag{2.9}$$

where the metric g_{ij} is given by

$$g_{ij} = \delta_{ij} + \pi_i \pi_j / (f^2 - \vec{\pi}^2) . \tag{2.10}$$

Since the isospin subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is an ordinary global symmetry which is linearly realized, we anticipate no difficulty in maintaining explicit invariance under this subgroup. We are thus led to consider general coordinate transformations which keep the isospin subgroup linear:

$$\vec{\pi}(x) \to \vec{\pi}(x) H(\vec{\pi}^2(x)). \tag{2.11}$$

H is an arbitrary function of $\vec{\pi}^2$, subject only to the conditions required for the invariance of S-matrix elements: that H be nonsingular and that $H(0)$ be nonvanishing.[12] (Later we will find it useful to generalize this redefinition to involve space-time derivatives as well as the fields themselves.)

Under the transformation (2.11), the metric becomes

$$g_{ij} = a(\vec{\pi}^2)\delta_{ij} + b(\vec{\pi}^2)\pi_i\pi_j , \tag{2.12}$$

where a and b depend on H and its derivatives with respect to $\vec{\pi}^2$. Expanding a and b in power series with coefficients a_n/f^{2n} and b_n/f^{2n}, and absorbing, if necessary, a multiplicative constant into $\vec{\pi}$, we have the most general chirally invariant Lagrangian with, at most, two space-time derivatives[13]:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + \tfrac{1}{2}\sum_{n=1}^{\infty} f^{-2n}\left[a_n(\vec{\pi}^2)^n(\partial_\mu\vec{\pi})^2 + b_n(\pi^2)^{n-1}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2\right]. \tag{2.13}$$

Note that only one set of the coefficients a_n and b_n are independent; specifying the a_n, say, determines H, which then determines the b_n. (For example, $b_1 = 1 + 2a_1$.) The Lagrangian (2.13) is invariant under the usual linear isospin transformation on $\vec{\pi}$, and under the nonlinear boost transformations, which now take the form [replacing (2.5)]

$$\vec{\pi} \to \vec{\pi} + \vec{\epsilon} F(\vec{\pi}^2) + (\vec{\epsilon}\cdot\vec{\pi})\vec{\pi} G(\vec{\pi}^2), \tag{2.14}$$

where

$$F(\vec{\pi}^2) \equiv [(f^2/H^2) - \vec{\pi}^2]^{1/2} \tag{2.15}$$

and $G(\vec{\pi}^2)$ is given by[14]

$$G(\vec{\pi}^2) = \frac{1 + 2FF'}{F - 2\vec{\pi}^2 F'}, \quad F' = \frac{dF}{d\vec{\pi}^2}. \tag{2.16}$$

We remark that because the three-sphere S_3 is a symmetric space, the general covariance of the Lagrangian [Eq. (2.9)] automatically implies chiral [O(4)] invariance (and vice versa). Also note that to each of the above parametrizations, there corresponds a "linear" theory, in which the $\vec{\pi}$ field in (2.7) is redefined before the limit $\lambda \to \infty$ is taken. Finally, we emphasize that because the choices in (2.13) are related by redefinition of the $\vec{\pi}$ field, they all give the same results on the mass shell.

It is now possible to examine the structure of the divergences generated in the loop expansion. Because the model is nonrenormalizable, the divergences will not have the form of the original Lagrangian; instead, there will be a finite number of new "counterterms" (divergent structures) which are generated in a given number of loops.

A simple power-counting argument[15] determines the structure of the invariant counterterms at each order. First, we define dimensionless fields by writing the matrix M of (2.2) in terms of a unitary matrix U,

$$M = fU. \tag{2.17}$$

The Lagrangian (2.4) is then simply

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{tr}(\partial_\mu U)^\dagger \partial_\mu U . \tag{2.18}$$

Now let us imagine that we are computing with a dimensionful, chiral-invariant regulator Λ. (Λ might be the renormalized mass M_H of the linear model, or the inverse lattice spacing if lattice regularization is used.) We define D as the dimension of a counterterm which appears at L loops. Since the fields are dimensionless, D simply counts the number of derivatives. Let n be the number of powers of f^2 and r the number of powers of the regulator Λ which accompany this counterterm. Dimensional analysis gives

$$D + 2n + r = 4. \tag{2.19}$$

The number n is easily determined because f^2 multiplies the entire Lagrangian (2.17) and hence is a loop-counting parameter:

$$n=1-L. \tag{2.20}$$

We thus have

$$D=2+2L-r. \tag{2.21}$$

This determines the dimension of possible counterterms and thereby enables us to enumerate them. At L loops, new counterterms, with higher dimensions than those previously seen, will appear as logarithmic ($r=0$) divergences; counterterms that were first generated at $L-1$ loops will appear with quadratically divergent coefficients.

Focusing on the one-loop graphs, we see that there may be quartic divergences with $D=0$, quadratic divergences with $D=2$, and logarithmic divergences with $D=4$. There are, in fact, no quartic divergences since the only invariant with no derivatives is trivial:

$$\mathrm{tr}U^\dagger U=2. \tag{2.22}$$

The absence of quartic divergences in a chirally invariant perturbation calculation is well known.[8]

The only invariant $D=2$ structure is $\mathrm{tr}[(\partial_\mu U)^\dagger \partial_\mu U]$, which is of the form of the original Lagrangian. We would thus expect that the quadratic divergences may be absorbed into wave-function renormalization of the $\vec{\pi}$ field and a redefinition of the parameter f.

The new counterterms at one loop are, therefore, logarithmic divergences with $D=4$. Using the unitarity of U and trace theorems for SU(2), it can be shown that there are only three such structures:

$$\begin{aligned}\mathcal{L}_1&=\alpha_1\mathrm{tr}(\partial^\mu U^\dagger\partial_\mu U)\mathrm{tr}(\partial^\nu U^\dagger\partial_\nu U),\\ \mathcal{L}_2&=\alpha_2\mathrm{tr}(\partial^\mu U^\dagger\partial^\nu U)\mathrm{tr}(\partial_\mu U^\dagger)(\partial_\nu U),\\ \mathcal{L}_4&=\alpha_4\mathrm{tr}(\Box U^\dagger\Box U),\end{aligned} \tag{2.23}$$

where the numbering corresponds to the notation of Ref. 1. Equivalently, in the notation of Eq. (2.9), they take the form

$$\begin{aligned}\mathcal{L}_1&=(4\alpha_1/f^4)(g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j)^2,\\ \mathcal{L}_2&=(4\alpha_2/f^4)(g_{ij}\partial_\mu\pi^i\partial_\nu\pi^j)^2,\\ \mathcal{L}_4&=(2\alpha_4/f^4)[g_{ij}D^2\pi^iD^2\pi^j+(g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j)^2],\end{aligned} \tag{2.24}$$

where D^2 is the covariant d'Alembertian

$$D^2\pi^i=\Box\pi^i+\Gamma^i_{jk}\partial_\mu\pi^j\partial_\mu\pi^k \tag{2.25}$$

with Γ^i_{jk} the Christoffel symbol,

$$\Gamma^i_{jk}=\tfrac{1}{2}g^{il}\left(\frac{\partial g_{kl}}{\partial\pi^j}+\frac{\partial g_{jl}}{\partial\pi^k}-\frac{\partial g_{jk}}{\partial\pi^l}\right). \tag{2.26}$$

We have thus enumerated all possible chirally invariant one-loop counterterms.

