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Equivalence of nonrenormalizable theories to renormalizable theories in a composite ($Z=0$) limit

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A graphical parallelism between $(\bar{\psi}\psi)\phi^2$ and $(\bar{\psi}\chi)\phi$ interaction theories is investigated by means of skeleton diagrams. It is shown that the nonrenormalizable $(\bar{\psi}\psi)\phi^2$ theory can be made renormalizable, provided it contains a bound state expressed by the composite field $\chi=\psi\phi$. In order for the quartic interaction theory to be renormalizable it is necessary that the renormalization constant for the χ field vanishes in the corresponding Yukawa-type theory. A concrete example in which such a situation is realized is given by a new soluble model—a hybrid of the Lee model and the Ruijgrok–Van Hove model. The corresponding quartic interaction model is seen to be renormalizable due to the renormalization scheme presented here.

I. INTRODUCTION

In recent years several authors[1,2] have suggested that some types of nonrenormalizable theories, such as the Nambu–Jona-Lasinio theory,[3] can be made renormalizable, provided the theory contains bound states. The essential points common to those works are (1) they use the auxiliary field method and (2) if the four-fermion coupling constant G is expressed as $G=g^2/\delta\mu^2$ with a dimensionless constant g and a dimensional parameter $\delta\mu^2$, then g plays the role of Yukawa coupling constant in the Lagrangian for the auxiliary field.

Eguchi[1] observed the possible renormalizability of the Nambu–Jona-Lasinio theory by comparing it with the linear σ model. He compared formally the Lagrangians for the two cases, and concluded that the Nambu–Jona-Lasinio theory can be renormalizable if the renormalization constants for the composite field and for the four-meson vertex vanish.[4]

His formal comparison, however, suffers from obvious defects.

(1) The induced four-meson vertex coupling constant $\lambda^{(R)}$ has no corresponding parameter in the original Lagrangian; there exists no graphical parallelism at this point between the two theories.

(2) The comparison is ambigious due to the Fierz identity.[5]

Furthermore, it is an open question whether the vanishing of the renormalization constants can occur at all.

Tamvakis and Guralnik[2] developed a renormalization scheme of four-fermion theories by means of a mean-field expansion. They adopt a vacuum with a spontaneous breakdown so as to allow the massless fermion to possess a nonvanishing mass m proportional to the mean field. Using the path-integral technique they succeed in expressing the meson mass and the induced meson-meson coupling constants in terms of the fermion mass and the renormalized coupling constant $g^{(R)}$. Thus, their work is free from the defect (1) in Eguchi's paper. Instead, the renormalized Yukawa coupling constant $g^{(R)}$ depends on the cutoff parameter.

In this paper we investigate in detail the graphical parallelism between the Yukawa theory and a quartic interaction theory. To avoid the problem of the ambiguity due to the Fierz identity, we adopt a nonrenormalizable model with the Lagrangian density

$$\mathcal{L}=\bar{\psi}(i\partial\!\!\!/-m_1)\psi+\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-\mu^2\phi^2)-\frac{g^2}{\delta m}\bar{\psi}\psi\phi^2\,. \tag{1.1}$$

We arrive at a conclusion similar to that suggested by Eguchi: In order for the theory to be renormalizable it is necessary that the renormalization constant in the corresponding Yukawa theory vanish. This vanishing guarantees that the theory is free from an ultraviolet cutoff. It is our belief that our criterion of renormalizability is valid for more general cases.

To accomplish the renormalization we must assume the vanishing of the renormalization constant. It is, however, a hard task, since the problem should be treated in a nonperturbative way. Therefore, to obtain a vanishing renormalization constant, we present here a soluble toy model—a hybrid of the Lee model[6] and the Ruijgrok–Van Hove model.[7] In this toy model the renormalization constant for the composite field does vanish if the renormalized charges satisfy a specific relation, and the corresponding quartic interaction model turns out to be renormalizable due to our renormalization scheme.

Unfortunately, one of the elementary fields should be a ghost, and we need the indefinite metric.

II. ANALOGY IN THE SKELETON GRAPHS

The quartic interaction theory (the "Q theory") described by the Lagrangian density (1.1) is not renormalizable in accordance with simple power counting. Nevertheless, if we regard the product $\psi\phi$ as a composite field, the theory is graphically analogous to the renormalizable Yukawa-type theory (the "Y theory") with the Lagrangian

$$\begin{aligned}\mathcal{L}' &= \bar{\psi}(i\partial\!\!\!/ - m_1)\psi + \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2)\\ &= \bar{\chi}(i\partial\!\!\!/ - m_2)\chi + \bar{\psi}\phi\chi + \bar{\chi}\phi\psi\,. \end{aligned} \tag{2.1}$$

In Eqs. (1.1) and (2.1) we have suppressed the ϕ^4 term which is not essential in the following discussions though it is necessary for the theory to be renormalizable.

