Thermodynamics of the Schwinger model

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We study the finite-temperature Schwinger model by explicitly calculating the exact fermion two-point function at finite temperature in the Coulomb gauge and thus obtain the canonical ensemble average of the Hamiltonian. The theory is found to be equivalent to an ensemble of noninteracting, neutral, massive, Bose particles. Thus the particle content is identical to that obtained at zero temperature.

The study of exactly soluble field theories in 1 space and 1 time dimension has received a good deal of attention in the hope that they might shed some light on the more realistic theories of 3 space and 1 time dimensions. One such theory is massless 1-space–1-time quantum electrodynamics. This model was first solved by Schwinger[1] and has subsequently been the subject of extensive investigation.[2-5]

Although the model contains massless fermions as fundamental fields, the explicit solution possesses no massless quanta, but only a neutral, massive single-particle state of Bose character. This is the well-known result at zero temperature. In this note, we address the solution of the Schwinger model at finite temperature. Explicitly, we calculate the exact fermion two-point function at finite temperature in the Coulomb gauge. The thermal ensemble average of the Hamiltonian is then shown to be equivalent to an ensemble of neutral, massive, noninteracting Bose particles with all massless excitations absent. Thus the particle content of the finite-temperature theory is identical to that obtained at zero temperature. Similar conclusions have previously been deduced by Dolan and Jackiw[6] and Fischler, Kogut, and Susskind.[7] Dolan and Jackiw[6] calculate the one-loop vacuum polarization at finite temperature finding the same result as at zero temperature. They thus conclude that the finite-temperature theory contains a massive boson just as at zero temperature. Fischler *et al.*[7] study the Schwinger model by writing the Coulomb-gauge Hamiltonian in a Bose-equivalent form, thus showing its equivalence to a noninteracting massive field theory.

The above result is markedly different from that obtained in those gauge theories in which the gauge boson acquires a mass at zero temperature via spontaneous symmetry breaking achieved with the use of scalar bosons. In that case, there exists a critical temperature above which the symmetry is restored in the sense that the scalar vacuum expectation value is zero and the gauge bosons are massless.[8] In the Schwinger model, of course, the Bose particle is not fundamental, but a fermion-antifermion bound state. Furthermore, while the scalar nonzero vacuum expectation value is required to set the scale of the zero-temperature vector-boson mass in the spontaneously broken gauge theories, the mass scale in the Schwinger model is set by the coupling constant, which because of the dimensionality of space-time, carries dimension mass.

Let us commence by reviewing some of the salient features in the solution of the zero-temperature Schwinger model. This will also allow us to introduce some notation. The Lagrangian for massless electrodynamics in 1 space and 1 time dimension can be written as

$$\mathcal{L} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\psi\alpha^\mu\partial_\mu\psi + ej^\mu A_\mu, \quad (1)$$

where[9]

$$j^\mu = \tfrac{1}{2}\psi\alpha^\mu q\psi. \quad (2)$$

Here $\psi(x)$ is a Hermitian Fermi field which has a two-dimensional multiplicity due to the nature of space-time and an additional two-dimensional multiplicity due to the charge degrees of freedom. The real, symmetric Dirac-Majorana matrices α^μ can be taken as

$$\alpha^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3)$$

while the two-dimensional antisymmetric, imaginary matrix operating in charge space is

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (4)$$

We employ a metric convention so that the space-time interval is $x^2 = x^\mu x_\mu = -t^2 + r^2$. From this Lagrangian follows the Dirac equation

$$\alpha^\mu(\partial_\mu - ieqA_\mu)\psi = 0 \quad (5)$$

and the anticommutation relation

$$[\psi(r,t),\ \psi(r',t)]_+ = \delta(r-r'). \tag{6}$$

Since the Lagrangian carries dimension mass squared, it follows that the coupling e has the dimension of mass and will set the scale for any mass which might appear in the model. Further, as with any gauge theory, we must fix a gauge. We choose the radiation or Coulomb gauge and thus fix

$$A_1 = 0. \tag{7}$$

The remaining component of the vector potential, A^0, is determined from the Maxwell equations to satisfy

$$\partial_1{}^2 A^0 = -ej^0. \tag{8}$$

The solution to this equation has been discussed in detail by Hagen.[10] He finds that for the charge-zero sector, which is the sector relevant to the present investigation, the solution is given by

$$A^0(r,t) = e\int_{-\infty}^{\infty} dr'\,\mathfrak{D}(r-r')j^0(r't), \tag{9}$$

where

$$-\frac{d^2}{dr^2}\mathfrak{D}(r-r') = \delta(r-r'), \tag{10}$$

so that

$$\mathfrak{D}(r) = \int_{-\infty}^{\infty} dk_1 e^{ik_1 r}\,\frac{1}{2}\left[\frac{1}{(k_1+i\epsilon)^2} + \frac{1}{(k_1-i\epsilon)^2}\right]. \tag{11}$$