Surprisingly, explicit calculations show—as we will see in Sec. III—that when the external lines are off the mass shell, the one-loop logarithmic divergences cannot be canceled by the counterterms (2.23) or (2.24). This fact has been known for some time.[2-5] A plausibility argument for the existence of the phenomenon was given by Honerkamp,[2] who pointed out that ordinary perturbation theory is a noncovariant procedure in this context. Imagine calculating loop diagrams by the background-field method (which is equivalent to ordinary perturbation theory). The expansion of the Lagrangian (2.9) around the background field involves the derivatives of $\mathcal{L}$: $\delta\mathcal{L}/\delta\pi^i$, $\delta^2\mathcal{L}/\delta\pi^i\delta\pi^j$, ... Because these derivatives are ordinary, not covariant, derivatives with respect to π, the calculation produces apparently noninvariant results. Honerkamp then went on to show that one can replace the usual background-field method with a procedure involving covariant derivatives which gives the same results for physical amplitudes on the mass shell and is invariant for all values of external momenta. Similar methods have been suggested by other authors.[4] We emphasize that these procedures are not equivalent (off the mass shell) to ordinary perturbation theory for any parametrization of the original Lagrangian.

In Sec. III, we suggest another approach to the problem. Instead of redefining perturbation theory, we merely redefine (order by order) the $\vec{\pi}$ field (or equivalently, the chiral transformations). Such "field-dependent renormalizations" can appear even in the renormalizable two-dimensional nonlinear model[6]; the only unusual aspect of the present redefinition is the fact that the new fields are functions not only of the old fields, but of their space-time derivatives as well.

In Sec. V, we will have occasion to refer to a version of the nonlinear model in which the $\mathrm{SU}(2)_L$ group is gauged. This model is discussed in great detail in Ref. 1; here, we merely list a few relevant features. The gauge field is $W_\mu=\vec{W}_\mu\cdot\vec{\tau}/2i$ and the gauge-invariant Lagrangian is

$$\mathcal{L}_{\mathrm{inv}}=\tfrac{1}{2}\mathrm{tr}(F_{\mu\nu})^2+\tfrac{1}{4}\mathrm{tr}(D_\mu M)^\dagger D^\mu M, \tag{2.27}$$

where $F_{\mu\nu}=\partial_\mu W_\nu-\partial_\nu W_\mu+g[W_\mu,W_\nu]$ and $D_\mu=\partial_\mu+gW_\mu$. As before, the constraint is $M=fU$, with U unitary. To $\mathcal{L}_{\mathrm{inv}}$ must, of course, be added ghost and gauge fixing terms. We will always work in the Landau gauge; where $\partial_\mu W^\mu$ vanishes, the ghosts do not couple directly to the scalars and there is no quadratic term mixing $\vec{\pi}$ and W_μ. In terms of the σ and $\vec{\pi}$ fields of (2.1) and (2.3), we have

$$\mathcal{L}_{\rm inv}=\tfrac{1}{2}\,{\rm tr}(F_{\mu\nu})^2+\tfrac{1}{2}M_W{}^2(\vec{W}_\mu)^2+\tfrac{1}{2}(\partial_\mu\vec{\pi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2 +\frac{g}{2}\vec{W}_\mu\cdot(\vec{\pi}\partial_\mu\sigma-\sigma\partial_\mu\vec{\pi}+\vec{\pi}\times\partial_\mu\vec{\pi}), \tag{2.28}$$

where

$$M_W{}^2\equiv\frac{g^2f^2}{4}=\frac{g^2}{4}(\sigma^2+\vec{\pi}^{\,2}). \tag{2.29}$$

III. THE NONLINEAR σ MODEL AT ONE LOOP

In this section, we examine some of the one-loop structure of the nonlinear σ model as revealed by direct computation with dimensional regularization. Comparison with the counterterms (2.23) allowed by the symmetries of the model will show that divergent pieces exist off the mass shell which do not have the structure of these counterterms. It will then be shown that the additional pieces can be eliminated through a (cutoff-dependent) field redefinition, generalized [from (2.11)] to include space-time derivatives.

In order to determine the coefficients in all three of the counterterms (2.23) and to exhibit the additional off-shell divergences, it suffices to compute the 2π and 4π Green's functions. For simplicity, the original form (2.6) of the nonlinear Lagrangian is used, corresponding to the choice $a_n=0$ in Eq. (2.13). The background-field discussion in the next section will, however, make use of the more general form (2.13).

The only one-loop contribution to the 2π Green's function is shown in Fig. 1. The apparent divergence (nonzero at $q=0$ and therefore violating the Adler condition) is well known[8] not to be present

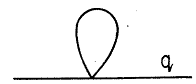

FIG. 1. The one-particle-irreducible 2π Green's function at one loop.

in chiral-invariant perturbation theory. The quadratically divergent term proportional to q^2 is defined to be zero with dimensional regularization. (In a regularization scheme with a dimensional cutoff, such as regularization by the linear model, this term will be nonzero but it will amount only to a renormalization of the original Lagrangian—see Sec. V.) It is clear from Fig. 1 that there can be no piece proportional to q^4 (logarithmically divergent). Thus the counterterm $\mathcal{L}_4$ (2.23), which produces a 2π vertex with four powers of momentum, has a vanishing coefficient at one loop.

The one-loop contributions to the 4π Green's function are shown in Fig. 2. The quartic and quadratic divergences are dealt with as they were in the 2π Green's function. It is the remaining logarithmic divergences, proportional to four powers of the external momentum, which are of central interest. Those which do not vanish on the mass shell require the use of the counterterms $\mathcal{L}_1$ and $\mathcal{L}_2$ (2.23) and signal the nonrenormalizability of the theory. In order to separate off these pieces and then focus attention on the remaining off-shell parts, it is helpful first to record the structure of the 4π vertices generated by $\mathcal{L}_1$ and $\mathcal{L}_2$. Using the external line nomenclature of Fig. 2, the counterterms are

$$\frac{8i\alpha_1}{f^4}V^{(1)}_{a_1\ldots a_4}=\frac{8i\alpha_1}{f^4}\delta_{a_1a_2}\delta_{a_2a_4}[-st-su+(q_1{}^2+q_2{}^2)(q_3{}^2+q_4{}^2)]+\text{perms}, \tag{3.1}$$

$$\frac{4i\alpha_2}{f^4}V^{(2)}_{a_1\ldots a_4}=\frac{4i\alpha_2}{f^4}\delta_{a_1a_2}\delta_{a_3a_4}[-2ut-st-su+2(q_1{}^2q_2{}^2+q_3{}^2q_4{}^2)+(q_1{}^2+q_2{}^2)(q_3{}^2+q_4{}^2)]+\text{perms}. \tag{3.2}$$

The logarithmically divergent part of the one-loop amplitude is

$$\frac{1}{16\pi^2}\frac{2i}{3\epsilon f^4}(V^{(1)}_{a_1\ldots a_4}+V^{(2)}_{a_1\ldots a_4})+\frac{1}{16\pi^2}\frac{i}{\epsilon f^4}\delta_{a_1a_2}\delta_{a_3a_4}[2s(s+t+u)-(q_1{}^2+q_2{}^2)(q_3{}^2+q_4{}^2)]+\text{perms}, \tag{3.3}$$

where $\epsilon=4-n$ and $s+t+u=\sum_i a_i{}^2$. We first consider expression (3.3) on the mass shell. The term in square brackets then vanishes and a comparison with the counterterms (3.1) shows that the cutoff dependence can be canceled by the choice

$$\alpha_1=-\frac{1}{16\pi^2}\frac{1}{12\epsilon}, \qquad \alpha_2=-\frac{1}{16\pi^2}\frac{1}{6\epsilon}. \tag{3.4}$$

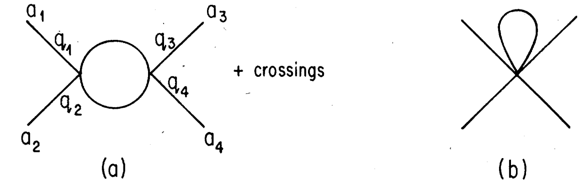

FIG. 2. Graphs contributing to the one-loop 4π Green's function.

