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Gauge dependence of the effective action

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The gauge dependence of the effective action of quantum non-Abelian gauge theories is studied. An alternative effective action is proposed and its equivalence with the usual effective action is discussed, as well as the equivalence with 't Hooft's effective action.

I. INTRODUCTION

The background field is a device which has been used to aid the discussion of formal properties of quantum field theories[1] and there has been some discussion of its use as an alternative to introducing external sources.[2] In the case of field theories with no gauge invariance or at most an Abelian gauge group, there seems to be no particular advantage to formulating the theory in terms of a background field (although one must use a background field to determine the ground state of the system.)

In quantum field theories, a classical source is generally used for one of two purposes. First, variations of the source yield matrix elements of the operator to which the source is coupled. In this context, matrix elements in the presence of the source form generating functionals of the various operator matrix elements and, formally at least, one can deal with all matrix elements simultaneously. Second, the source may represent another large (classical) system which responds weakly to the quantum system being studied. In the case of a non-Abelian gauge field (either those having internal symmetries such as the grand unified theories or coordinate-invariant general relativity), the complexities associated with the operator gauge invariance render either use of an external source less attractive. In the first case, one is calculating the generating functional of matrix elements of operators which depend upon the choice of operator gauge. These matrix elements have no direct physical significance and the process of obtaining gauge-invariant answers to physical questions, though well understood in principle, is fraught with opportunities for error. In the second case, the physical significance of a specified source depends upon both the global topology and the local values of the field to which the source gives rise: one does not know what one has wrought in specifying the source until one has specified the associated field as well.[3] (The source is determined by its components in a given set of basis vector fields but the relationship between the basis vectors at different points is determined by the gauge field itself. Different choices of, e.g., operator gauge or boundary conditions can drastically change the physical significance of the predetermined components of the source.)

Superficially, one avoids these problems by working in the background-field formulation. Instead of studying how the system responds to a change in a source (external current or stress-energy tensor), one studies how the system responds to a change in the background field. Nothing essential is lost by the change because one is dealing directly with the matrix element of the field and, by studying the effects of small changes in the gauge field at large space-time separation, one may still recover all the scattering information which one usually obtains from the Green's functions. In the case of a specified background, there are no problems of interpretation because one is specifying both the global topology and the local geometry at the same time.

The background-field method has been used extensively since Lee and Zinn-Justin[4] first discussed the gauge and renormalization properties of the non-Abelian gauge field coupled to scalar fields with spontaneous symmetry breaking. The effective Lagrangian is a functional of the expectation value of the field which, in turn, may be taken to be the background field. This development is reviewed in Sec. II and the definition of the effective action and its dependence upon the choice of operator gauge are reviewed. The effective action is a functional of the background field, $\overline{\phi}$, which, when expanded as a power series in $\overline{\phi}$, is the generating functional of the single-particle-irreducible vertices of the theory. These vertices depend upon the choice of operator gauge in a nontrivial way, but the effective action, when evaluated at a field corresponding to a vanishing source, is independent of the choice of gauge.

The resultant effective action determines the classical gauge of the background field as well as the operator gauge. One would like an alternative procedure which is invariant under classical gauge transformations and operator gauge transformations, and which describes the same physics as the original theory. In a series of lectures, 't Hooft[2] has proposed an alternative effective action which exhibits these properties with the exception that the equivalence of his formulation with others is unclear and the procedure for calculating his effective action is somewhat vague. In Sec. III an alternative procedure is outlined which is clearly equivalent to the usual theory. The procedure for calculating the effective action is exactly the same as for the usual theory except that the operator gauge condition is taken to depend on the background field and some explicit additional terms appear. A general discussion of the gauge-transformation properties of the action is given and it is shown that the action and all physical matrix elements are independent of both the classical and the quantum gauge choices. It is shown that the field equations derived from the gauge-invariant effective action may have additional solutions which are unphysical and a criterion for physical acceptability of the solutions is provided. The form of the resultant gauge-invariant effective action depends upon the operator gauge choice as do the solutions of the field equations; however, the action when evaluated at a solution does not depend on either the operator or the classical gauge.

In Sec. IV the relation of 't Hooft's procedure to the procedure described in Sec. III is briefly outlined. The effective action derived by 't Hooft's procedure is quite different but possesses the same solutions and describes the same physical situation.

While this work was underway, I received a report from B. DeWitt[5] which discusses the general problem of the background-field method in considerable detail and addresses many of the same questions.

II. GAUGE-THEORY QUANTIZATION

A quantum gauge theory is the quantum theory based on the classical action

$$W_{\rm cl} = \int dx\, \mathcal{L}(\phi)\,, \tag{1}$$

which is invariant under a local gauge transformation

$$\begin{aligned}\delta_\xi \phi^a(x) &= (D^a{}_i(\phi)\delta\xi^i)(x)\\ &= \int dy < x|D^a{}_i|y> \delta\xi^i(y)\\ &= <ax|(D\delta\xi)\,,\end{aligned} \tag{2}$$

where the $\delta\xi^i(y)$ are the infinitesimal local gauge functions. The index a denotes the field index (i, μ for gauge fields or $\mu\nu$ for general relativity) and the index i denotes the group index (i or μ). The fields $\{\phi\}$ include all the fields in the theory and may be taken to include Fermi fields as well as Bose fields but in this paper only Bose fields will be explicitly considered. In the case of a gauge group based on a non-Abelian internal-symmetry group, the vector gauge fields transform as