In order to see the graphical analogy between the two theories we compare their generating functionals. For the Q theory we have

$$\begin{aligned}W_1[J,\xi,\bar{\xi},\eta,\bar{\eta}] = \int \mathfrak{D}\psi\mathfrak{D}\bar{\psi}\mathfrak{D}\phi \exp\left(i\int \mathcal{L}\,d^4x\right) &\times\exp\bigg[i(J\phi)+i(\bar{\eta}\psi)+i(\bar{\psi}\eta)\\ &-i\left(\bar{\xi}\frac{1}{\delta m}\psi\phi\right)-i\left(\bar{\psi}\phi\frac{1}{\delta m}\xi\right)\\ &-i\left(\bar{\xi}\frac{1}{\delta m}\xi\right)\bigg],\end{aligned} \tag{2.2}$$

where we introduced the source terms for the products $\psi\phi$ and $\bar{\psi}\phi$ to investigate the dynamics of the collective motion of this type.[8] Here and afterward the parentheses, e.g., $(J\phi)$, stand for integration over space-time. They will also be used for integrations including a two-point function:

$$(A\Omega B) = \int d^4x\,d^4y\,A(x)\Omega(x,y)B(y)\,. \tag{2.3}$$

With the same notation we can write for the Y theory

$$\begin{aligned}W_2[J,\xi,\bar{\xi},\eta,\bar{\eta}] = \int \mathfrak{D}\psi\mathfrak{D}\bar{\psi}\mathfrak{D}\chi\mathfrak{D}\bar{\chi}\mathfrak{D}\phi \exp\left(i\int \mathcal{L}'d^4x\right) &\times\exp[i(J\phi)+i(\bar{\eta}\psi)+i(\bar{\psi}\eta)\\ &+i(\bar{\xi}\chi)+i(\bar{\chi}\xi)].\end{aligned} \tag{2.4}$$

We may also express the actions in the same way:

$$\begin{aligned}S &= \int \mathcal{L}\,d^4x\\ &= -(\bar{\psi}S_{1_F}{}^{-1}\psi) - \tfrac{1}{2}(\phi\Delta_F{}^{-1}\phi) - \left(\bar{\psi}\phi\frac{g^2}{\delta m}\psi\phi\right),\end{aligned} \tag{2.5}$$

$$\begin{aligned}S' &= \int \mathcal{L}'d^4x\\ &= -(\bar{\psi}S_{1_F}{}^{-1}\psi) - (\bar{\chi}S_{2_F}{}^{-1}\chi) - \tfrac{1}{2}(\phi\Delta_F{}^{-1}\phi)\\ &\quad + g(\bar{\psi}\phi\chi) + g(\bar{\chi}\phi\psi)\,,\end{aligned} \tag{2.6}$$

where S_{1_F}, S_{2_F}, and Δ_F are the free propagators:

$$S_{iF}(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,\frac{m_i + p\!\!\!/}{m_i{}^2 - p^2 - i\epsilon}\,e^{-ip(x-y)} \quad (i=1,2), \tag{2.7}$$

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{\mu^2 - k^2 - i\epsilon}\,e^{-ik(x-y)}\,. \tag{2.8}$$

To compare the two generating functionals we perform the path integration with respect to the χ field in W_2 to obtain

$$\begin{aligned}W_2 = \int \mathfrak{D}\psi\mathfrak{D}\bar{\psi}\mathfrak{D}\phi \exp[&-\tfrac{1}{2}i(\phi\Delta_F{}^{-1}\phi) - i(\bar{\psi}S_{1_F}{}^{-1}\psi)\\ &+ ig^2(\bar{\psi}\phi S_{2_F}\phi\psi)]\\ \times\exp[&i(J\phi)+i(\bar{\psi}\eta)+i(\bar{\eta}\psi)+ig(\bar{\xi}S_{2_F}\psi\phi)\\ &+ig(\bar{\psi}\phi S_{2_F}\xi)+i(\bar{\xi}S_2\ \ \xi)]\,.\end{aligned} \tag{2.9}$$

Obviously, the quantity $-1/\delta m$ in W_1 plays the same role as S_{2_F} in W_2.

For the Yukawa-type theory we know that the dressed inverse propagators can be expressed by means of proper self-energy parts

$$S_{1_F}'{}^{-1}(p) = m_1{}^{(R)} - p\!\!\!/ + \Sigma_1(p) - \delta m_1\,, \tag{2.10a}$$

$$S_{2_F}'{}^{-1}(p) = m_2{}^{(R)} - p\!\!\!/ + \Sigma_2(p) - \delta m_2\,, \tag{2.10b}$$

and

$$\Delta_F'{}^{-1}(k^2) = \mu^{(R)2} - k^2 + \Pi(k^2) - \delta\mu^2\,. \tag{2.10c}$$

The proper vertex part $\Gamma_{12}(p,p')$, as illustrated in Fig. 1, has the structure

$$\Gamma_{12}(p,p') = 1 + \sum_{S^i}\Lambda_{12}(S^i) \tag{2.10d}$$

$$= \Gamma_{21}(p',p)\,, \tag{2.10e}$$

where $\Lambda_{12}(S^i)$ ia a functional of dressed propagators and vertices inserted into a skelton diagram S^i. If S^i has $2n+1$ vertices, $\Lambda_{12}(S^i)$ is proportional to $(g^2)^n$ and contains n S_{1_F}' functions, n S_{2_F}' functions, n Δ_F' functions, and $2n+1$ Γ_{12} functions.