Without the mass-shell restriction, the counterterms now remove the first term of (3.3), leaving the second term as a logarithmically divergent contribution to the 4π Green's function.

Since this piece vanishes on shell, it poses no problem for one-loop computations. However, it would, at the very least, complicate higher-order computations and so it is important to understand its origin. We shall show that the extra term corresponds to a Lagrangian reparametrization, generalized from (2.11) to include space-time derivatives.

We start with a power-series expansion of (2.11),

$$\vec{\pi} \to \vec{\pi}\left(1 + \frac{a_1}{2f^2}\,\vec{\pi}^2 + \cdots\right), \tag{3.5}$$

which gives the Lagrangian (2.13) from (2.6). To discuss a 4π Green's function, it suffices to work to order π^3 in the series expansion, a fact which remains true even when generalizing to include space-time derivatives. The residual piece in (3.3) can in fact be understood in terms of the following generalization of (3.5):

$$\vec{\pi} \to \vec{\pi}\left(1 + \frac{a_1}{2f^2}\,\vec{\pi}^2 + \frac{c_1}{2f^4}\,\vec{\pi}\cdot\Box\vec{\pi} + \frac{d_1}{2f^4}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \cdots\right). \tag{3.6}$$

To see this, we first observe that the residual piece in (3.3) corresponds to an effective interaction Lagrangian

$$\Delta\mathcal{L}^{4\pi} = +\frac{1}{16\pi^2\epsilon f^4}\left[\tfrac{3}{2}(\vec{\pi}\cdot\Box\vec{\pi})^2 + 2(\partial_\mu\vec{\pi})^2(\vec{\pi}\cdot\Box\vec{\pi})\right]. \tag{3.7}$$

Now consider the change of variables (3.6) with a_1 set to zero in order to focus on the derivative terms. It precisely cancels the incremental piece (3.7) providing

$$c_1 = \frac{3}{16\pi^2\epsilon}, \quad d_1 = \frac{1}{4\pi^2\epsilon}. \tag{3.8}$$

What this result suggests is that the term in square brackets in (3.3) is not really noninvariant. Such a structure would, in fact, have been present in the original Lagrangian had the more general parametrization, given by (3.6), been employed. Although this discussion has been limited to the 2π and 4π Green's functions and to the initial parametrization (2.6), those restrictions can easily be lifted. This will be done in the next section, and then some of the problems and implications of this sort of derivative reparametrization will be discussed.

IV. A BACKGROUND-FIELD COMPUTATION

Perhaps the simplest way to obtain the full one-loop effective Lagrangian is the background-field method.[16] By letting

$$\pi_i = \tilde{\pi}_i + \phi_i, \tag{4.1}$$

where $\tilde{\pi}_i$ is a classical background field and ϕ_i is a quantum field, the Lagrangian can be expanded in quantum fields about the background field. The Lagrangian is taken to be (2.9), with the metric $g_{ij}(\pi)$ given generally by (2.12). The zeroth-order term in ϕ_i gives the original Lagrangian, the sum of all tree graphs, and the linear term gives the equation of motion.

The quadratic term yields the one-loop effective Lagrangian. The logarithmically divergent part can be calculated simply using 't Hooft's algorithm,[17] and that has in fact been done by Tataru.[3] After the correction of some canceling sign errors, the result for the divergent part (not the counterterm) is

$$\Delta\mathcal{L} = \frac{1}{16\pi^2\epsilon}\,\mathrm{tr}\left(\tfrac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \tfrac{1}{2}X^2\right), \tag{4.2}$$

where $Y_{\mu\nu}$ and X are given in terms of the π fields by

$$\begin{aligned} X^i{}_j &\equiv R^i{}_{jkl}(\partial_\mu\pi^k\partial^\mu\pi^l) + g^{ik}\Gamma^m_{kj}S_m, \\ (Y_{\mu\nu})^i{}_j &\equiv R^i{}_{jkl}\partial_\mu\pi^k\partial_\nu\pi^l. \end{aligned} \tag{4.3}$$

Here Γ^m_{kj} is the Christoffel symbol (2.26), and $R^i{}_{jkl}$ is the curvature tensor given in a symmetric space by[18]

$$R_{ijkl} = \frac{1}{f^2}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right). \tag{4.4}$$

Finally, the factor S_m is

$$S_m = -g_{mi}(\pi)\Box\pi^i - \Gamma_{m,ij}\partial_\mu\pi^i\partial^\mu\pi^j. \tag{4.5}$$

The field equation is simply

$$S_m = 0, \tag{4.6}$$

so that this piece contributes only off the mass shell.[19]

With a little algebra, $\Delta\mathcal{L}$ can be written in the form

$$\Delta\mathcal{L} = +\frac{1}{16\pi^2}\frac{1}{f^2}\frac{1}{3\epsilon}\left[(g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j)^2 + 2(g_{ij}\partial_\mu\pi^i\partial_\nu\pi^j)^2\right] + \Delta\mathcal{L}', \tag{4.7}$$

where $\Delta\mathcal{L}'$ contains S_m as a factor and therefore does not contribute on the mass shell. The mass-shell part agrees completely with the counterterms (2.24), where the coefficients α_1 and α_2 are given by (3.4). (Note that the counterterms are defined to have the opposite sign from $\Delta\mathcal{L}$.) The improvement over the previous section is that the full one-loop effective Lagrangian, not just the 2π and 4π Green's functions, is given and it agrees on the mass shell with the counterterm structure. In addition, there is no need to specify a particular parametrization of the metric g_{ij} in the original Lagrangian (2.9).

Similar remarks apply to the additional piece $\Delta\mathcal{L}'$. It is the complete one-loop result and it is computed for a general g_{ij}. Comparison with the direct 4π computation (3.7) can be made by keeping only the 4π terms in $\Delta\mathcal{L}'$. The result is

$$\Delta\mathcal{L}'^{(4\pi)} = +\frac{1}{16\pi^2\epsilon}\frac{1}{f^4}\left[\left(\tfrac{9}{2}a_1^2+5a_1+\tfrac{3}{2}\right)(\vec{\pi}\cdot\Box\vec{\pi})^2+(3a_1+2)(\partial_\mu\vec{\pi})^2(\vec{\pi}\cdot\Box\vec{\pi})+a_1^2\vec{\pi}^2(\Box\vec{\pi})^2-2a_1(\Box\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial^\mu\vec{\pi})\right]$$
$$+\text{ a total derivative} \qquad (4.8)$$

and it agrees exactly with (3.7) for the choice $a_1=0$.

More generally, it can be seen that for any choice of metric $g_{ij}(\pi)$, the incremental divergent piece $\Delta\mathcal{L}'$ can be canceled by a field transformation involving two space-time derivatives. To exhibit the transformation to order π^3, it suffices to generate (4.8), the 4π piece of $\Delta\mathcal{L}'$, and the requisite transformation is

$$\pi_i \to \pi_i + \frac{c_1}{2f^4}\pi_i\vec{\pi}\cdot\Box\pi + \frac{d_1}{2f^4}\pi_i(\partial_\mu\vec{\pi})^2 + \frac{e_1}{2f^4}(\partial_\mu\pi_i)(\vec{\pi}\cdot\partial^\mu\vec{\pi}) + \frac{f_1}{2f^4}(\vec{\pi})^2\Box\pi_i + \cdots, \qquad (4.9)$$

where

$$c_1=\frac{1}{16\pi^2\epsilon}(3+10a_1+9a_1^2),$$
$$d_1=\frac{1}{8\pi^2\epsilon}(2+3a_1), \qquad (4.10)$$
$$e_1=-\frac{a_1}{4\pi^2\epsilon},$$
$$f_1=\frac{a_1^2}{8\pi^2\epsilon},$$

and, again, this reduces to (3.7) for the choice $a_1=0$.