$$\delta_\xi A_i^\mu = (\partial^\mu \delta_{ij} - igc_{ikj}A_k^\mu)\delta\xi_j\,, \tag{3}$$

where the matrices c_{ikj} are totally antisymmetric, imaginary, and form the structure constants of the internal-symmetry group. In the case of general relativity, the gauge field is the metric itself which transforms as

$$\begin{aligned}\delta_\xi g_{\mu\nu}(x) &= \delta\xi_{\mu;\nu} + \delta\xi_{\nu;\mu}\\ &= g_{\mu\lambda}\delta\xi^\lambda{}_{,\nu} + g_{\lambda\nu}\delta\xi^\lambda{}_{,\mu} + g_{\mu\nu,\lambda}\delta\xi^\lambda\,.\end{aligned} \tag{3$'$}$$

Because the classical action is invariant under gauge transformations, the functional integral defining the quantum theory cannot be taken over all fields; it must be restricted to fields which are gauge inequivalent. The gauge condition which picks out the fields over which the integral is evaluated will be taken to be linear,

$$C^i{}_a\phi^a = \zeta^i\,, \tag{4}$$

where the gauge-fixing operators $C^i{}_a$ may be taken to be either algebraic (e.g., the axial gauge $A_a^3 = 0$) or differential (e.g., the Lorentz gauge $\partial_\mu A_a^\mu = 0$) and are arbitrary provided that they uniquely determine the gauge. The gauge condition may involve scalar fields as well as the gauge field but we will not explicitly consider such conditions. The choice of operator C defines the operator gauge and the functional measure defining the field theory is[4,6]

$$\int [d\phi] \exp\left\{ i \int dx [\mathcal{L}(\phi) - \tfrac{1}{2}\phi C^T C\phi]\right\} \Delta[\phi, C]\,, \tag{5}$$

where the Faddeev-Popov determinant[7]

$$\Delta[\phi, C] = \mathrm{Det}\, CD(\phi) \tag{6}$$

is the Fredholm determinant of the operator CD which results from the change in $C\phi$ under a gauge transformation,

$$\delta_\xi{}^i C\phi = {}^iC_a D^a{}_j \delta\xi^j\,. \tag{7}$$

The time-ordered matrix element of an operator functional $F[\phi]$ is given by

$$\langle \text{out} | T(F[\phi]) | \text{in} \rangle = \langle \text{out} | \text{in} \rangle \langle F[\phi] \rangle$$
$$= \int [d\phi] F[\phi] \exp\left\{ i \int [\mathcal{L}(\phi) - \tfrac{1}{2}\phi C^T C \phi] \right\} \Delta[\phi, C]$$
$$= \int [d\phi][d\chi][d\tilde{\chi}] F[\phi] \exp\left\{ i \int [\mathcal{L}(\phi) - \tfrac{1}{2}\phi C^T C \phi + \chi C D \tilde{\chi}] \right\}, \quad (8)$$

where the Faddeev-Popov determinant has been written as an integral over the anticommuting variables χ and $\tilde{\chi}$.[7] The in and out states are determined by the values chosen for ϕ on the initial and final surfaces. If the operator F is chosen to be

$$F[\phi] = \exp\left[i \int dx\, \phi^a(x) J_a(x) \right] \equiv \exp\left(i \int \phi J \right), \quad (9)$$

then the matrix element of any operator may be obtained by functional differentiation of the matrix element of F, that is,

$$\left\langle \text{out} \left| \exp\left(i \int \phi J \right) \right| \text{in} \right\rangle \equiv \langle \text{out} | \text{in} \rangle^J$$
$$= \exp(iW[J, C])$$
$$= \int [d\phi][d\chi][d\tilde{\chi}] \exp\left\{ i \int [\mathcal{L}(\phi) - \tfrac{1}{2}\phi C^T C \phi + \chi C D(\phi) \tilde{\chi} + J\phi] \right\} \quad (10)$$

is a generating functional for all matrix elements of ϕ and J is the external source. Except for the $\exp(i \int J\phi)$ factor, the integral is invariant under the Becchi-Rouet-Stora (BRS) transformation[7]

$$\delta_{\text{BRS}}\, \phi^a = D^a(\phi)_i \delta\lambda\, \tilde{\chi}^i ,$$
$$\delta_{\text{BRS}}\, \chi^i = \delta\lambda (\phi C^T)^i , \quad (11)$$
$$\delta_{\text{BRS}}\, \tilde{\chi}^k = -\tfrac{1}{2}\, \mathcal{C}(i,j,k) \delta\lambda\, \tilde{\chi}^i \tilde{\chi}^j ,$$

where the indices (i,j,k) in the operator $\mathcal{C}$ include the space-time indices which are also summed over and the parameter $\delta\lambda$ is an anticommuting number. The operator $\mathcal{C}$ is the structure function of the gauge group and is defined by

$$\left[\frac{\delta}{\delta \xi^i(x)}, \frac{\delta}{\delta \xi^j(y)} \right] = \int dz\, \mathcal{C}(i,x;j,y;k,z) \frac{\delta}{\delta \xi^k(z)}, \quad (12)$$

where

$$\frac{\delta}{\delta \xi^i(x)} = \langle x |_i D^{Ta} \frac{\delta}{\delta \phi^a} = \int dy \langle x |_i D^{Ta} | y \rangle \frac{\delta}{\delta \phi^a(y)} .$$