The resemblance in the structure of the two generating functionals W_1 and W_2 suggests that we write

$$S_{BF}'{}^{-1}(p) = \Sigma_B(p) - \delta m \tag{2.11a}$$

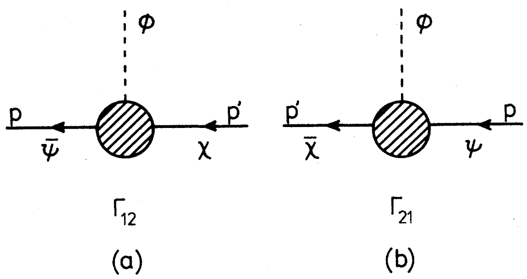

FIG. 1. The vertex parts (a) Γ_{12} and (b) Γ_{21}.

for the propagator of the composite field in the nonrenormalizable Q theory. For the other functions in that theory we write

$$\bar{S}'^{-1}_{1_F}(p) = m_1^{(R)} - \not{p} - \bar{\Sigma}_1(p) - \delta m_1\,, \tag{2.11b}$$

$$\Delta'^{-1}_F(k^2) = \mu^{(R)^2} - k^2 + \Pi(k^2) - \delta\mu^2, \tag{2.11c}$$

and

$$\bar{\Gamma}_{1B}(p, p') = 1 + \sum_{S^i} \bar{\Lambda}_{1B}(S^i) \tag{2.11d}$$

$$= \bar{\Gamma}_{B1}(p', p)\,, \tag{2.11e}$$

where $\bar{\Gamma}_{1B}$ and $\bar{\Gamma}_{B1}$ stand for the vertices illustrated in Fig. 2.

We note that the functions $\bar{\Sigma}_1$, Σ_B, $\bar{\Pi}$, and $\bar{\Gamma}_{1B}$ should have the same structure as Σ_1, Σ_2, Π_1 and Γ_{12}, respectively, with respect to the dressed propagators and vertices.

III. RENORMALIZATION AND THE NECESSITY OF THE VANISHING OF THE RENORMALIZATION CONSTANT

It is well known that any Yukawa-type theory has the renormalization scheme by means of the Ward-type identities. In the case of our Y theory we can write

$$\Gamma_1^\mu \equiv \frac{\partial}{\partial p_\mu}(S'^{-1}_{1_F}) = -\gamma^\mu + \Lambda_1^\mu\,, \tag{3.1a}$$

$$\Gamma_2^\mu \equiv \frac{\partial}{\partial p_\mu}(S'^{-1}_{2_F}) = -\gamma^\mu + \Lambda_2^\mu\,, \tag{3.1b}$$

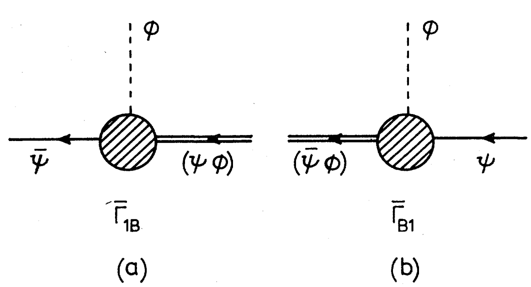

FIG. 2. The vertex parts (a) $\bar{\Gamma}_{1B}$ and (b) $\bar{\Gamma}_{B1}$.

and

$$W^\mu \equiv \frac{\partial}{\partial k_\mu}(\Delta'^{-1}_F) = -2k^\mu + \Delta^\mu\,, \tag{3.1c}$$

where

$$\Lambda_1^\mu \equiv \frac{\partial}{\partial p_\mu}\Sigma_1\,, \tag{3.2a}$$

$$\Lambda_2^\mu \equiv \frac{\partial}{\partial p_\mu}\Sigma_2\,, \tag{3.2b}$$

and

$$\Delta^\mu \equiv \frac{\partial}{\partial k_\mu}\Pi\,. \tag{3.2c}$$

With the help of the identity

$$\frac{\partial}{\partial p_\mu}S_F(p) = S_F(p)\gamma^\mu S_F(p) \tag{3.3}$$

the auxiliary vertex parts Λ_1^μ, Λ_2^μ, and Δ^μ can be expanded into the skeltons

$$\Lambda_1^\mu = \sum_{S^i}\Lambda_1^\mu(S^i)\,, \tag{3.4a}$$

$$\Lambda_2^\mu = \sum_{S^i}\Lambda_2^\mu(S^i)\,, \tag{3.4b}$$

and

$$\Delta^\mu = \sum_{S^i}\Delta^\mu(S^i)\,. \tag{3.4c}$$

For a skeleton diagram S^i with $4n+3$ vertices, $\Lambda_1^\mu(S^i)$ contains a factor g^{4n+2}, $2n$ S'_{1_F} functions, $2n+2$ S'_{2_F} functions, $(4n+2)$ functions, $2n+1$ Δ'_F functions, and one Γ_2^μ function. If the number of vertices is $4n+1$, $\Lambda_1^\mu(S^i)$ contains a factor g^{4n}, $2n$ S'_{1_F} functions, $2n$ S'_{2_F} functions, $4n$ Γ_{12} functions, $2n$ Δ'_F functions, and one Γ_1^μ function. It is easy to give similar statements about Λ_2^μ and Δ^μ.

For the Q theory we write, corresponding to Eqs. (3.1), (3.2), and (3.4),

$$\bar{\Gamma}_1^\mu \equiv \frac{\partial}{\partial p_\mu}(\bar{S}'^{-1}_{1_F}) = -\gamma^\mu + \bar{\Lambda}_1^\mu\,, \tag{3.5a}$$

$$\Gamma_B^\mu \equiv \frac{\partial}{\partial p_\mu}(S'^{-1}_{BF}) = \Lambda_B^\mu\,, \tag{3.5b}$$

and

$$\bar{W}^\mu \equiv \frac{\partial}{\partial k_\mu}(\Delta'^{-1}_F) = -2k^\mu + \bar{\Delta}^\mu\,, \tag{3.5c}$$

where

$$\bar{\Lambda}_1^\mu \equiv \frac{\partial}{\partial p_\mu}\bar{\Sigma}_1 = \sum_{S^i}\bar{\Lambda}_1^\mu(S^i)\,, \tag{3.6a}$$

$$\Lambda_B^\mu \equiv \frac{\partial}{\partial p_\mu}\Sigma_B = \sum_{S^i}\Lambda_B^\mu(S^i)\,, \tag{3.6b}$$

and

$$\overline{\Delta}^{\mu} \equiv \frac{\partial}{\partial k_{\mu}} \Pi = \sum_{S^{i}} \overline{\Delta}^{\mu}(S^{i}) . \tag{3.6c}$$

From the graphical consideration, $\overline{\Lambda}_1^{\mu}$, Λ_B^{μ}, and Δ^4 have the same structure as Λ_1^{μ}, Λ_2^{μ}, and Δ^{μ}, respectively, regarded as functionals of the dressed propagators and vertices.