To summarize, the Green's functions of the nonlinear σ model at the one-loop level in the usual perturbation expansion cannot be rendered finite even by the inclusion of the new (nonrenormalizable) counterterms allowed by the original symmetries of the theory. Additional logarithmically divergent pieces remain off shell, which can only be eliminated by a field redefinition involving space-time derivatives. Alternatively, the additional pieces may be viewed as invariant under a redefined symmetry transformation, where H in (2.11) and (2.14)–(2.16) involves space-time derivatives. The transformation will remain local order by order in the loop expansion. We conclude this section with some comments on this result.

(1) It is clear from (4.8) that there does not exist a choice of a_1, that is, a choice of metric $g_{ij}(\pi)$, which makes all the pieces of $\Delta\mathcal{L}'$ vanish. The phenomenon exists with any coordinate choice.

(2) In higher orders, divergent structures with more than four powers of the momentum will be encountered. Those that do not correspond to the allowed counterterms will presumably correspond to higher terms in the transformation (4.9), involving more than two space-time derivatives. A two-loop computation, for example, would proceed first by rendering the one-loop subgraphs finite. This would involve the use of the counterterms (2.24) along with the transformation (4.9) to eliminate the additional off-shell one-loop divergences. The full two-loop graph could then be rendered finite through the use of the new two-loop counterterms and the higher-derivative terms in (4.9).

(3) As far as we know, there is no discussion in the literature of field transformations in quantum field theory which involve space-time derivatives. The classic theorems on the invariance of the on-shell S matrix[12] are restricted to point transformations (no derivatives). However, in a functional integral context, a finite number of derivatives poses no special problems and the on-shell S matrix should again be invariant. A finite number is sufficient to any finite order in the loop expansion and thus the transformation will remain local and will not change the unitarity properties of the theory. Whether these transformations could be viewed as canonical transformations is not clear to us.

(4) It seems likely that the phenomenon discussed here is a general feature of nonlinear quantum field theories. With some standard parametrization to begin with, the perturbation expansion will, in a sense, induce reparametrizations involving space-time derivatives. Since this never happens in a linear theory, some insight into the phenomenon can be gotten by reexamining the nonlinear theory as the $\lambda\to\infty$ limit of the linear theory (2.7). This is done in the next section; in the Conclusion, a comparison is made between these considera-

tions and the generalized background-field approach of Honerkamp and others.[2-4]

V. RELATION TO THE LINEAR MODEL

In the three parts of this section, we examine how the nonlinear model emerges as the $\lambda\to\infty$ limit of the renormalized linear theory. We focus our attention, in particular, on the apparently noninvariant terms of the nonlinear theory and expalin, in subsection A, how such terms can arise despite the Goldstone-realized symmetry of the linear model. In subsection B, we show why the noninvariant terms do not contribute to on-mass-shell quantities. Finally, subsection C briefly discusses the regularization of the gauged nonlinear model. We demonstrate that the linear model provides a dimensionful cutoff M_H, which preserves the chiral Ward identities (in the case of the one-loop quadratic divergences). An alternative, a simple cutoff in momentum space, is shown to be inconsistent.

A. Generation of "noninvariant" terms

The Lagrangian of the linear model is given by (2.7); its formal nonlinear limit, by (2.6). We will work only with this explicit O(4) parametrization of the models. Since the symmetry of the linear theory is spontaneously broken, one makes the shift

$$\sigma=\sigma'+f. \tag{5.1}$$

The effective action $\Gamma'_{\rm LIN}(\sigma',\vec\pi)$ (the generator of one-particle irreducible graphs) is then invariant under the transformation of the classical fields

$$\vec\pi\to\vec\pi+\vec\epsilon(\sigma'+f),\quad \sigma'\to\sigma'-\vec\epsilon\cdot\vec\pi, \tag{5.2}$$

as well as simple isospin rotations of $\vec\pi$. Equivalently, we may define a function of the unshifted field σ, $\Gamma_{\rm LIN}(\sigma,\vec\pi)\equiv\Gamma'_{\rm LIN}(\sigma',\vec\pi)$; the transformation law may then be expressed simply as

$$\vec\pi\to\vec\pi+\vec\epsilon\sigma,\quad \sigma\to\sigma-\vec\epsilon\cdot\vec\pi. \tag{5.3}$$

To generate the nonlinear effective action $\Gamma_{\rm NL}(\vec\pi)$ from $\Gamma'_{\rm LIN}(\sigma',\vec\pi)$ is a two-step process. First, one must sum up all "trees" which can be made from $\Gamma'_{\rm LIN}(\sigma',\vec\pi)$ by connecting σ' lines, with no σ' lines left external. This is because $\Gamma_{\rm NL}(\vec\pi)$ is defined to be irreducible with respect to $\vec\pi$ lines only; σ' is determined in terms of $\vec\pi$ and does not appear explicitly in the nonlinear theory. As is well known, the sum over trees for any action functional may be simply obtained by evaluating the action at its stationary point. The second step consists of taking the limit as the coupling constant λ goes to infinity, with the parameter f held fixed. Of course, since higher and higher powers of λ appear as the order in the loop expansion increases, the limit $\lambda\to\infty$ does not exist in perturbation theory. We will only be interested in comparing the linear and nonlinear models *at a given number of loops*; in particular we will concentrate on what happens at one loop. To go beyond a loop-by-loop analysis, one would have to treat the models in some nonperturbative way.[20]

Some comments are necessary here about the renormalization of the linear model. With dimensional continuation, the parameters λ and f^2 are multiplicatively renormalized.[21] (Equivalently, one may replace f^2 with $m^2\equiv-\lambda f^2$, the negative mass-squared term of the unshifted theory.) We define λ and f^2 to be renormalized quantities made finite by minimal subtraction. For notational convenience, we will describe the theory in terms of the parameters f^2 and M_H^2, where M_H^2 is defined by (2.8). (Note, however, that M_H is only equal to the mass of the σ' particle in the tree approximation.) Our operating assumption is then that the nonlinear theory is generated by the limit $M_H\to\infty$ with f^2 fixed; we will show that this is correct at least through the one-loop terms which grow with M_H. We may now summarize the relation of the two models by writing

$$\Gamma_{\rm NL}(\vec\pi)=\lim_{M_H\to\infty}\Gamma_{\rm LIN}(\tilde\sigma,\vec\pi), \tag{5.4}$$

where $\tilde\sigma$ is defined by

$$\left.\frac{\delta\Gamma_{\rm LIN}(\sigma,\vec\pi)}{\delta\sigma}\right|_{\sigma=\tilde\sigma}=\left.\frac{\delta\Gamma'_{\rm LIN}(\sigma',\vec\pi)}{\delta\sigma'}\right|_{\sigma'=\tilde\sigma-f}=0. \tag{5.5}$$