In the case of a non-Abelian internal-symmetry group,

$$\mathcal{C}(i,x;j,y;k,z) = -igc_{ijk} \delta(x-y)\delta(y-z) ,$$

while in the general-relativity case,

$$\mathcal{C}(\mu,x;\nu,y;\lambda,z) = (\partial_{y\nu}\delta^\lambda_\mu - \partial_{x\mu}\delta^\lambda_\nu)\delta(x-y)\delta(y-z) .$$

The Lee-Slavnov identities[4,8] may then be derived using the invariance of $\langle | \chi^i | \rangle$ under the Becchi-Rouet-Stora transformation, which is simply a change of integration variable:

$$0 = \delta_{\text{BRS}} \langle \text{out} | \chi^j(y) | \text{in} \rangle / \delta\lambda$$
$$= \langle \text{out} | C^j{}_a \phi^a(y) | \text{in} \rangle$$
$$+ i \int dx\, J_a(x) \langle \text{out} | T(D^a{}_i \tilde{\chi}^i(x) \chi^j(y)) | \text{in} \rangle , \quad (13)$$

or, defining the ghost propagator $\mathcal{G}$,

$$i\langle T(\tilde{\chi}^i(x)\chi^j(y)) \rangle \equiv \mathcal{G}^{ij}(x,y) ,$$

and the vertex Σ,

$$i\langle T(D^a{}_i \tilde{\chi}^i(x)\chi^j(y)) \rangle \equiv i\Sigma^a{}_i\, \mathcal{G}^{ij}(x,y) , \quad (14)$$

$$C^j{}_a \frac{\delta W}{\delta J_a}(x) = C^j{}_a \langle \phi^a \rangle(x) = (-i)(J_a \Sigma^a{}_i\, \mathcal{G}^{ij})(x)$$
$$= (-i)^j (\mathcal{G}^T \Sigma^T J)(x) . \quad (15)$$

The expectation value of ϕ^a does not satisfy the gauge condition unless the external source vanishes. The operator C is arbitrary and should have no physical significance. A variation of C yields

$$\delta_C \exp(iW[J,C]) = i \left\langle \text{out} \left| \int [\chi \delta C D(\phi) \tilde{\chi} - \phi \delta C^T C \phi] \right| \text{in} J \right\rangle, \quad (16)$$

but invariance under the Becchi-Rouet-Stora transformation implies that

$$0=-i\delta_{\mathrm{BRS}}\langle\mathrm{out}|\,T(\chi^j(x)(\delta C^i\phi)(y))\,|\,\mathrm{in}\rangle/\delta\lambda$$

$$=-i\left\langle\mathrm{out}\left|T\left[((C^j\phi)(x)(\phi\delta C^{Ti})(y))-(\chi^j(x)(\delta C^i D\tilde{\chi})(y))-i\left(\chi^j(x)(\phi\delta C^{T^i})(y)\int dz\,J_a(z)(D^a\tilde{\chi})(z)\right)\right]\right|\mathrm{in}\right\rangle, \qquad (17)$$

which implies, when (x,i) and (y,i) are set equal and summed over,

$$0=-i\left\langle\mathrm{out}\left|\int\phi\delta C^T C\phi\right|\mathrm{in}\right\rangle+i\left\langle\mathrm{out}\left|\int\chi\delta CD\tilde{\chi}\right|\mathrm{in}\right\rangle$$
$$-i\int\left(\frac{1}{i}\frac{\delta}{\delta J}\delta C^T\right)^i(x)dx\left\langle\mathrm{out}\left|T\left(\chi^i(x)\int JD\tilde{\chi}\right)\right|\mathrm{in}\right\rangle, \qquad (18)$$

where the operator $\delta/\delta J$ does *not* act on the explicit J.

This result, when subtracted from the variation of $\exp(iW)$ with C yields, since $\delta W/\delta J=\langle\phi\rangle$,

$$\delta_C W[J,C]=-\int\left[\langle\phi\rangle\delta C^T\mathcal{G}^T\Sigma^T J-i\left(\frac{1}{i}\frac{\delta}{\delta J}\delta C^T\mathcal{G}^T\Sigma^T\right)J\right]$$
$$=-\int\left[\langle\phi\rangle\delta C^T C\langle\phi\rangle-i\left(\frac{1}{i}\frac{\delta}{\delta J}\delta C^T\mathcal{G}^T\Sigma^T\right)J\right]. \qquad (19)$$

The effective action W is independent of C provided J is first set equal to zero. The final term may be rewritten using the ghost equation motion which implies

$$C\Sigma\mathcal{G}=i$$

and defining Γ by

$$\frac{\delta\Sigma^c{}_i}{\delta J_a(x)}=\int dy\,G^{ab}(x,y)\Gamma^c_{bi}(y),$$

where the coordinates associated with c and i have been suppressed, and where G is the gauge field propagator, to find

$$\delta_C W=-\int\langle\phi\rangle C^T\delta C\langle\phi\rangle+\int\mathrm{Tr}[\delta CK^a]J_a, \qquad (20)$$

where

$$\langle b|K^a|i\rangle$$
$$=G(b,b')\langle a'|\Gamma(b')|i\rangle\left\{\delta_{a'a}-\left\langle a'\left|\frac{1}{i}C^T\mathcal{G}^T\Sigma^T\right|a\right\rangle\right\}$$
$$\equiv G(b,b')\Gamma^{a'b'}{}_i\{_{a'}P_1{}^{\Sigma}\} \qquad (21)$$

with coordinate indices suppressed and repeated indices summed over. The factor in curly brackets is a projection operator orthogonal to C^T on the right and orthogonal to Σ^T on the left.