Notice that due to its antisymmetry, $F_{\mu\nu}$ has only one vanishing component F_{01} which corresponds to a longitudinal electric field, and hence is not an independent degree of freedom. In this gauge, the Hamiltonian density can be identified with

$$\mathcal{H} = -\tfrac{1}{2}\psi(r,t)\alpha^1\partial_r\psi(r,t) + \frac{e^2}{2}j^0(r,t)\int dr'\,\mathfrak{D}(r-r')j^0(r',t), \tag{12}$$

which clearly exhibits the purely Coulombic nature of the interaction. By using the expression for A^0 [Eq. (9)], the Dirac equation (5) becomes

$$\alpha\partial\psi(r,t) = -ieq\psi(r,t)\int dr'\,\mathfrak{D}(r-r')j^0(r',t), \tag{13}$$

and the Hamiltonian can be recast in the convenient form

$$\mathcal{H} = -\tfrac{1}{4}\psi(r,t)\tilde{\alpha}\partial\psi(r,t), \tag{14}$$

where $\tilde{\alpha}^\mu = (\alpha^0, -\alpha^1) = (1, -\alpha^1)$.

The complete solution of the Schwinger model is secured by expressing the complete set of Green's functions in closed analytic form. These functions are defined at zero temperature by

$$G(x_1,\ldots,x_{2n}) = \langle 0|T(\psi(x_1)\cdots\psi(x_{2n}))|0\rangle, \tag{15}$$

where one must remember to include a factor of (−1) for each permutation needed to put the fermion fields in the required time ordering. Although the complete set of Green's functions at zero temperature has been obtained,[2,5] the vacuum expectation value of $\mathcal{H}$ only requires knowledge of the two-point function $G(x_1,x_2)$. Likewise, we shall only need the finite-temperature two-point function to obtain the thermal ensemble average of the Hamiltonian. Thus we shall concentrate on $G(x_1,x_2)$.

By using the Dirac equations as given by Eq. (13) in conjunction with the fact that the electromagnetic current satisfies the massive Klein-Gordon equation

$$\left(-\partial^2 + \frac{e^2}{\pi}\right)j^\mu(x) = 0, \tag{16}$$

Brown[2] has shown that the two-point function satisfies the equation

$$(\alpha\partial)_1 G(x_1,x_2) = \delta^2(x_1-x_2) + (\alpha\partial)_1 F(x_1-x_2)q_1q_2G(x_1,x_2), \tag{17}$$

where

$$F(x_1-x_2) = -ie^2\int\frac{d^2k}{(2\pi)^2}\Delta_+(k;0)\times\frac{1}{2}\left[\frac{1}{(k_1-i\epsilon)^2}+\frac{1}{(k_1+i\epsilon)^2}\right]\times\Delta_+\left(k;\frac{e^2}{\pi}\right)(\tilde{\alpha}k)^2[e^{ik(x_1-x_2)}-1], \tag{18}$$

and

$$\Delta_+(k;\mu^2) = \frac{1}{k^2+\mu^2-i\epsilon} \tag{19}$$

is the massive Klein-Gordon propagator at zero temperature. The solution to Eq. (17) is readily secured as

$$G(x_1,x_2) = \exp[F(x_1-x_2)q_1q_2]G^{(0)}(x_1-x_2), \tag{20}$$

where $G^{(0)}(x_1-x_2)$ is the free-field zero-temperature Green's function.