Let us make a loop expansion of $\Gamma_{\rm NL}$, $\Gamma_{\rm LIN}$ and σ:

$$\begin{aligned}\Gamma_{\rm NL}(\vec\pi)&=\Gamma^0_{\rm NL}(\vec\pi)+\Gamma^1_{\rm NL}(\vec\pi)+\cdots,\\ \Gamma_{\rm LIN}(\sigma,\vec\pi)&=\Gamma^0_{\rm LIN}(\sigma,\vec\pi)+\Gamma^1_{\rm LIN}(\sigma,\vec\pi)+\cdots,\\ \tilde\sigma&=\tilde\sigma_0+\tilde\sigma_1+\cdots,\end{aligned} \tag{5.6}$$

where $\Gamma^0_{\rm LIN}$ is of course just the classical action of the theory and $\tilde\sigma_0$ is therefore the solution to the classical equation of motion:

$$\left.\frac{\delta\Gamma^0_{\rm LIN}(\sigma,\vec\pi)}{\delta\sigma}\right|_{\sigma=\tilde\sigma_0}=-\Box\tilde\sigma_0-\frac{M_H^2}{2f^2}(\tilde\sigma_0^2+\vec\pi^2-f^2)\tilde\sigma_0=0. \tag{5.7}$$

Putting (5.5) into (5.3) then gives

$$\Gamma^0_{\rm NL}(\vec\pi)=\lim_{M_H\to\infty}\Gamma^0_{\rm LIN}(\tilde\sigma_0,\vec\pi), \tag{5.8a}$$

$$\Gamma^1_{\rm NL}(\vec\pi)=\lim_{M_H\to\infty}\Gamma^1_{\rm LIN}(\tilde\sigma_0,\vec\pi) \tag{5.8b}$$

where, by (5.7), a term proportional to $\tilde\sigma_1$ in (5.8b) vanishes. Also from (5.7), it is straightforward to find $\tilde\sigma_0$ as a power series in f^2/M_H^2:

$$\tilde\sigma_0=\Sigma_0+\frac{f^2}{M_H^2}\Sigma_1+\frac{f^4}{M_H^4}\Sigma_2+\cdots, \tag{5.9}$$

where

$$\Sigma_0 = (f^2 - \vec{\pi}^2)^{1/2}, \tag{5.10a}$$

$$\Sigma_1 = -\frac{\Box\Sigma_0}{\Sigma_0^2}, \tag{5.10b}$$

$$\Sigma_2 = -\frac{2\Box\Sigma_1}{\Sigma_0^2} - \frac{3}{2}\frac{(\Box\Sigma_0)^2}{\Sigma_0^5}. \tag{5.10c}$$

It is now easy to see how the apparently noninvariant terms of the nonlinear model are generated. The nonlinear transformation (2.5) are just the linear transformations (5.3) with σ replaced by Σ_0, and if $\tilde{\sigma}_0$ in (5.8) were just equal to Σ_0, then $\Gamma^0_{\rm NL}(\vec{\pi})$ and $\Gamma^1_{\rm NL}(\vec{\pi})$ would be explicitly invariant. For $\Gamma^0_{\rm NL}$, one immediately discovers that the correction terms in (5.9) do not contribute in the limit $M_H \to \infty$; one simply gets

$$\Gamma^0_{\rm NL} = \int d^4x [\tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + \tfrac{1}{2}(\partial_\mu \Sigma_0)^2] = \int d^4x \mathcal{L}_{\rm NL}, \tag{5.11}$$

where $\mathcal{L}_{\rm NL}$ is just the nonlinear Lagrangian (2.6) and is clearly invariant under (2.5). However, $\Gamma^1_{\rm LIN}$ will contain terms growing as powers of M_H^2 which can combine with the higher-order terms in $\tilde{\sigma}_0$ to produce divergent, apparently noninvariant results. In other words, we may write, from (5.8b),

$$\Gamma^1_{\rm NL}(\vec{\pi}) = \lim_{M_H\to\infty} [\Gamma^1_{\rm LIN}(\Sigma_0, \vec{\pi}) + \hat{\Gamma}^1(\vec{\pi})], \tag{5.12}$$

where

$$\hat{\Gamma}^1(\vec{\pi}) \equiv \frac{f^2}{M_H^2}\left(\frac{\delta\Gamma^1_{\rm LIN}}{\delta\sigma}\right)_{\Sigma_0}\Sigma_1 + \frac{f^4}{M_H^4}\left[\left(\frac{\delta\Gamma^1_{\rm LIN}}{\delta\sigma}\right)_{\Sigma_0}\Sigma_2 + \frac{1}{2}\left(\frac{\delta^2\Gamma^2_{\rm LIN}}{\delta\sigma^2}\right)_{\Sigma_0}\Sigma_1^2\right] + \cdots, \tag{5.13}$$

with integration over space-time variables implied. Although $\Gamma^1_{\rm LIN}(\Sigma_0, \vec{\pi})$ is invariant under (2.5), $\hat{\Gamma}^1(\vec{\pi})$ is not and furthermore does not necessarily vanish as $M_H \to \infty$ because of the powers of M_H^2 that appear in $\Gamma^1_{\rm LIN}$. (In higher orders of the loop expansion, noninvariances can come both from terms in $\tilde{\sigma}_0$ that are nonleading in f^2/M_H^2—as occurs here—and from higher-loop corrections to $\tilde{\sigma}$.)

As an example, we calculate in this way the noninvariant divergent terms in the 2π and 4π functions of $\Gamma^1_{\rm NL}$. We expect to reproduce the result of Sec. III, (3.7), for the logarithmically divergent terms; in addition, we will find noninvariant quadratic divergences.

When expanded in powers of the π field, Σ_1 and Σ_2 start at order π^2; we therefore only need $\Gamma^1_{\rm LIN}$ to order π^2 to get $\hat{\Gamma}^1$ to order π^4. $\Gamma^1_{\rm LIN}(\sigma, \vec{\pi})$ $[\equiv \Gamma'^1_{\rm LIN}(\sigma', \vec{\pi})]$ is the sum of all one-loop graphs, irreducible with respect to $\vec{\pi}$ and the shifted field σ'. According to (5.13), we evaluate the derivatives of $\Gamma'^1_{\rm LIN}$ at $\sigma' = \Sigma_0 - f$, which also starts at order π^2. We thus need only calculate the following irreducible Green's functions: $G_{\sigma'}$, $G_{\sigma'\sigma'}$, and $G_{\sigma'\pi\pi}$, where the subscripts indicate the number and type of external lines. The graphs for these Green's functions are depicted in Figs. 3, 4, and 5. Including only those terms which will contribute to divergences in $\Gamma^1_{\rm NL}$, we calculate with dimensional regularization and minimal subtraction in the linear model,

$$G_{\sigma'} = -\frac{3i}{32\pi^2}\frac{M_H^4}{f}(\ln M_H^2 - 1 + \xi),$$

$$G_{\sigma'\sigma'}(k) = -\frac{3i}{32\pi^2}\frac{M_H^4}{f^2}[4\ln M_H^2 - 3 + 5\xi + \ln(-k^2)] + \cdots, \tag{5.14}$$

$$G_{\sigma'\pi\pi}(k,q) = -\frac{3i}{32\pi^2}\frac{M_H^4}{f^3}[3\ln M_H^2 - 2 + 4\xi + \ln(-k^2)] - \frac{i}{16\pi^2}\frac{M_H^2}{f^3}\ln M_H^2(q^2 + k\cdot q) + \cdots,$$

where $\xi \equiv c + \ln\pi$ (c is Euler's constant) and $\cdots$ represents terms of order M_H^2 or less. $\Gamma'^1_{\rm LIN}$ is then given by

$$i\Gamma'^1_{\rm LIN}(\sigma', \vec{\pi}) = G_{\sigma'}\sigma'(0) + \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\sigma'(k)\sigma'(-k)G_{\sigma'\sigma'}(k) + \tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\sigma'(k)\vec{\pi}(q)\cdot\vec{\pi}(-k-q)G_{\sigma'\pi\pi}(k,q) + \cdots. \tag{5.15}$$