To summarize, the general variation of W is given by

$$\delta W[J,C]=\int\{\delta J_a\langle\phi^a\rangle-\langle\phi^a\rangle\delta C_a^{T\,i}C^i{}_b\langle\phi^b\rangle$$
$$+\delta C^i{}_a K^{ab}{}_i J_b\} \qquad (22)$$

and

$$C^i{}_a\langle\phi^a\rangle=-i\mathcal{G}^{Tij}\Sigma_j^{Tb}J_b=-i(\mathcal{G}^T\Sigma^T J)^i. \qquad (23)$$

The Lee-Slavnov identities, Eq. (23), and the variation of W with changes in C are the basic gauge-transformation properties of the theory. Any physical quantity must and will be invariant under such transformations when the external current J is set equal to zero.

The usual effective action may be obtained by observing that

$$\langle\phi^a\rangle=\frac{\delta W}{\delta J_a} \qquad (24)$$

may be solved for J as a functional of $\langle\phi\rangle$ and performing a Legendre transformation. The effective action is then defined as

$$\Gamma[\langle\phi\rangle,C]=W[J(\langle\phi\rangle),C]-J_a\langle\phi^a\rangle. \qquad (25)$$

The general variation of the effective action is

$$\delta\Gamma=\delta W-\delta J_a\langle\phi^a\rangle-J_a\delta\langle\phi\rangle$$
$$=-J_a(\langle\phi\rangle)\delta\langle\phi^a\rangle-\langle\phi^a\rangle\delta C_a^{T\,i}C^i{}_b\langle\phi^b\rangle$$
$$+\delta C^i{}_a K^{ab}{}_i J_b. \qquad (26)$$

The variation of Γ with respect to $\langle\phi\rangle$ holding C constant yields, when set equal to $-J$, the effective field equation for $\langle\phi\rangle$. The equations possess the full range of solutions with whatever boundary conditions are appropriate. If the gauge is varied, W and Γ do change because the functional form of Γ changes. The expansion of Γ as a power series $\langle\phi\rangle$ yields the single-particle-irreducible vertices each one of which depends on C. However, when evaluated at a solution to the source-free field equation,

$$\frac{\delta\Gamma}{\delta\langle\phi^a(x)\rangle}=0, \qquad (27)$$

the resultant action is independent of C. Hence the associated matrix element, $\exp(i\Gamma)$, is independent of C and no physical quantity depends upon C.

III. THE GAUGE-INVARIANT EFFECTIVE ACTION

As a prelude to the treatment of the gauge-theory background field, consider a scalar theory

$$\exp(iW[J]) = \int [d\phi] \exp\left\{i \int [\mathcal{L}(\phi) + J\phi]\right\}. \quad (28)$$

Solving

$$\langle\phi\rangle = \delta W[J]/\delta J \quad (29)$$

for J and defining the effective action by

$$\Gamma[\langle\phi\rangle] = W[J[\langle\phi\rangle]] - \int J[\langle\phi\rangle]\langle\phi\rangle , \quad (30)$$

we obtain the field equation

$$\delta\Gamma/\delta\langle\phi\rangle = -J . \quad (31)$$

An alternative way to obtain the same result is to introduce a background field $\overline{\phi}$ and only couple J to the field $\phi' = \phi - \overline{\phi}$. Then

$$\begin{aligned}\exp(i\tilde{W}[J, \overline{\phi}]) &= \int [d\phi'] \exp\left\{i \int [\mathcal{L}(\overline{\phi} + \phi') + J\phi']\right\} \\ &= \int [d\phi] \exp\left\{i \int [\mathcal{L}(\phi) + J(\phi - \overline{\phi})]\right\} \\ &= \exp\left[i\left(W[J] - \int J\overline{\phi}\right)\right] . \end{aligned} \quad (32)$$

If the external current J is determined by requiring that $\tilde{W}$ be an extremum under variations of J,

$$0 = \delta\tilde{W}[J, \overline{\phi}]/\delta J = (\delta W[J]/\delta J) - \overline{\phi} = \langle\phi\rangle - \overline{\phi} , \quad (33)$$

then the background field $\overline{\phi}$ is the expectation value of the field $\langle\phi\rangle$, and J is the current which produces the expectation value $\langle\phi\rangle = \overline{\phi}$. As a result, if $\tilde{W}$ is evaluated at the stationary value $J[\overline{\phi}]$, we obtain

$$\tilde{W}[J[\overline{\phi}], \overline{\phi}] = \Gamma[\overline{\phi}] ,$$

the same effective action that results from the standard Legendre transformation.

If the same procedure is followed for a gauge theory, one again obtains an effective action. The action is *not* the same as one obtains from the Legendre transformation. It does, however, yield the same physics. To show this, define $\tilde{W}[J, \overline{\phi}, C]$ by

$$\exp(i\tilde{W}[J, \overline{\phi}, C]) = \int [d\phi'][d\chi][d\tilde{\chi}] \exp\left\{i \int [\mathcal{L}(\overline{\phi} + \phi') - \tfrac{1}{2}\phi' C^T C \phi' + \chi C D(\overline{\phi} + \phi')\tilde{\chi} + J\phi']\right\}, \quad (34)$$

where the gauge condition is on ϕ' rather than $\phi = \overline{\phi} + \phi'$ and the source couples only to ϕ'.