At finite temperature, the prescription is to replace the vacuum expectation value with the ensemble average. For any operator A, the canonical ensemble average is defined[11] by

$$\langle A\rangle_{(T)} = (\mathrm{Tr}\,e^{-\beta H})^{-1}\,\mathrm{Tr}\,e^{-\beta H}A, \tag{21}$$

where β is the inverse temperature, $\beta = T^{-1}$, and H is the Hamiltonian. The Tr in this expression dictates a sum over a complete set of states of the system. The cyclic property of the trace and the Heisenberg equation of motion

$$A(r,t) = e^{iHt} A(r,0) e^{-iHt} \tag{22}$$

yields the symmetry condition

$$\langle A(r,t) B(r',t')\rangle_{(T)} = \langle B(r',t') A(r, t+i\beta)\rangle_{(T)}, \tag{23}$$

which holds for all operators A and B. This periodicity condition serves as the appropriate finite-temperature boundary condition. For instance, if we knew the ensemble average of the commutator or anticommutator of any two operators

$$\langle [A(x), B(x')]_{\mp}\rangle_{(T)} = \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} g(k), \tag{24}$$

then the ensemble average of the product of the operator is obtained with the aid of the periodicity condition (23) as

$$\langle A(x) B(x')\rangle_{(T)} = \int \frac{d^2k}{(2\pi)^2} e^{ik(x-x')} \frac{g(k)}{1 \mp e^{-\beta k^0}}. \tag{25}$$

Here, the minus (plus) sign in the denominator on the right-hand side corresponds to $g(k)$ being associated with the commutator (anticommutator) of Eq. (24).

The complete solution of the finite-temperature model is obtained by solving for the finite-temperature Green's functions

$$G(x_1, \ldots, x_{2n})_{(T)} = \langle T(\psi(x_1) \cdots \psi(x_{2n}))\rangle_{(T)}. \tag{26}$$

Restricting attention to the two-point function $G(x_1, x_2)_{(T)}$, we proceed in complete analogy to the zero-temperature case, except everywhere replacing the vacuum expectation value with the canonical ensemble average. Thus, by again exploiting the fact that the electromagnetic current satisfies the massive Klein-Gordon equation (16), we obtain the finite-temperature version of Eq. (17)

$$(\alpha\partial)_1 G(x_1, x_2)_{(T)} = \delta^2(x_1 - x_2) + (\alpha\partial)_1 F(x_1 - x_2)_{(T)} G(x_1, x_2)_{(T)}. \tag{27}$$

Here, the function $F(x_1 - x_2)_{(T)}$ has the same form as in Eq. (18) except that the zero-temperature function $\Delta_+(k;\mu^2)$ is replaced by its finite-temperature version $\Delta_+(k;\mu^2)_{(T)}$, which satisfies

$$(-\partial^2 + \mu^2)\Delta_+(x;\mu^2) = (-\partial^2 + \mu^2) \int \frac{d^2k}{(2\pi)^2} e^{ikx} \Delta_+(k;\mu^2) = \delta^2(x) \tag{28}$$

with finite-temperature boundary conditions. Explicitly, we have

$$\Delta_+(k;\mu^2)_{(T)} = \frac{1}{k^2 + \mu^2 - i\epsilon} + \frac{2\pi i\, \delta(k^2 + \mu^2)}{\exp[\beta(k_1^2 + \mu^2)^{1/2}] - 1}. \tag{29}$$

The first term is the zero-temperature expression, while the second term serves as a finite-temperature modification. In terms of this finite-temperature Klein-Gordon propagator, the function $F(x_1 - x_2)_{(T)}$ is given by [cf. Eq. (18)]

$$F(x_1 - x_2)_{(T)} = -ie^2 \int \frac{d^2k}{(2\pi)^2} \Delta_+(k;0)_{(T)} \times \frac{1}{2}\left[\frac{1}{(k_1 - i\epsilon)^2} + \frac{1}{(k_1 + i\epsilon)^2}\right] \times \Delta_+\left(k; \frac{e^2}{\pi}\right)_{(T)} (\tilde{\alpha} k)^2 [e^{ik(x_1 - x_2)} - 1]. \tag{30}$$