Putting (5.15), (5.14), and (5.10) into (5.13), one gets, after some algebra,

$$\hat{\Gamma}^1(\vec{\pi}) = \frac{3}{32\pi^2}\frac{M_H^2}{f^4}(\ln M_H^2 - 1 + \xi)\int d^4x(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \frac{\ln M_H^2}{32\pi^2 f^4}\int d^4x\{\tfrac{3}{2}(\vec{\pi}\cdot\Box\vec{\pi})^2 + 2(\partial_\mu\vec{\pi})^2(\vec{\pi}\cdot\Box\vec{\pi}) + \tfrac{1}{2}[(\partial_\mu\vec{\pi})^2]^2\} + O(\pi^6) + \text{finite as } M_H \to \infty. \tag{5.16}$$

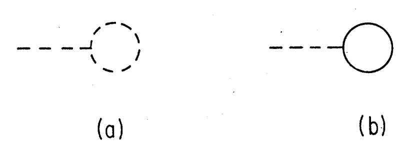

FIG. 3. Graphs contributing to $G_{\sigma'}$. Dashed lines are σ' fields; solid lines are π fields. With dimensional regularizations, graph (b) vanishes.

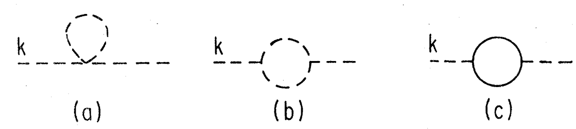

FIG. 4. Graphs contributing to $G_{\sigma'\sigma'}(k)$.

Let us first discuss the terms in (5.16) which grow like $\ln M_H^2$ as $M_H \to \infty$ (the terms which are logarithmic divergences of the nonlinear theory). Note that with the identification

$$\ln M_H \to 1/\epsilon \,, \tag{5.17}$$

we generate the residual term (3.7), which is not of the form of any invariant counterterms. In addition, however, there is a logarithmically divergent term in (5.16), proportional to $[(\partial_\mu \vec{\pi})^2]^2$, which does contribute to an invariant structure [specifically, to $\mathcal{L}_1$—see (2.24)]. This implies that we cannot simply ignore $\hat{\Gamma}^1(\vec{\pi})$ in constructing $\Gamma^1_{\rm NL}(\vec{\pi})$ through (5.12) even if we are only interested in on-mass-shell quantities. $\hat{\Gamma}^1(\vec{\pi})$ contains all the one-loop noninvariant structures but not *only* these structures.

Terms growing like M_H^2 or $M_H^2 \ln M_H^2$ (quadratic divergences) are new to us since they were defined away when the nonlinear model was dimensionally regulated in Secs. III and IV. Note that the quadratic divergence of (5.16) is of the form of a term in the original Lagrangian (2.6). Nevertheless, it is not invariant under the transformation law (2.5). Recall that the only invariant structures with two powers of derivatives is proportional to $\mathcal{L}_{\rm NL}$; since the quadratic divergence in $\hat{\Gamma}^1$ has a term with four π fields, but no term with two π fields, it cannot be proportional to $\mathcal{L}_{\rm NL}$. Alternatively, if this divergence is removed by defining renormalized quantities $\vec{\pi}_r$ and f_r,

$$\vec{\pi}_r = \sqrt{Z}\,\vec{\pi}\,, \qquad f_r = \sqrt{Z'}\,f\,, \tag{5.18}$$

then one immediately sees that the transformation law (2.5) is violated for the renormalized quantities since $Z \neq Z'$. (In particular, $\hat{\Gamma}^1$ contributes to Z' but not to Z.)

To obtain $\Gamma^1_{\rm NL}(\vec{\pi})$ in its entirety, not just the apparently noninvariant terms, one must, by (5.12), simply add the explicitly invariant $\Gamma^1_{\rm LIN}(\Sigma_0, \vec{\pi})$ to $\hat{\Gamma}^1(\vec{\pi})$. We have done this to find all divergent terms in the 2π and 4π nonlinear model irreducible Green's functions. A straightforward, but lengthy, calculation shows that (a) as expected,[8] there are no quartic (M_H^4 or $M_H^4 \ln M_H^2$) divergences; (b) after making the identification (5.17), the logarithmic divergences are precisely those found in Sec. III [Eq. (3.3)] by dimensional regularization of the nonlinear model; (c) the quadratic divergences are given by

$$\Gamma^1_{\rm NL,quad} = \lim_{M_H \to \infty} \frac{1}{32\pi^2} \frac{M_H^2}{f^2} \int d^4x \left[\tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + (6 \ln M_H^2 + 6\xi - 5) \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{2f^2} \right] + O(\pi^6) \,. \tag{5.19}$$

Comparing (5.19) with quadratic divergences in $\hat{\Gamma}^1$ [Eq. (5.16)], one easily sees that the additional terms coming from $\Gamma^1_{\rm LIN}(\Sigma_0, \vec{\pi})$ are proportional to the original Lagrangian and hence are indeed invariant under (2.5). For future reference, we note that all of the one-loop quadratic divergences may be canceled with the choice

$$Z = 1 - \frac{1}{32\pi^2} \frac{M_H^2}{f^2} + \cdots \,, \qquad Z' = 1 + \frac{1}{32\pi^2} \frac{M_H^2}{f^2} (6 \ln M_H^2 + 6\xi - 7) + \cdots \,, \tag{5.20}$$

where $\cdots$ represents terms of two-loop order which would have a coefficient f^{-4}.

Parenthetically, we remark that the quadratic divergences appear in a straightforward way in our calculations, in contrast to previous work.[10] By a judicious finite renormalization of the linear model before the $M_H \to \infty$ limit is taken, one could avoid encountering these divergences explicitly in the nonlinear model; however, we see no particular reason to do this. At two-loop order, the quadratic divergences would not be of the form of the original Lagrangian and therefore could not be eliminated by such a procedure.

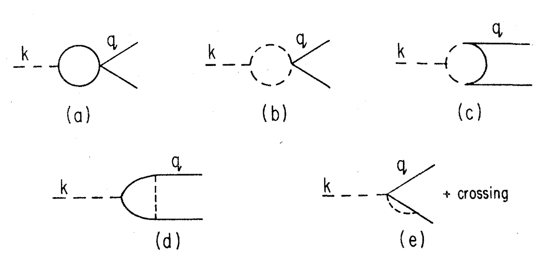

FIG. 5. Graphs contributing to $G_{\sigma'\pi\pi}(k, q)$.

B. On-shell invariance

Although the exposition thus far has shown how the apparently noninvariant structures of the nonlinear model are generated from the linear model, it has not made clear why the explicit nonlinear invariance is maintained on shell. To answer this question, we adopt a technique similar to the background-field method described in Sec. IV.

We start with the functional integral expression for the generating functional of π-field Green's functions in the linear model $W_{\rm LIN}(\vec{J})$:

$$\exp[i W_{\rm LIN}(\vec{J})] = \int [d\vec{\pi}][d\sigma]\exp\left\{i\int d^4x[\mathcal{L}_{\rm LIN}(\sigma,\vec{\pi})+\vec{J}\cdot\vec{\pi}]\right\}, \tag{5.21}$$

where $\mathcal{L}_{\rm LIN}$ is given by (2.7). We expect that the nonlinear-model generating functional $W_{\rm NL}(\vec{J})$ is related to $W_{\rm LIN}(\vec{J})$ simply by

$$W_{\rm NL}(\vec{J}) = \lim_{M_H\to\infty} W_{\rm LIN}(\vec{J}). \tag{5.22}$$

The linear-model equations of motion, which determine the stationary points of the action in (5.21) are

$$\Box\sigma+\frac{M_H^2}{2f^2}(\sigma^2+\vec{\pi}^2-f^2)\sigma=0, \tag{5.23}$$

$$\Box\vec{\pi}-\frac{(\Box\sigma)}{\sigma}\vec{\pi}=\vec{J}, \tag{5.24}$$

where we have used (5.23) to simplify (5.24).