As with the scalar theory, one may translate ϕ' by $-\overline{\phi}$ to obtain

$$\begin{aligned}\exp(i\tilde{W}[J, \overline{\phi}, C]) &= \int [d\phi][d\chi][d\tilde{\chi}] \exp\left\{i \int [\mathcal{L}(\phi) - \tfrac{1}{2}\phi C^T C \phi + \chi C D(\phi)\tilde{\chi} (J + \overline{\phi} C^T C)\phi - J\overline{\phi} - \tfrac{1}{2}\overline{\phi} C^T C \overline{\phi}]\right\} \\ &= \exp\left[i\left(W[J + C^T C\overline{\phi}, C] - \int (J\overline{\phi} + \tfrac{1}{2}\overline{\phi} C^T C \overline{\phi})\right)\right] \end{aligned} \quad (35)$$

or

$$\tilde{W}[J, \overline{\phi}, C] = W[J + \overline{\phi} C^T C, C] - \int (J\overline{\phi} + \tfrac{1}{2}\overline{\phi} C^T C \overline{\phi}) . \quad (36)$$

Under an arbitrary variation of J, $\overline{\phi}$, and C, the results of the standard theory, Eq. (20), yield

$$\delta\tilde{W} = \int \{\delta J_a(\langle\phi^a\rangle - \overline{\phi}^a) + \delta\overline{\phi}^a[-J_a + C^{T}{}_a{}^i C^i{}_b(\langle\phi^b\rangle - \overline{\phi}^b)] - (\langle\phi^a\rangle - \overline{\phi}^a)\delta C^{T}{}_a{}^i C^i{}_b(\langle\phi^b\rangle - \overline{\phi}^b) + \delta C^i{}_a K^{ab}{}_i J_b\} , \quad (37)$$

where the functions K and $\langle\phi\rangle$ are functionals of the current $J + \overline{\phi} C^T C$. Only J appears in the K term because of the projection operator orthogonal to C. The Lee-Slavnov identities imply that

$$C^i{}_a\langle\phi^a\rangle = -i g^{Tij}\Sigma^T{}_j{}^b(J_b + C^T{}_b{}^k C^k{}_c\overline{\phi}^c)$$

or (38)

$$C^i{}_a(\langle\phi^a\rangle - \overline{\phi}^a) = -i g^{Tij}\Sigma^T{}_j{}^b J_b ;$$

the gauge condition is on $\langle\phi'\rangle$ rather than $\langle\phi\rangle$.

The quantity $\tilde{W}$ evaluated at the J extremum is a functional of $\overline{\phi}$ and C which may be related to the usual effective action

$$\begin{aligned}\tilde{W}[J, \overline{\phi}, C] &= W[J + \overline{\phi} C^T C, C] - \int (J\overline{\phi} + \tfrac{1}{2}\overline{\phi} C^T C \overline{\phi}) \\ &= \left\{W[J + \overline{\phi} C^T C, C] - \int (J + \overline{\phi} C^T C)\overline{\phi}\right\} \\ &\quad + \tfrac{1}{2}\int \overline{\phi} C^T C \overline{\phi} . \end{aligned} \quad (39)$$

Hence $\tilde{W}$, when evaluated at the $J + \phi C^T C$ such that

$=\overline{\phi}$, becomes

$$\tilde{W}=\tilde{\Gamma}[\overline{\phi},C]=\Gamma(\overline{\phi},C)+\tfrac{1}{2}\int\overline{\phi}C^TC\overline{\phi}\,. \tag{40}$$

If $\tilde{\Gamma}$ is an appropriate effective action, it must yield the field equations when its variation is set equal to $-J$. It does since

$$0=\delta\tilde{\Gamma}/\delta\overline{\phi}^a+J_a=\delta\Gamma/\delta\overline{\phi}^a+(J_a+C_a{}^{Ti}C^i{}_b\overline{\phi}^b) \tag{41}$$

is the old equation with the current $J+C^TC\overline{\phi}$. If a physical solution is sought, J must be set equal to zero and the equations are identical if the further gauge condition $C\overline{\phi}=0$ is imposed.

In the original formulation, the gauge condition was imposed on ϕ; in the new formulation, it is imposed on $\phi'=\phi-\overline{\phi}$, but *not* on $\overline{\phi}$. As a result, one may choose any classical gauge for $\overline{\phi}$. To understand this, note that if a gauge transformation is made on $\overline{\phi}$, then the *same* gauge transformation may be made on the integration variables $\phi=\overline{\phi}+\phi'$, χ and $\tilde{\chi}$. Under that transformation, if $J=0$, the path integral is invariant except for the gauge-fixing quantities involving C; however, under a gauge transformation,

$$\delta_\xi\phi'=D(\overline{\phi}+\phi')\delta\xi-D(\overline{\phi})\delta\xi=T^i\phi'\delta\xi^i \tag{42}$$

and $D(\overline{\phi}+\phi')\tilde{\chi}$ transforms the same way as ϕ',

$$\delta D(\overline{\phi}+\phi')\tilde{\chi}=T^i(D\tilde{\chi})\delta\xi^i\,. \tag{43}$$