Next we introduce the multiplicative renormalizations. For the Y theory we set

$$S_{1_F} = Z_1 S_{1_F}^{(R)} , \tag{3.7a}$$

$$S_{2_F} = Z_2 S_{2_F}^{(R)} , \tag{3.7b}$$

$$\Delta_F = Z_{\phi} \Delta_F^{(R)} , \tag{3.7c}$$

$$\Gamma_{12} = Z_{12}^{-1} \Gamma_{12}^{(R)} , \tag{3.7d}$$

and

$$g^2 \frac{Z_1 Z_2 Z_{\phi}}{Z_{12}^{\;2}} = g^{(R)2} , \tag{3.7e}$$

and

$$\Gamma_1^{\mu} = Z_1^{-1} \Gamma_1^{(R)\mu} , \tag{3.8a}$$

$$\Gamma_2^{\mu} = Z_2^{-1} \Gamma_2^{(R)\mu} , \tag{3.8b}$$

and

$$W^{\mu} = Z_{\phi}^{-1} W^{(R)\mu} . \tag{3.8c}$$

Equations (3.1) and (2.10d) are now rewritten as

$$\Gamma_1^{(R)\mu} = - Z_1 \gamma^{\mu} + \Lambda_1^{\mu}(g^{(R)}; S_{1_F}^{(R)}, S_2^{(R)}, \Delta_F^{(R)}, \Gamma_{12}^{(R)}, \Gamma_1^{(R)\mu}, \Gamma_2^{(R)\mu}) , \tag{3.9a}$$

$$\Gamma_2^{(R)\mu} = - Z_2 \gamma^{\mu} + \Lambda_2^{\mu}(g^{(R)}; S_{1_F}^{(R)}, S_{2_F}^{(R)}, \Delta_F^{(R)}, \Gamma_{12}^{(R)}, \Gamma_1^{(R)\mu}, \Gamma_2^{(R)\mu}), \tag{3.9b}$$

$$W^{(R)\mu} = - 2 Z_{\phi} k^{\mu} + \Delta^{\mu}(g^{(R)}; S_{1_F}^{(R)}, S_{2_F}^{(R)}, \Delta_F^{(R)}, \Gamma_{12}^{(R)}, \Gamma_1^{(R)\mu}, \Gamma_2^{(R)\mu}) , \tag{3.9c}$$

and

$$\Gamma_{12}^{(R)} = Z_{12} + \Lambda_{12}(g^{(R)}; S_{1_F}^{(R)}, S_{2_F}^{(R)}, \Delta_F^{(R)}, \Gamma_{12}^{(R)}) . \tag{3.9d}$$

These equations, the subtractions on the mass shell, and the differential equations

$$\frac{\partial}{\partial p_{\mu}} S_{1_F}^{(R)-1}(p) = \Gamma_1^{(R)\mu}(p) , \tag{3.10a}$$

$$\frac{\partial}{\partial p_{\mu}} S_{2_F}^{(R)-1}(p) = \Gamma_2^{(R)\mu}(p) , \tag{3.10b}$$

$$\frac{\partial}{\partial k_{\mu}} \Delta_F^{(R)-1}(k) = W^{(R)\mu}(k) \tag{3.10c}$$

completely determine the divergent constants Z_1, Z_2, Z_{ϕ}, and Z_{12} and all the renormalized propagators and vertex functions as finite quantities.

We now try to follow the same procedure in the Q theory. We set

$$\overline{S}_{1_F} = \mathfrak{z}_1 \overline{S}_{1_F}^{(R)} , \tag{3.11a}$$

$$S_{BF} = \mathfrak{z}_B S_{BF}^{(R)} , \tag{3.11b}$$

$$\overline{\Delta}_F = \mathfrak{z}_{\phi} \overline{\Delta}_F^{(R)} , \tag{3.11c}$$

$$\overline{\Gamma}_{1B} = \mathfrak{z}_{1B}^{-1} \overline{\Gamma}_{1B}^{(R)} , \tag{3.11d}$$

and

$$g^2 \frac{\mathfrak{z}_1 \mathfrak{z}_B \mathfrak{z}_{\phi}}{\mathfrak{z}_{1B}^{\;2}} = g^{(R)2} . \tag{3.11e}$$