If one were interested only in the linear model, one might now proceed by shifting σ and $\vec{\pi}$ in (5.21) by the solutions to (5.23) and (5.24). However, the previous discussion has shown that the separation of explicitly invariant structures from apparently noninvariant ones is performed most easily in terms of the quantity $\Sigma_0\equiv(f^2-\vec{\pi}^2)^{1/2}$ and not the solution to (5.23) or (5.7), $\bar{\sigma}_0$. We therefore make the shift

$$\vec{\pi}\to\vec{\pi}(\vec{J})+\vec{p}, \qquad \bar{\sigma}\to\Sigma_0(\vec{J})+s, \tag{5.25}$$

where $\Sigma_0(\vec{J})\equiv[f^2-\vec{\pi}^2(\vec{J})]^{1/2}$ and $\vec{\pi}(\vec{J})$ is the solution of the equation

$$\Box\vec{\pi}-\frac{(\Box\Sigma_0)}{\Sigma_0}\vec{\pi}=\vec{J}. \tag{5.26}$$

Note that $\vec{\pi}(\vec{J})$ and $\Sigma_0(\vec{J})$ satisfy the linear equations of motion only in the limit $M_H\to\infty$; (5.26) is, however, the exact nonlinear equation of motion.

Putting (5.25) into (5.21), separating off the terms which are independent of the new integration variables $\vec{p}$ and s, and using (5.26) on the remainder gives

$$W_{\rm LIN}(\vec{J}) = \int d^4x[\mathcal{L}_{\rm LIN}(\Sigma_0(\vec{J}),\vec{\pi}(\vec{J}))+\vec{J}\cdot\vec{\pi}(\vec{J})] + \Omega_{\rm LIN}(\vec{\pi}(\vec{J})), \tag{5.27}$$

with the following definitions:

$$\exp[i\Omega_{\rm LIN}(\vec{\pi})]\equiv\int[d\vec{p}][ds]\exp\left[i\int d^4x\,\tilde{\mathcal{L}}(s,p;\vec{\pi})\right], \tag{5.28}$$

$$\tilde{\mathcal{L}}(s,p;\vec{\pi})\equiv\tfrac{1}{2}(\partial_\mu\vec{p})^2+\tfrac{1}{2}(\partial_\mu s)^2 -\frac{M_H^2}{2f^2}(\vec{p}^2+s^2+2\Sigma_0 s+2\vec{\pi}\cdot\vec{p})^2 + sK_s(\vec{\pi})+\vec{p}\cdot\vec{K}_p(\vec{\pi}), \tag{5.29}$$

$$K_s(\vec{\pi})\equiv-\Box\Sigma_0, \tag{5.30}$$

$$\vec{K}_p(\vec{\pi})\equiv-\frac{(\Box\Sigma_0)}{\Sigma_0}\vec{\pi}. \tag{5.31}$$

Note that $\tilde{\mathcal{L}}$ contains linear terms in s and $\vec{p}$, with "sources" K_s and $\vec{K}_p$, because $\Sigma_0(\vec{J})$ and $\vec{\pi}(\vec{J})$ are not exact solutions of the linear equations of motion.

Now, since $\mathcal{L}_{\rm LIN}(\Sigma_0,\vec{\pi})=\mathcal{L}_{\rm NL}(\vec{\pi})$, and since $\vec{\pi}(\vec{J})$ is the solution to the nonlinear equation of motion, the first term on the right in (5.27) just generates all the nonlinear connected tree diagrams. Therefore $\Omega_{\rm NL}(\vec{\pi})$, defined by

$$\Omega_{\rm NL}(\vec{\pi}) = \lim_{M_H\to\infty}\Omega_{\rm LIN}(\vec{\pi}), \tag{5.32}$$

must generate all the nonlinear connected loop diagrams. At one loop, in fact, $\Omega_{\rm NL}(\vec{\pi})$ is equal to the effective action $\Gamma_{\rm NL}(\vec{\pi})$; however in higher loops $\Omega_{\rm NL}(\vec{\pi})$ will contain one-particle-reducible graphs such as Fig. 6, which is not in $\Gamma_{\rm NL}(\vec{\pi})$. Evaluating $\Omega_{\rm NL}(\vec{\pi})$ at $\vec{\pi}=\vec{\pi}(\vec{J})$ just replaces each external line by the sum of all trees.

A confusing point in this development is the following: $\Omega_{\rm NL}(\vec{\pi})$ is supposed to generate all diagrams with at least one loop; yet $\Omega_{\rm LIN}(\vec{\pi})$, as defined in (5.28), clearly contains some tree diagrams since $\tilde{\mathcal{L}}$ has linear terms in s and $\vec{p}$. However, these extra trees vanish as $M_H\to\infty$ (they are just the difference between the linear and nonlinear trees) as can be easily checked in graphs with low numbers of external π lines.

Let us now discuss how $\Omega_{\rm LIN}(\vec{\pi})$ and $\Omega_{\rm NL}(\vec{\pi})$ transform under the nonlinear transformation

$$\vec{\pi}\to\vec{\pi}+\vec{\epsilon}\Sigma_0, \qquad \Sigma_0\to\Sigma_0-\vec{\epsilon}\cdot\vec{\pi}. \tag{2.5}$$

A linear change of variables in (5.28),

$$\vec{p}\to\vec{p}+\vec{\epsilon}s, \qquad s\to s-\vec{\epsilon}\cdot\vec{p}, \tag{5.33}$$

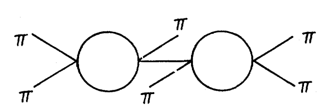

FIG. 6. A graph contributing to $\Omega_{\rm NL}(\vec{\pi})$ which does not contribute to $\Gamma_{\rm NL}(\vec{\pi})$.

makes clear that the noninvariance under (2.5) of $\Omega_{LIN}(\vec{\pi})$—and hence of $\Omega_{NL}(\vec{\pi})$—is due only to the presence of the source terms sK_s and $\vec{p}\cdot\vec{K}_p$ in $\tilde{\mathcal{L}}$. Explicitly, the variation in $\Omega_{LIN}(\vec{\pi})$ under (2.5) is given by

$$\Omega_{LIN}(\vec{\pi}) \to \Omega_{LIN}(\vec{\pi}) + \int d^4x\, \vec{\epsilon}\cdot\left(\Box\vec{\pi} - \frac{\vec{\pi}\Box\Sigma_0}{\Sigma_0}\right)\left\langle s + \frac{\vec{p}\cdot\vec{\pi}}{\Sigma_0}\right\rangle, \tag{5.34}$$

where the symbol $\langle\ \rangle$ denotes the connected vacuum expectation value calculated with the Lagrangian $\tilde{\mathcal{L}}$.