Thus, the change in the C-dependent terms is

$$\delta\int(\chi CD\tilde{\chi}-\tfrac{1}{2}\phi'C^TC\phi')=\int(\chi\delta_\xi CD\tilde{\chi}-\phi'C^T\delta_\xi C\phi')\,, \tag{44}$$

where

$$\delta_\xi C=CT^i\delta\xi^i\,.$$

We find that $\tilde{\Gamma}$ is invariant under the simultaneous classical gauge transformation of $\overline{\phi}$ *and* the change of operator gauge $\delta_\xi C$. Thus, given a solution $\overline{\phi}$, to $\delta\tilde{\Gamma}/\delta\overline{\phi}=0$ for a given C, the classical gauge transformation of $\overline{\phi}$ will be a solution for a *different* C. However, $\tilde{\Gamma}$, when evaluated at a solution, is independent of C. Hence one may find a solution for any classical gauge choice, $\overline{C}\cdot\overline{\phi}=0$; the solution will *not* be the gauge transform of the solution for a different $\overline{C}$ but $\tilde{\Gamma}$ evaluated at that solution will be independent of $\overline{C}$ and C.

In order to obtain full invariance under classical gauge transformations of the background field, the operator gauge must transform covariantly under classical gauge transformations. This may be achieved by making C a functional of a new gauge field $\tilde{\phi}$, for example,

$$C\phi'\to(\delta^i{}_c\partial_\mu-igC_{ibc}\tilde{A}_\mu{}^b)\phi_c{}'^\mu \tag{45}$$

in the case of a non-Abelian gauge field or

$$C\phi'\to(\sqrt{-\tilde{g}}\tilde{g}^{\lambda\mu}h_{\mu\nu})_{,\lambda}-\tfrac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\mu\lambda}h_{\mu\lambda,\nu}$$
$$=\sqrt{-\tilde{g}}(h_{\mu\nu}{}^{;\mu}-\tfrac{1}{2}g^{\mu\lambda}h_{\mu\lambda;\nu})\,, \tag{46}$$

where the covariant derivatives are taken relative to the background metric $\tilde{g}$. The gauge choice is now covariant under simultaneous classical gauge transformations of $\overline{\phi}$ and $\tilde{\phi}$ [in the general-relativity case, the sum over gauge conditions $\overline{\phi}C^TC\overline{\phi}$ must include a factor of $(\tilde{g}^{\lambda\sigma}/\sqrt{-\tilde{g}})$ to make the gauge term a scalar,

$$\int\overline{\phi}C^TC\overline{\phi}\to\int dx\sqrt{-\tilde{g}}\tilde{g}^{\lambda\sigma}(h_{\lambda\mu}{}^{;\mu}-\tfrac{1}{2}\tilde{g}^{\mu\tau}h_{\mu\tau;\lambda})$$
$$\times(h_{\sigma\nu}{}^{;\nu}-\tfrac{1}{2}\tilde{g}^{\nu\rho}h_{\nu\rho;\sigma})\,.$$

The field $\tilde{\phi}$ is, in principle, distinct from $\overline{\phi}$ but should be topologically equivalent to $\overline{\phi}$ so that it may be continuously deformed into $\overline{\phi}$. The theory which results is then invariant under simultaneous gauge transformations of $\overline{\phi}$ and $\tilde{\phi}$,

$$\Gamma[\overline{\phi},C(\tilde{\phi})]=\Gamma[\overline{\phi}+D(\overline{\phi})\delta\xi,C(\tilde{\phi}+D(\tilde{\phi})\delta\xi)]\,. \tag{47}$$

The two fields, $\overline{\phi}$ and $\tilde{\phi}$, may now be identified. Then, the field equation results from varying $\overline{\phi}$ while holding $\tilde{\phi}$ fixed. However the effective action $\overline{\Gamma}$ is

$$\overline{\Gamma}[\overline{\phi}]=\tilde{\Gamma}[\overline{\phi},C(\overline{\phi})]\,, \tag{48}$$

which is a gauge-invariant function of $\overline{\phi}$ and every solution to the field equation, Eq. (41), for $J=0$ is also a solution to the field equation,

$$\delta\overline{\Gamma}[\overline{\phi}]/\delta\overline{\phi}^a=0\,. \tag{49}$$

To see this observe that

$$\delta_{\overline{\phi}}\overline{\Gamma}=\delta_{\overline{\phi}}\tilde{\Gamma}+\delta_C\tilde{\Gamma}=\int(-\delta\overline{\phi}^aJ_a+\delta C^i{}_aK^{ab}{}_iJ_b)$$

or

$$\delta\overline{\Gamma}/\delta\overline{\phi}^a=-J_a+(\delta C^i{}_c/\delta\overline{\phi}^a)K^{cb}{}_iJ_b\,. \tag{50}$$

Given a solution $\overline{\phi}$, to $\delta\tilde{\Gamma}/\delta\overline{\phi}^a=0$, $J_a=0$ and, therefore $\delta\overline{\Gamma}/\delta\overline{\phi}^a=0$.