Alternatively, using the relation

$$\Delta_+(k;0)_{(T)} \Delta_+(k;\mu^2)_{(T)} = \frac{1}{\mu^2}[\Delta_+(k;0)_{(T)} - \Delta_+(k;\mu^2)_{(T)}], \tag{31}$$

which follows immediately from Eq. (29), $F(x_1 - x_2)_{(T)}$ can be written as

$$F(x_1 - x_2)_{(T)} = -i\pi \int \frac{d^2k}{(2\pi)^2} \left[\Delta_+(k;0)_{(T)} - \Delta_+\left(k; \frac{e^2}{\pi}\right)_{(T)}\right] \times \frac{1}{2}\left[\frac{1}{(k_1 - i\epsilon)^2} + \frac{1}{(k_1 + i\epsilon)^2}\right] \times (\tilde{\alpha} k)^2 [e^{ik(x_1 - x_2)} - 1]. \tag{32}$$

We note for future reference that

$$\lim_{x_2 \to x_1} F(x_1 - x_2)_{(T)} = i\pi \int \frac{d^2k}{(2\pi)^2} \left[\Delta_+(k;0)_{(T)} - \Delta_T\left(k; \frac{e^2}{\pi}\right)_{(T)}\right] \times \frac{1}{2}\left[\frac{1}{(k_1 - i\epsilon)^2} + \frac{1}{(k_1 + i\epsilon)^2}\right] (\tilde{\alpha} k)^2 \frac{[k(x_1 - x_2)]^2}{2}, \tag{33}$$

which vanishes as $(x_1 - x_2)^2$. Having secured the function $F(x_1 - x_2)_{(T)}$, we now solve Eq. (27) for the temperature two-point function and obtain

$$G(x_1, x_2)_{(T)} = \exp[F(x_1 - x_2)_{(T)} q_1 q_2] G^{(0)}(x_1 - x_2)_{(T)}. \tag{34}$$

Here, $G^{(0)}(x_1 - x_2)_{(T)}$ is the finite-temperature free-field two-point function satisfying

$$(\partial\alpha)_1 G^{(0)}(x_1 - x_2)_{(T)} = \delta^2(x_1 - x_2), \tag{35}$$

with finite-temperature boundary conditions. This function is readily obtained since the free fermion field anticommutator $[\psi^{(0)}(x_1), \psi^{(0)}(x_2)]_+$ (and hence its ensemble average) is the c number

$$[\psi^{(0)}(x_1),\psi^{(0)}(x_2)]_+ = \langle[\psi^{(0)}(x_1),\psi^{(0)}(x_2)]_+\rangle_{(T)} = -\int \frac{d^2k}{(2\pi)^2} e^{ik(x_1-x_2)} 2\pi\tilde{\alpha}k\,\delta(k^2) \times[\theta(k^0)-\theta(-k^0)]. \quad (36)$$

Using this anticommutator and Eq. (25), the free-field temperature two-point function is given by

$$G^{(0)}(x_1-x_2)_{(T)} = \langle T(\psi^{(0)}(x_1)\psi^{(0)}(x_2))\rangle_{(T)} = i\int \frac{d^2k}{(2\pi)^2} e^{ik(x_1-x_2)}\tilde{\alpha}k \times\left[\frac{1}{k^2-i\epsilon} - \frac{2\pi i\,\delta(k^2)}{e^{\beta|k_1|}+1}\right]. \quad (37)$$

Once again, the first term is the zero-temperature result and the second term is a finite-temperature correction. Note that in the limit in which the two space-time points become coincident, only the zero-temperature contribution survives, and we find

$$\lim_{x_2\to x_1} G^{(0)}(x_1-x_2)_{(T)} = \frac{-i}{2\pi}\frac{\tilde{\alpha}(x_1-x_2)}{(x_1-x_2)^2}. \quad (38)$$

With all the necessary apparatus now assembled, we are in a position to study the canonical ensemble average of the Schwinger model Hamiltonian [Eq. (14)]. Using the definition of the two-point temperature Green's function, the ensemble average of $\mathcal{H}$ can be expressed as

$$\langle\mathcal{H}\rangle_{(T)} = -\frac{i}{4}\lim_{x_2\to x_1^+} \mathrm{tr}\,\tilde{\alpha}\partial_1 G(x_1,x_2)_{(T)}. \quad (39)$$