Corresponding to Eqs. (3.9) we have

$$\overline{\Gamma}_1^{(R)\mu} = - \mathfrak{z}_1 \gamma^{\mu} + \overline{\Lambda}_1^{\mu}(g^{(R)}; \overline{S}_{1F}^{(R)}, S_{BF}^{(R)}, \overline{\Delta}_F^{(R)}, \overline{\Gamma}_{1B}^{(R)}, \Gamma_1^{(R)\mu}, \Gamma_B^{(R)\mu}) , \tag{3.12a}$$

$$\Gamma_B^{(R)\mu} = \Lambda_B^{\mu}(g^{(R)}; \overline{S}_{1F}^{(R)}, S_{BF}^{(R)}, \overline{\Delta}_F^{(R)}, \overline{\Gamma}_{1B}^{(R)}, \overline{\Gamma}_1^{(R)\mu}, \Gamma_B^{(R)\mu}) \equiv \Lambda_B^{(R)\mu}(\not{p}, g^{(R)}) , \tag{3.12b}$$

$$\overline{W}^{(R)\mu} = - 2 \mathfrak{z}_{\phi} k^{\mu} + \overline{\Delta}^{\mu}(g^{(R)}; \overline{S}_{1F}^{(R)}, S_{BF}^{(R)}, \overline{\Delta}_F^{(R)}, \overline{\Gamma}_{1B}^{(R)}, \overline{\Gamma}_1^{(R)\mu}, \Gamma_B^{(R)\mu}) , \tag{3.12c}$$

and

$$\overline{\Gamma}_{1B}^{(R)} = \mathfrak{z}_{1B} + \overline{\Lambda}_{1B}(g^{(R)}; \overline{S}_{1F}^{(R)}, S_{BF}^{(R)}, \overline{\Delta}_F^{(R)}, \overline{\Gamma}_{1B}^{(R)}) . \tag{3.12d}$$

We also have the differential equations corresponding to Eqs. (3.10).

Owing to the graphical parallelism, the only discrepancy between the two theories consists in the term $- \mathfrak{z}_B \gamma^{\mu}$ in Eq. (3.12b). Therefore, Eq. (3.12b) does not determine the constant $\mathfrak{z}_B$ as Eq. (3.9b) does for Z_2. Instead, the requirement of the on-mass-shell condition for (3.12b) gives a relation between the renormalized coupling constant $g^{(R)}$ and the bound-state mass m_B. Since the perturbative expansion of $\Lambda_B^{(R)\mu}$ contains ultraviolet divergences in each order of $g^{(R)}$, this relation between $g^{(R)}$ and m_B depends on the cutoff parameter Λ, giving rise to the nonrenormalizability of the Q theory.

If, however, the condition

$$\lim_{\Lambda \to \infty} \Lambda_B^{(R)\mu}(m_B, g^{(R)}) = - \gamma^{\mu} \tag{3.13}$$

happens to be fulfilled for some finite values of

m_B and $g^{(R)}$, the renormalization procedure turns out to be valid as in the case of the Y theory.

Comparison of (3.9b) and (3.12b) shows that the condition (3.13) is equivalent to the condition

$$\lim_{\Lambda\to\infty} Z_2 = 0 \tag{3.14}$$

in the Y theory. Namely, our Q theory is renormalizable, if (3.14) can be fulfilled for physically admissible values of the renormalized masses and charges.

IV. GENERALIZED SOLUBLE MODEL

In this and Sec. V and VI we attempt to construct a soluble toy model in which the vanishing of the renormalization constant is realized for one of the participating fields.

There exists a class of well-known soluble Yukawa-type models: the neutral scalar model,[9] the Lee model,[6] and the Ruijgrok–Van Hove (RVH) model.[7] These models are soluble in the sense that one can explicitly construct the one-particle states.

Among these models the vanishing of the renormalization constant is realized in the neutral scalar model and the RVH model. However, they are not adequate for our purpose: In the neutral scalar model the Z vanishes for the nucleon which is the only fermion in the theory; in the RVH model the vanishing of Z occurs simultaneously for all the participating fermions. In both cases there is no room for the elementary fermions. Therefore, we seek another soluble model which is compatible with our renormalization scheme.

A. A generalized soluble model

We consider a system of n different spinless fermions (nucleons) interacting with a neutral scalar particle (meson). We assume that the nucleons are heavy, and neglect the momentum dependence of their energy. The Hamiltonian is given by

$$H = \psi^\dagger M_0 \psi + (a^\dagger \Omega a) - \psi^\dagger [G^\dagger (af) + G(a^\dagger f)]\psi , \tag{4.1}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix} , \tag{4.2}$$

$$M_0 = \begin{pmatrix} m_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & m_n \end{pmatrix} , \tag{4.3}$$

$$(a^\dagger \Omega a) = \int d^3\vec{k}\, a^\dagger(\vec{k})\omega(\vec{k})a(\vec{k}) ,$$
$$\omega(\vec{k}) = (\mu^2 + \vec{k}^2)^{1/2} , \tag{4.4}$$

and

$$(af) = \int d^3\vec{k}\, a(\vec{k}) f(\vec{k}) ,$$
$$f(\vec{k}) = \frac{1}{(2\pi)^{3/2}[2\omega(\vec{k})]^{1/2}} \tag{4.5}$$

The nucleon fields ψ_μ and $\psi_\mu^\dagger$ ($\mu = 1, 2, \ldots, n$) and the meson field operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ satisfy the conventional commutation rules of the annihilation and creation operators.

This model, characterized by the arbitrary coupling matrix G, contains all the soluble models mentioned above as special cases.