The solution $J(\overline{\phi})=0$ may not be the only solution. From the definition of the operator K^a, Eq. (20), there is a projection operator

$$P_1{}^\Sigma=1+iC^T\mathcal{G}^T\Sigma^T$$

acting to the left on the a index. It will be convenient to define other projection operators,

$$P_0{}^\Sigma=1-P_1{}^\Sigma$$
$$P_1{}^D=1+C^T\mathcal{G}_0{}^TD^T\,, \tag{51}$$

and

$$P_0{}^D=1-P_1{}^D=-C^T\mathcal{G}^T{}_0D^T\,,$$

where $\mathcal{G}_0$ satisfies

$$-D^T C^T \mathcal{G}_0 = 0 .$$

From the Lee-Slavnov identity, Eq. (38), J, when determined by the condition $\langle \phi \rangle - \overline{\phi} = 0$, must satisfy

$$\Sigma^T J = 0 \tag{52}$$

and, since $\overline{\Gamma}$ is invariant under classical gauge transformations, it must satisfy

$$0 = D^T \delta\overline{\Gamma}/\delta\overline{\phi} = -D^T(1 - LP_1{}^{\Sigma})J = 0 , \tag{53}$$

where

$$(LP_1{}^{\Sigma})_a{}^b = (\delta C^i{}_c/\delta\overline{\phi}^a)(K^{cb'}{}_i)(P_1{}^{\Sigma})_{b'}{}^b .$$

The condition, Eq. (52), on J is the covariant conservation requirement, including the quantum effects, while the covariant conservation of $\delta\overline{\Gamma}/\delta\overline{\phi}$ is the statement of gauge invariance, which implies covariant conservation of the source $(1 - LP_1{}^{\Sigma})J$. In the classical limit, $\Sigma = (-i)D$ and the operator L vanishes; the two statements are equivalent. Because $\Sigma J = 0$, the invariance requirement, Eq. (53), implies

$$D^T(1 - LP_1{}^{\Sigma}) = AP_0{}^{\Sigma} .$$

Multiply by $P_1{}^{\Sigma}$ from the right to find

$$D^T(1 - L)P_1{}^{\Sigma} = 0$$

or since $P_1{}^D$ is orthogonal to D^T from the left,

$$(1 - L)P_1{}^{\Sigma} = P_1{}^D B P_1{}^{\Sigma} . \tag{54}$$

In the classical limit, $L = 0$ and $P_1{}^D = P_1{}^{\Sigma}$, hence

$$B = 1 - \Pi ,$$

where Π vanishes in the classical limit and may be taken to satisfy

$$P_1{}^D \Pi = \Pi = \Pi P_1{}^{\Sigma} .$$

Then,

$$LP_1{}^{\Sigma} = P_0{}^D P_1{}^{\Sigma} + \Pi ,$$

and the field equation for $\overline{\phi}$ reads

$$\delta\overline{\Gamma}/\delta\overline{\phi} = -[1 - P_0{}^D P_1{}^{\Sigma} - \Pi]J = 0$$

or

$$(P_0{}^D P_1{}^{\Sigma} + \Pi)J = J . \tag{55}$$

Since

$$P_1{}^{\Sigma} J = J ,$$

$$\Pi J = (1 - P_0{}^D)J = P_1{}^D J ,$$

and since

$$P_1{}^{\Sigma} P_1{}^D J = P_1{}^{\Sigma} J = J ,$$

we have

$$\Pi^2 J = \Pi P_1{}^D J = \Pi J .$$

Hence a solution for which J does not vanish must be an eigenvector of Π with eigenvalue $\Pi' = 0, 1$. For $\Pi' = 0$,

$$0 = \Pi J = P_1{}^D J ,$$

but

$$0 = P_1{}^{\Sigma} P_1{}^D J = P_1{}^{\Sigma} J = J .$$

In this case J vanishes and the solution is the usual solution. For $\Pi' = 1$,

$$P_1{}^{\Sigma} J = J = \Pi J = P_1{}^D J$$

or

$$0 = (P_1{}^{\Sigma} - P_1{}^D)J = P_0{}^D J = -C^T \mathcal{G}_0{}^T [D^T J] .$$

Thus,

$$0 = D^T J = (D^T - i\Sigma^T)J .$$

The difference between D^T and $i\Sigma^T$ is simply the radiative corrections to the conservation equation.

I know of no reason why such a solution should not exist. Such a solution, if one does exist, is nonclassical. The operator Π is a quantum correction, the leading term of which is of one-loop order. Thus a $\overline{\phi}$ is required such that $\Pi[\overline{\phi}]$ have a unit eigenvalue *and* that $J(\overline{\phi})$ be the unit eigenvector. Such a solution is nevertheless unphysical and must be rejected. The variation of $\overline{\Gamma}$ under changes of C must still vanish and, in particular, one may consider the variation of the $\overline{\phi}$ dependence of C, or

$$0 = \frac{\delta C}{\delta\overline{\phi}} \frac{\delta\overline{\Gamma}}{\delta C}\bigg|_{\overline{\phi}} = -\frac{\delta\Gamma}{\delta\overline{\phi}}\bigg|_C = J \neq 0$$

for the hypothesized solution. Such anomalous solutions cannot occur in perturbation theory for $\overline{\phi}$ (they may appear if one calculates $\overline{\Gamma}$ perturbatively and solves for $\overline{\phi}$ nonperturbatively). The anomalous solutions, if they exist, may be eliminated by imposing reasonable classical boundary conditions or by calculating $\tilde{\Gamma}[\overline{\phi}, C(\tilde{\phi})]$, varying $\overline{\phi}$ and then setting $\tilde{\phi} = \overline{\phi}$. The resulting non-gauge-invariant equation need not be solved, the resultant $J(\overline{\phi})$ may be evaluated to check that a solution to $\delta\overline{\Gamma}/\delta\overline{\phi} = 0$ is also a solution to $J(\overline{\phi}) = 0$.

IV. 'T HOOFT'S METHOD

In his work,[2] 't Hooft used a slightly different but essentially equivalent procedure. In this section, the two procedures will be compared.