Here tr implies a trace over the two-dimensional Dirac space and the two-dimensional charge space. The limiting procedure $x_2\to x_1^+$ dictates taking the limit $t_2\to t_1^+$ followed by $r_2\to r_1$.[12] Using the explicit solution for $G(x_1,x_2)_{(T)}$ [Eq. (34)], and expanding $\exp[F(x_1-x_2)q_1q_2]$ for $x_2\to x_1$ [cf. Eq. (33)], we find

$$\langle\mathcal{H}\rangle_{(T)} = -\frac{i}{4}\lim_{x_2\to x_1^+}\mathrm{tr}\,\tilde{\alpha}\partial_1 G^{(0)}(x_1-x_2)_{(T)} - \frac{i}{4}\lim_{x_2\to x_1^+}\mathrm{tr}\,q_1q_2\tilde{\alpha}\partial_1[F(x_1-x_2)_{(T)}G^{(0)}(x_1-x_2)_{(T)}]. \quad (40)$$

The first term is the ensemble average of the free fermion field Hamiltonian, $\langle\mathcal{H}^{(0)}\rangle_{(T)}$, which using the explicit form for $G^{(0)}(x_1-x_2)_{(T)}$, is given by

$$\langle\mathcal{H}^{(0)}\rangle_{(T)} = \langle\mathcal{H}^{(0)}\rangle_{(T=0)} + 2\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{|k_1|}{e^{\beta|k_1|}+1}. \quad (41)$$

To evaluate the remaining contribution to $\langle\mathcal{H}\rangle_{(T)}$, we use the limiting forms for $F(x_1-x_2)_{(T)}$ and $G^{(0)}(x_1-x_2)_{(T)}$ [Eqs. (31) and (38), respectively] as well as the properties of the Dirac matrices [Eq. (3)] to obtain

$$\frac{i}{4}\lim_{x_2\to x_1^+}\mathrm{tr}\,q_1q_2\tilde{\alpha}\partial_1[F(x_1-x_2)_{(T)}G^{(0)}(x_1-x_2)_{(T)}] = \frac{i}{4}\mathrm{tr}\,q_1q_2\int\frac{d^2k}{(2\pi)^2}\left[\Delta_+(k;0)_{(T)} - \Delta_+\left(k;\frac{e^2}{\pi}\right)_{(T)}\right](k^0)^2. \quad (42)$$

Furthermore, the antisymmetry of the charge matrix, in conjunction with trace in charge space implies that q_1q_2 can be effectively replaced with negative the identity matrix. Using the expression for $\Delta_+(k;\mu^2)_{(T)}$ [Eq. (29)] and recalling that the tr runs over Dirac and charge space so that $\mathrm{tr}1=4$, we find, after combining all the pieces, the ensemble average

$$\langle\mathcal{H}\rangle_{(T)} = \langle\mathcal{H}\rangle_{(T=0)} + 2\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{|k_1|}{e^{\beta|k_1|}+1} - \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{|k_1|}{e^{\beta|k_1|}-1} + \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{(k_1^2+e^2/\pi)^{1/2}}{\exp[\beta(k_1^2+e^2/\pi)^{1/2}]-1}. \quad (43)$$

Here we have extracted the zero-temperature piece and have explicitly written only the finite-temperature modifications. Moreover, since

$$2\int_0^\infty dx\,\frac{x}{e^x+1} = \int_0^\infty dx\,\frac{x}{e^x-1}, \quad (44)$$

the above simplifies to

$$\langle\mathcal{H}\rangle_{(T)} = \langle\mathcal{H}\rangle_{(T=0)} + \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{(k_1^2+e^2/\pi)^{1/2}}{\exp[\beta(k_1^2+e^2/\pi)^{1/2}]-1}, \quad (45)$$

which is the ensemble average of a Hamiltonian corresponding to a collection of noninteracting, neutral bosons with mass $e/\sqrt{\pi}$.

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[9]The electromagnetic current as naively defined by Eq. (2) is a singular operator. It can be more suitably defined using a point-separation method as $j^\mu(x) = \frac{1}{2}\psi(x+\epsilon)\alpha^\mu q\psi(x-\epsilon)$, where ϵ^μ is a purely spatial vector and the symmetrical limit $\epsilon \to 0$ is to be taken after all calculations have been performed. In general, in order to preserve gauge invariance, the point-separation definition of the current must be modified by an exponential factor of the line integral of the vector potential. However, in the Coulomb gauge, this extra factor is just unity because of the gauge condition and the fact that ϵ^μ is purely spatial. The necessity of using the point-separation definition for j^μ manifests itself in the calculation of the divergence of the axial-vector current, Eq. (19). For additional details, see Ref. 2.

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