(i) The neutral scalar model:

$$n = 1, \quad G = g . \tag{4.6}$$

(ii) The Lee model:

$$n = 2, \quad G = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} . \tag{4.7}$$

(iii) The RVH model of order n:

$$G = \begin{pmatrix} 0 & g_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & g_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & 0 \\ 0 & \cdot & & & & 0 & g_n \\ g_1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix} . \tag{4.8}$$

B. The solutions in the one-nucleon sector

Since we are interested only in the bound states with the nucleon number $N = 1$, we confine ourselves to the one-nucleon sector. It is then convenient to write the Hamiltonian (4.1) as a matrix in the nucleon charge space:

$$H = M_0 + (a^\dagger \Omega a) - G^\dagger (af) - G(a^\dagger f) . \tag{4.9}$$

We define a new meson operator $\alpha(\vec{k})$ which is a matrix in the nucleon space

$$\alpha(\vec{k}) \equiv a(\vec{k}) - Gf(\vec{k})/\omega(\vec{k}) \tag{4.10}$$

$$= e^{G(a^\dagger \Omega^{-1} f)} a(\vec{k}) e^{-G(a^\dagger \Omega^{-1} f)} , \tag{4.11}$$

and rewrite H into a much simpler form:

$$H = M + (\alpha^\dagger \Omega \alpha) , \tag{4.12}$$

where

$$M \equiv M_0 - \Delta G^\dagger G \tag{4.13}$$

and

$$\Delta = (f\Omega^{-1} f) = \frac{1}{(2\pi)^3} \int d^3\vec{k} \frac{1}{2\omega(\vec{k})^2} \tag{4.14}$$

is a divergent integral.

In order to make the model soluble, it is necessary to restrict the renormalized mass matrix M to be a multiple of the unit matrix[10]:

$$M = mI \tag{4.15}$$

If the bare nucleon masses are so chosen to guarantee (4.15), we only have to solve the eigenvalue problem of the second term of the Hamiltonian (4.12).

Since the operator $(\alpha^\dagger \Omega \alpha)$ is a non-negative definite, we can easily see that the states $|\tilde{\psi}\rangle_\mu$ $(\mu = 1, 2, \ldots, n)$ defined by the condition

$$\alpha(\vec{k}) |\tilde{\psi}\rangle_\mu = 0 \quad (\mu = 1, 2, \ldots, n) \tag{4.16}$$

are the degenerate ground states of (4.12) representing the physical one-nucleon states with mass m.

In view of Eq. (4.11) the general solutions of (4.16) are found to be

$$|\tilde{\psi}\rangle_\mu = e^{G(a^\dagger \Omega^{-1} f)} |\psi\rangle_\nu U^\nu_\mu \tag{4.17}$$

where $|\psi\rangle_\nu$'s are the bare nucleon states satisfying

$$a(\vec{k}) |\psi\rangle_\nu = 0 \quad (\text{for all } \vec{k}) . \tag{4.18}$$

C. The wave-function renormalization

The matrix U in (4.17) should be determined so that the $|\tilde{\psi}\rangle_\mu$'s form an orthonormal basis:

$$^\mu\langle \tilde{\psi} | \tilde{\psi}\rangle_\nu = \delta^\mu_\nu . \tag{4.19}$$

(The raising of the index here implies adoption of a general metric. In the next section we shall need an indefinite one.)

If we define the "renormalization matrix" Z by

$$(Z^{-1})^\mu_\nu = {}^\mu\langle \psi | e^{G^\dagger (a\Omega^{-1} f)} e^{G(a^\dagger \Omega^{-1} f)} |\psi\rangle_\nu , \tag{4.20}$$

the orthonormal condition (4.19) reads

$$U^\dagger Z^{-1} U = I \tag{4.21}$$

or

$$Z = UU^\dagger . \tag{4.22}$$

The matrix U can be chosen to be Hermitian: $U^\dagger = U$.

The right-hand side of (4.20) can be calculated to be

$$(Z^{-1})^\mu_\nu = \sum_{n=0}^{\infty} \frac{L^n}{n!} (G^{\dagger n} G^n)^\mu_\nu , \tag{4.23}$$

where L is a divergent integral:

$$L = (f\Omega^{-2} f) = \frac{1}{(2\pi)^3} \int d^3\vec{k} \frac{1}{2\omega(\vec{k})^3} . \tag{4.24}$$

Let λ_i $(i = 1, 2, \ldots, n)$ be the eigenvalues of G. For simplicity we assume that they are nondegenerate. Then we have the spectral decomposition of G:

$$G = \sum_{i=1}^{n} \lambda_i \Lambda_i , \tag{4.25}$$

where

$$\Lambda_i = \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{\lambda_j - G}{\lambda_j - \lambda_i} \tag{4.26}$$

is the projection to the eigensubspace belonging to λ_i.[11]

The matrix Z^{-1} can be expressed in terms of the Λ_i's as

$$Z^{-1} = \sum_{ij} e^{L \bar{\lambda}_i \lambda_j} \Lambda_i^\dagger \Lambda_j . \tag{4.27}$$

D. The charge renormalization

The renormalized coupling matrix $G^{(R)}$, defined by

$$G^{(R)\mu}_\nu = {}^\mu\langle \tilde{\psi} | G | \tilde{\psi}\rangle_\nu , \tag{4.28}$$

can be easily calculated to be

$$G^{(R)} = U^{-1} G U , \tag{4.29}$$

owing to Eqs. (4.17), (4.20), and (4.22). Using Eqs. (4.27) and (4.29) we obtain the expression for Z in terms of the renormalized coupling constants (see Appendix):

$$Z = \sum_{ij} e^{-L \bar{\lambda}_i \lambda_j} \Lambda_i^{(R)\dagger} \Lambda_j^{(R)} , \tag{4.30}$$

where $\Lambda_i^{(R)}$'s are the eigenprojection of $G^{(R)}$:

$$\Lambda_i^{(R)} = U^{-1} \Lambda_i U . \tag{4.31}$$

V. A SPECIAL SOLUBLE MODEL AND RELATED QUARTIC MODEL

In this section we are to specify the coupling matrix G so that the generalized soluble model discussed in the previous section may have the desired properties. Our requirements are the following:

(1) The matrix $G^\dagger G$ should be diagonal, in order that the condition (4.16) may be satisfied.