Instead of requiring that the J be determined by the vanishing of the variation of $W[J, \overline{\phi}, C(\overline{\phi})]$ with respect to J holding $\overline{\phi}$ and $C(\overline{\phi})$ constant, 't Hooft regards J as an unknown functional of $\overline{\phi}$. The equation for J is then

$$\frac{\Delta\tilde{W}}{\Delta\phi^a} = -J_a[\overline{\phi}] = \frac{\delta\tilde{W}}{\delta\overline{\phi}^a}\Big|_{J,C} + \frac{\delta C^i_b}{\delta\overline{\phi}^a}\frac{\delta\tilde{W}}{\delta C^i{}_b}\Big|_{J,\overline{\phi}} + \frac{\delta J_b}{\delta\overline{\phi}^a}\frac{\delta W}{\delta J_b}\Big|_{\overline{\phi},C}. \tag{56}$$

Apparently, this is a straightforward equation for J; however, $\tilde{W}$ depends explicitly on J. If the general variation of $\tilde{W}$, Eq. (37), is used to evaluate $\Delta\tilde{W}/\Delta\overline{\phi}^a$, Eq. (56) reads

$$\begin{aligned}-J_a = -J_a + {}_aC^TC(\langle\phi\rangle - \overline{\phi}) + (\delta J_b/\delta\overline{\phi}^a)\{\langle\phi^b\rangle - \overline{\phi}^b\}\\ - (\langle\phi^b\rangle - \overline{\phi}^b)(\delta C^T{}_b{}^i/\delta\overline{\phi}^a)C^i{}_c(\langle\phi^c\rangle - \overline{\phi}^c)\\ + (\delta C^i{}_b/\delta\overline{\phi}^a)K^{bc}iJ_c\,,\end{aligned} \tag{57}$$

and the current appears explicitly on the right. The current is presumed to be determined uniquely by $\overline{\phi}$, hence the matrix $\delta J/\delta\overline{\phi}$ must be nonsingular, and, using the Lee-Slavnov identities,

$$\begin{aligned}\delta J_b/\delta\overline{\phi}^a(\langle\phi^b\rangle - \overline{\phi}^b) = i_a(C^T\mathcal{G}^T\Sigma^TJ)\\ - i(\langle\phi\rangle - \overline{\phi})(\delta C^T/\delta\overline{\phi}^a)\mathcal{G}^T\Sigma^TJ\\ - {}_a(LJ)\,.\end{aligned} \tag{58}$$

The right side vanishes for $J=0$ and the equation is then equivalent to simply varying J, but in general the equation is not equivalent to the $\delta\tilde{W}/\delta J=0$ equation. Eventually, the current J will be set equal to zero, hence it may be regarded as small and $\overline{\phi}$ must be chosen to be such that for $\langle\phi\rangle \sim \overline{\phi}$ J is small. Then, J will be a functional of $\overline{\phi}$ such that $\langle\phi\rangle$ will be equal to $\overline{\phi}$ with correction terms which vanish as J goes to zero. When $\tilde{W}$ is evaluated at the solution, J_H, to Eq. (58), it becomes the effective action

$$\Gamma_H[\overline{\phi}] = \tilde{W}[J(\overline{\phi}), \overline{\phi}, C(\overline{\phi})]. \tag{59}$$

Then,

$$0 = (\delta\Gamma_H[\overline{\phi}]/\delta\overline{\phi}) = -J\,, \tag{60}$$

which immediately implies $\langle\phi\rangle = \overline{\phi}$ and that $\overline{\phi}$ satisfies the same equation as before: the theories are the same. However, before J is set equal to zero, $\tilde{\Gamma}$ is a different functional of $\overline{\phi}$ than is Γ_H. The two effective actions are physically equivalent in that they produce the same solutions and the same matrix elements when evaluated at a solution to the source-free equation.

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[1]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); J. Honerkamp, Nucl. Phys. B48, 269 (1972); R. Jackiw, Phys. Rev. D 9, 1686 (1974); R. Kallosh, Nucl. Phys. B78, 293 (1974); I. Ya. Aref'eva, A. A. Slavnov, and L. D. Faddeev, Teor. Mat. Fiz. 21, 311 (1974) [Theor. Math. Phys. 21, 1165 (1974)]; M. T. Grisaru, P. Van Nieuwenhuizen, and C. C. Wu, Phys. Rev. D 12, 3203 (1975); R. Akhoury and W. I. Weisberger, *ibid.* 22, 959 (1980).

[2]G. 't Hooft in Acta Universitatis Wratislavensis No. 38 XIIth Winter School of Theoretical Physics in Karpacz, 1975, Vol. I, 1976 (unpublished); Functional and Probabilistic Methods in Quantum Field Theory, Vol. I, 1976 (unpublished).

[3]See, for example, P. Sikivie and N. Weiss, Phys. Rev. D 18, 3809 (1978); R. Jackiw and P. Rossi, *ibid.* 21, 426 (1980).

[4]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972); 8, 4654(E) (1973).

[5]B. S. DeWitt, in *Quantum Gravity II*, edited by C. Isham, R. Penrose, and D. Sciama (Oxford University Press, New York, to be published).

[6]L. D. Faddeev and V. W. Popov, Phys. Lett. 25B, 29 (1967).

[7]C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. 42, 127 (1975).

[8]A. A. Slavnov, Teor. Mat. Fiz. 10, 153 (1972) [Theor. Math. Phys. 10, 99 (1972)].