To make connection with the methods of Sec. V A consider dividing $\Omega_{LIN}(\vec{\pi})$ into two sets of graphs: those that do not contain the sources K_s and $\vec{K}_p$, and those that do. An explicit one-loop computation shows that the former set gives precisely the invariant contribution $\Gamma^1_{LIN}(\Sigma_0, \vec{\pi})$ in (5.12), while the latter gives the noninvariant contribution $\hat{\Gamma}^1(\vec{\pi})$. This is as expected, since it is the difference between Σ_0 and the exact solution of the linear-model equation of motion which leads to the presence both of $\hat{\Gamma}^1(\vec{\pi})$ in (5.12) and of the source terms here—generating noninvariant structures in each case.

It is now not hard to see why apparently noninvariant terms in $\Omega_{NL}(\vec{\pi})$ do not contribute to on-mass-shell quantities. To calculate the S matrix, one must first set $\vec{\pi}=\vec{\pi}(\vec{J})$ to "dress" Ω_{NL} with external trees; by (5.26), the variation in Ω_{NL} in (5.34) is then proportional to $\vec{J}$. On the mass shell, the variation now vanishes because the quantity $\langle s+\vec{p}\cdot\vec{\pi}/\Sigma_0\rangle$ has no poles at $k^2=0$: The s field is massive (mass $\simeq M_H$) and the second term is at least quadratic in massless fields. Note that our argument holds to all orders of the loop expansion.

We conclude this discussion with two remarks:

(1) What has been shown here is that the variation of Ω_{NL} vanishes on the mass shell. As de Wit and Grisaru point out,[5] this does not automatically guarantee that Ω_{NL} is a sum of an invariant term plus a term which vanishes on the mass shell; Ω_{NL} could also have a noninvariant term which did not vanish on shell but whose variation did. However, we do not expect this possibility to be realized in this model—it certainly does not occur at one loop, as the explicit calculations in this section and the previous sections show.

(2) We have discussed here the loop functional $\Omega_{NL}(\vec{\pi})$ which is not equal to the effective action $\Gamma_{NL}(\vec{\pi})$ except at one loop. It may be possible to extend the result of on-shell invariance to $\Gamma_{NL}(\vec{\pi})$ by an induction argument since the reducible graphs which contribute to Ω and not Γ are always made of irreducible parts, each of which has fewer loops than the whole graph. However, one would have to worry about the internal lines which may be off shell.

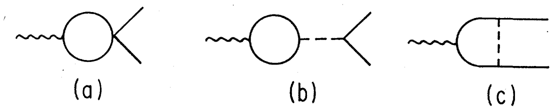

FIG. 7. Graphs contributing to the $M_H{}^2$ divergence of the W-2π vertex. Wiggly lines are W fields; solid lines are π fields; dashed lines are σ' fields.

C. Comments on the gauged model

Finally, we discuss the regularization of the gauged nonlinear model (2.27). The renormalization of this model in two dimensions has been discussed by Bardeen and Shizuya[22]; here we make some simple comments about the four-dimensional case. Suppose, first, we regularize by using the renormalized linear model. [The Lagrangian of the gauged linear theory is just (2.28) without the constraint $\sigma^2+\vec{\pi}^2=f^2$, with the potential term $-(\lambda/4)(\sigma^2+\vec{\pi}^2-f^2)^2$ added, and with the σ field shifted in the usual way.] The presence of the gauge field does not lead to any new quadratic ($M_H{}^2$) divergences in the 2π or 4π Green's functions, so we still expect the quadratic divergences to be removed by renormalization of $\vec{\pi}$ and f as in (5.18), with Z and Z' given by (5.20). Equations (2.28) and (2.29) then imply that the quadratic divergences in the 2π-W vertex and in $M_W{}^2$ should also be removed by Z and Z', respectively. Computation of the appropriate graphs (Figs. 7 and 8) shows that this is indeed the case. Ward identities relating the different vertices are thus preserved; this is true despite the fact that $Z\neq Z'$ (which we have described as an apparent noninvariance).

On the other hand, if the nonlinear model is regularized by a momentum-space cutoff on Feynman integrals, one easily finds that the quadratic divergences coming from the 2π, 4π, $2W$, and 2π-W Green's functions are inconsistent with the Ward identities. This is presumably because such a cutoff is akin to a Pauli-Villars π-mass term and therefore breaks chiral invariance.

The logarithmic divergences of the gauged nonlinear theory can be studied directly with dimensional regularization; there is no need to regular-

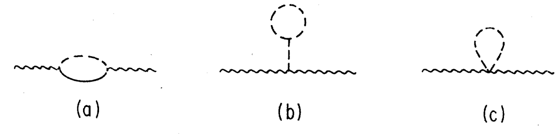

FIG. 8. Graphs contributing to the $M_H{}^2$ divergence of the W mass.

ize with the linear model. Once the redefinitions of the π field (as described in Secs. III and IV) have been performed, the chiral symmetry appears explicitly. See Ref. 1 for a detailed discussion.

VI. CONCLUSION

We have analyzed some features of the loop expansion of the nonlinear σ model. The theory is nonrenormalizable so that new counterterms are required at each order to define the theory. At one loop, they are $\mathcal{L}_1$ and $\mathcal{L}_2$ of Eq. (2.24), both of which must absorb logarithmic cutoff dependence. In addition to nonrenormalizability, however, a new problem arises in the nonlinear model which was our main concern in this paper. Even the new counterterms which are allowed by the nonlinear symmetry are not sufficient to remove all of the cutoff dependence of off-mass-shell Green's functions. These apparently noninvariant, logarithmically divergent pieces, which vanish on shell, are exhibited for the one-loop 4π Green's function in the second line of Eq. (3.3). They can also be seen in the full one-loop effective Lagrangian [Eq. (4.2)]. In Eq. (4.7), the off-shell piece of this effective Lagrangian, not removable by the invariant counterterms, has been separated into the third term $\Delta\mathcal{L}'$.

It was pointed out that these terms correspond to (or, equivalently, can be eliminated by) a π-field redefinition involving space-time derivatives. The most general form of the transformation, needed through one loop, is shown in Eq. (4.9). Several issues involved in making transformations of this sort were discussed at the end of Sec. IV.

In order to elucidate further the properties of the nonlinear model, we examined in detail the way it emerges from the linear model in the limit of large M_H (σ mass). In particular, we identified the origin of the off-shell, divergent one-loop terms (now meaning terms which grow with M_H), which are not removed by the invariant counterterms. It was also shown that these terms do not contribute to mass-shell quantities in any order of the loop expansion.

The entire discussion was in the context of conventional perturbation theory: The direct computation of Green's functions in the nonlinear model (Sec. III), the use of the background-field method to generate the full, divergent one-loop effective Lagrangian (Sec. IV), and the analysis of the large-M_H limit of the linear model (Sec. V). A rather different approach to dealing with the apparently noninvariant divergent terms was suggested by Honerkamp[2] and others.[3,4] They treat only the nonlinear model and employ a modification of the usual background-field method in order to maintain the nonlinear general covariance at all stages. This leads to a modified set of Feynman rules off shell, which in turn lead to one-loop Green's functions whose divergent parts correspond precisely to the allowed counterterms. The offending off-shell divergences simply do not appear.

While this second approach seems elegant and natural from the point of view of the nonlinear theory, its use of modified Feynman rules, and the resulting form of the Green's functions, make the connections to the linear model more remote. If the linear model is used, as in the heavy-Higgs-boson analysis of Ref. 1, to regulate and therefore define the nonlinear model in a physical way, ordinary perturbation theory is certainly appropriate. In the limit of large M_H, the apparently noninvariant terms arise but, as we have shown, they do not contribute to on-shell quantities. Our method of eliminating them even off shell through a π-field transformation would then be applied to the linear model before taking the $M_H \to \infty$ limit. In that sense, the approach suggested here gains physically what it perhaps loses in geometric elegance.

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