(2) We regard one of the nucleon fields, say χ, as corresponding to the composite field $\psi\phi$ in the theory with a quartic interaction $(\psi^\dagger\psi)\phi^2$. According to the postulate of the graphical parallelism the interaction of χ should be of the form $(\psi^\dagger\chi)\phi$. This means that the $\chi^\dagger - \chi$ diagonal matrix element of G should be zero.

(3) The renormalization constant Z for χ should tend to zero as $L\to\infty$, while the other Z's remain finite.

If $n=2$, we cannot find any model satisfying these requirements. Among many possibilities in the case of $n=3$, a choice of special interest is obtained by choosing G as

$$(G_\mu^\nu)=\begin{pmatrix}0 & 0 & g_3\\ 0 & 0 & g_2\\ 0 & g_1 & 0\end{pmatrix}\quad (g_1g_2\equiv g^2>0)\,. \tag{5.1}$$

It is now straightforward to calculate the renormalization matrix Z. It will be seen then that Z is diagonal, and that Z_2 becomes negative if we take the limit $L\to\infty$ with the renormalized charges held fixed. To remedy this defect we introduce in advance an indefinite-metric tensor

$$(\eta^{\mu\nu})=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{5.2}$$

Accordingly the commutation rules of the nucleon operators are modified to

$$\{\psi_\mu,\psi_\nu^\dagger\}=\eta_{\mu\nu}\,. \tag{5.3}$$

In the one-nucleon sector we replace the Hermitian conjugation by the adjoint operation

$$\bar{A}=\eta A^\dagger\eta\,. \tag{5.4}$$

For example, we have

$$(\bar{G}_\mu^\nu)=((\eta G^\dagger\eta)_\mu^\nu)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -g_1\\ g_3 & -g_2 & 0\end{pmatrix}. \tag{5.5}$$

The eigenvalues λ_i of G and the corresponding projections Λ_i are given by

$$\lambda_1=0,\quad \lambda_2=g,\quad \lambda_3=-g\quad (g=\sqrt{g_1g_2})\,, \tag{5.6}$$

$$\begin{aligned}\Lambda_1&=(G-g)(G+g)/(-g^2)\,,\\ \Lambda_2&=G(G+g)/2g^2\,,\\ \Lambda_3&=G(G-g)/2g^2\,.\end{aligned} \tag{5.7}$$

The Z^{-1} matrix calculated by means of (4.27) turns out to be diagonal:

$$((Z^{-1})_\mu^\nu)=\begin{pmatrix}Z_1^{-1} & 0 & 0\\ 0 & Z_2^{-1} & 0\\ 0 & 0 & Z_3^{-1}\end{pmatrix} \tag{5.8}$$

with the eigenvalues

$$\begin{aligned}Z_1^{-1}&=1\,,\\ Z_2^{-1}&=\alpha^2+(1-\alpha^2)\cosh g^2L-\beta^2\sinh g^2L\,,\\ Z_3^{-1}&=\cosh g^2L+(\gamma^2-1/\beta^2)\sinh g^2L\,,\end{aligned} \tag{5.9}$$

where

$$\alpha=\frac{g_3}{g_2},\quad \beta=\left(\frac{g_1}{g_2}\right)^{1/2},\quad \gamma=\frac{g_3}{(g_1g_2)^{1/2}}\,. \tag{5.10}$$

The renormalization coupling constants given by (4.29) are

$$\begin{aligned}g_1^{(R)}&=\left(\frac{Z_2}{Z_3}\right)^{1/2}g_1\,,\\ g_2^{(R)}&=\left(\frac{Z_3}{Z_2}\right)^{1/2}g_2\,,\end{aligned} \tag{5.11}$$

and

$$g_3^{(R)}=\left(\frac{Z_3}{Z_1}\right)^{1/2}g_3\,.$$

Expressed in terms of these $g_i^{(R)}$'s the renormalization constants have the form

$$\begin{aligned}Z_1&=1\,,\\ Z_2&=\alpha^{(R)2}+(1-\alpha^{(R)2})\cosh g^2L\\ &\quad+\beta^{(R)2}\sinh g^2L\,,\end{aligned} \tag{5.12}$$

and

$$Z_3=\cosh g^2L+\left(\frac{1}{\beta^{(R)2}}-\gamma^{(R)2}\right)\sinh g^2L\,,$$

where $\alpha^{(R)}$, $\beta^{(R)}$, and $\gamma^{(R)}$ are defined by (5.10) with g_i's replaced by the $g_i^{(R)}$'s.

We now search for the desired situation in which one (and only one) of the Z's tends to zero as $L\to\infty$. This is realized by imposing the following restriction on the $g_i^{(R)}$'s:

$$\frac{1}{\beta^{(R)2}}-\gamma^{(R)2}=-1 \tag{5.13}$$

or

$$g_3^{(R)}=g_2^{(R)}(g_1^{(R)}+g_2^{(R)})\,. \tag{5.14}$$

Under this restriction we have

$$Z_2=1+\beta^{(R)2}-\beta^{(R)2}e^{-g^2L} \tag{5.15}$$

and

$$Z_3=e^{-g^2L} \tag{5.16}$$

so that

$$Z_1=1\,,\quad Z_2\to 1+\beta^{(R)2}\,,\quad Z_3\to 0 \tag{5.17}$$

as $L\to\infty$.

To conclude this section we construct a quartic interaction model in which the nucleon ψ_3 is produced as a bound state of ψ_1, ψ_2, and the scalar meson. For this purpose we write the Lagrangian

$$\begin{aligned} L&=L_0+L_I\,,\\ L_0&=-\bar\psi(t)\left(M_0-i\frac{d}{dt}\right)\psi(t)\\ &\quad-\left(a^\dagger(t)\left(\Omega-i\frac{d}{dt}\right)a(t)\right), \\ L_I&=\bar\psi(t)\bar G\psi(t)(a(t)f)+\bar\psi(t)G\psi(t)(a^\dagger(t)f)\end{aligned} \tag{5.18}$$

corresponding to the Hamiltonian (4.1) (but with the indefinite metric). This Lagrangian is subject to the graphical parallelism mentioned in Sec. II with the Lagrangian

$$\begin{aligned} L'&=L_0'+L_I'\,,\\ L_0'&=-\psi_1^\dagger(t)\left(m_1-i\frac{d}{dt}\right)\psi_1(t)\\ &\quad+\psi_2^\dagger(t)\left(m_2-i\frac{d}{dt}\right)\psi_2(t)\\ &\quad-\left(a^\dagger(t)\left(\Omega-i\frac{d}{dt}\right)a(t)\right), \\ L_I'&=[g_1\psi_2^\dagger(af)+(-g_2\psi_2^\dagger+g_3\psi_1^\dagger)(a^\dagger f)]\frac{-1}{\delta m}\\ &\quad\times[g_1\psi_2(a^\dagger f)+(-g_2\psi_2+g_3\psi_1)(af)]\,.\end{aligned} \tag{5.19}$$

The theory described by the Lagrangian L' is nonrenormalizable in the sense of power counting. However, if the theory contains a bound state which is expressed by the composite field

$$\chi=g_1\psi_2(a^\dagger f)+(-g_2\psi_2+g_3\psi_1)(af)\,, \tag{5.20}$$

the renormalizability is guaranteed according to the scheme developed in Sec. II and III.

VI. DISCUSSION

We have shown that the $(\bar\psi\psi)\phi^2$ theory is renormalizable provided it contains a bound state expressed by the composite field $\chi=\psi\phi$. The existence of the bound state means that the self-energy part of the composite particle is finite owing to mutual cancellation of the divergences appearing in the perturbative expansion. The graphical parallelism then shows that, in the corresponding Yukawa-type theory, the renormalization constant for the "composite" field vanishes, as was originally suggested by Eguchi.

Although we worked in this paper with the $(\bar\psi\psi)\phi^2$ theory, to avoid the troubles mentioned in Sec. I, it can be expected that our scheme is applicable to more general cases.

In Secs. IV and V we showed that in some type of renormalizable models the wave-function renormalization constant can vanish for one field while those for other fields remain finite in the large-cutoff limit.

An undesirable defect of our model is that it contains a ghost field. In the framework of our generalized soluble model we could not avoid the ghost for various choices of the coupling matrix G. Although it is quite difficult to give a general proof, the authors are inclined to believe that the existence of a ghost field is a general feature of the four-dimensional theories with vanishing renormalization constant. (If the space-time is two-dimensional there exist some possibilities of realizing a ghost-free theory.[12])

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APPENDIX

In this appendix we prove the equality (4.30). Since the similarity transformation (4.29),

$$G^{(R)}=U^{-1}GU\,, \tag{A1}$$

does not change the eigenvalues of the matrix, the eigenprojections for $G^{(R)}$ are given by (4.31):

$$\Lambda_i^{(R)}=U^{-1}\Lambda_iU\,. \tag{A2}$$

We choose U to be Hermitian and use (A2) to write (4.27):

$$Z^{-1}=\sum_{ij}e^{L\bar\lambda_i\lambda_j}U^{-1}\Lambda_i^{(R)\dagger}UU\Lambda_j^{(R)}U^{-1}\,, \tag{A3}$$

or, since $Z=U^2$,

$$1=\sum_{ij}e^{L\bar\lambda_i\lambda_j}\Lambda_i^{(R)\dagger}Z\Lambda_j^{(R)}. \tag{A4}$$

Multiplying (A4) by $\Lambda_i^{(R)\dagger}$ on the left and by $\Lambda_j^{(R)}$ on the right we get

$$\Lambda_i^{(R)\dagger}\Lambda_j^{(R)}=e^{L\bar\lambda_i\lambda_j}\Lambda_i^{(R)\dagger}Z\Lambda_j^{(R)} \tag{A5}$$

or

$$\Lambda_i^{(R)\dagger}Z\Lambda_j^{(R)}=e^{-L\bar\lambda_i\lambda_j}\Lambda_i^{(R)\dagger}\Lambda_j^{(R)}. \tag{A6}$$

On summing this equality with respect to i and j we obtain the result

$$Z=\sum_{ij}e^{-L\bar\lambda_i\lambda_j}\Lambda_i^{(R)\dagger}\Lambda_j^{(R)}. \tag{A7}$$

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[10]If G is triangular as in the Lee model, the model is soluble without this restriction.

[11]According to the conventional mathematical terminology the Λ_i's are not projections because they are not Hermitian. But they satisfy $\Lambda_i\Lambda_j=\delta_{ij}\Lambda_j$ and $\Sigma_{i=1}^{n}\Lambda_i=I$.

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