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The propagator of stochastic electrodynamics

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The "elementary propagator" for the position of a free charged particle subject to the zero-point electromagnetic field with Lorentz-invariant spectral density $\propto \omega^3$ is obtained. The nonstationary process for the position is solved by the stationary process for the acceleration. The dispersion of the position elementary propagator is compared with that of quantum electrodynamics. Finally, the evolution of the probability density is obtained starting from an initial distribution confined in a small volume and with a Gaussian distribution in the velocities. The resulting probability density for the position turns out to be equal, to within radiative corrections, to $\psi\psi^*$ where ψ is the Kennard wave packet. If the radiative corrections are retained, the present result is new since the corresponding expression in quantum electrodynamics has not yet been found. Besides preceding quantum electrodynamics for this problem, no renormalization is required in stochastic electrodynamics.

I. INTRODUCTION

From a purely classical point of view a background radiation field mainly due to distant matter must exist everywhere in the universe since

(i) electrons in atoms have an accelerated motion and therefore they must radiate,

(ii) the radiation field at a given point is the vector sum of the fields radiated by all the atoms of the universe because of the classical superposition principle,

(iii) the radiation field decreases with distance r as r^{-1} while the number of atoms contained in spherical shells concentric to the point considered and of constant thickness Δr increases as r^2 in a homogeneous universe.

Consequently, it is not possible to consider an "isolated atom" either in the intergalactic space (the background field is practically independent of the near matter) or in a superconducting cavity. Indeed the radiation does not bounce back on the cavity walls. Because of the superposition principle, the radiation emitted by the atoms of the walls superposes to the radiation entering the walls and produces changes and correlations in the random phases θ_s so that the oscillating dipoles in the walls undergo a force, the so-called Casimir effect (see the review paper by Boyer[1]). However, either inside or outside the cavity the electric field can be represented by

$$\vec{E}(\vec{x},t)=\int d^3k\vec{E}_0(k,t,\theta_s)\exp[(i\omega t-\vec{k}\cdot\vec{x})], \tag{1}$$

where the complex amplitude $\vec{E}_0(k,t,\theta_s)$ is such that

$$\langle\vec{E}_0(k,t,\theta_s)\rangle=0$$

but (2)

$$\langle E_0{}^2(k,t,\theta_s)\rangle\neq 0.$$

The mean value of the fluctuations is not changed by the screening (as is also implied by calculation of the Casimir effect).

From the requirement that the "background radiation" be the same for all inertial observers, it turns out[1] that its energy per normal mode must be proportional to the angular frequency ω, and, hence, its spectral energy density $G_E(\omega)$ must vary as ω^3. This leads to the identification of the background radiation with the so-called "quantum electromagnetic vacuum" or "zero-point electromagnetic (em) field", whose power spectrum is given by

$$G_E(\omega)=\hbar\omega^3/2\pi^2c^3, \tag{3}$$

where $\hbar$ is the reduced Planck's constant and c is the light speed. Recently, Surdin[2] has given a hint to obtain the constant of proportionality between $G_E(\omega)$ and ω^3 in terms of the average density of the universe and of the Hubble radius so that $\hbar$ would be a derived quantity. A more plausible justification of the Surdin approach will be given in a future paper. In this paper the proportionality constant $\hbar(2\pi^2c^3)^{-1}$ is taken as an initial assumption.

As a consequence of the zero-point em field a charged particle[3] is subjected to a rapid random action leading to a stochastic motion which could explain the stability of the atoms. In other words, an atom, on average, should radiate what it absorbs from all the other atoms.

The above considerations are the basis of the recent random (or stochastic) electrodynamics.[1] However, the present work will not lean on the papers quoted in Ref. 1, not even on other authors[4-7] who tried to obtain the Schrödinger equation assuming that all particles are subjected to a Brownian motion. Indeed a Brownian motion[8] is the one of a particle subject to random impulses whose correlation function is represented by a Dirac δ function and therefore their power spectrum is white[9] (i.e., independent of ω). Now a charged particle, in motion with respect to the observer which sees an isotropic radiation (or isotropic impulses), undergoes a friction because of the Doppler effect (the radiation, or the impulses, are more intense in the forward direction than in the rear direction). The expression of the braking force has been obtained by Einstein and Hopf[10] in the case of a charged harmonic oscillator of charge e, mass m, barycentric velocity $\vec{V}$, frequency ω_0 [for a more readable deduction see Eq. (51) of Appendix A by Boyer in Ref. 1]. It is (c is the light speed)

$$\vec{F} = -\tfrac{4}{5}\pi^2\vec{V}e^2(mc^2)^{-1}[G_E(\omega_0) - \tfrac{1}{3}\omega_0(dG_E/d\omega)_0], \quad (4)$$

which is zero only when the spectral density $G_E(\omega)$ is given by (3). The result (4) can be applied even to a free particle, as shown in Sec. II (see also Appendix A).

For a white spectrum, as the one of a Brownian motion, the second term of (4) is zero and the first term cannot be canceled. Consequently, a Brownian motion cannot be frictionless and is therefore unable to describe the reversible processes of quantum mechanics. Hence, all the preceding authors (who have tried to obtain the Schrödinger equation from a Brownian motion) have implicitly been forced to assume another hypothetical force balancing the friction force. One author[11] explicitly assumes the existence of particles which would produce the Brownian force without friction.[12] Besides the above drawback, tricks are contained in Ref. 5 and inconsistencies in Refs. 4, 6, 7, and 11, as shown in Appendix B.

The resulting motion of a charged particle[3] subject to the random radiation with spectrum (3) is stochastic but *not* Brownian since the spectral densities for acceleration, velocities, and positions (see Appendix A) are different from the corresponding spectra of a Brownian motion. An attempt for the derivation of the Schrödinger equation has been recently done by de la Pena-Auerbach and Cetto[13] who, however, obtained an equation in the phase space which they have not succeeded in solving.[14]

In the present paper a more limited problem will be treated but it will be completely solved by stochastic electrodynamics: the propagator of a free particle. This problem has been developed in an approximate way by Santos[15] in the case that the probability density $\rho(\vec{v}_p,t)$ in the velocity space is a Dirac δ function at the initial time t (i.e., for $\Delta t = 0$). The relevant propagator, calculated in Sec. II of the present paper in a rigorous way, is denoted as elementary propagator and its dispersion is compared with a similar result obtained in quantum electrodynamics.

In Sec. III the probability density of the position is obtained by the elementary propagator starting from an initial distribution confined in a small volume and also taking into account the initial distribution of the velocities. If the initial volume is much smaller than the considered final volume in which, for example, the minimum values of the position probability density are of the order of 10% of the maximum values, the final probability density is a "quasi-transition probability" and is denoted as "effective propagator". To within radiative corrections it turns out to be equal to the result obtainable by quantum theory.

II. THE ELEMENTARY PROPAGATOR OF STOCHASTIC ELECTRODYNAMICS

We study the stochastic motion of a charged particle[3] under the action of the zero-point em field but otherwise free. If we join successive points of its path by small segments of lengths λ_j, the stochastic motion is equivalent, to within distances of λ_j order, to a random walk. If $\lambda = \langle\lambda_j\rangle$ is chosen so that the average change of the direction of the particle velocity $\vec{v}_p$ is $\pi/2$, then λ is the mean free path of the equivalent process of random flights with isotropic scattering. We are interested in finding the distribution function (called probability density, if normalized) of the particle position after the particle has performed many equivalent free flights starting from an initial point $\vec{x}_0$. We can therefore exploit a fundamental result relevant to any random walk with a large number of flights. Such result, shown, for example, in Sec. IV D of Ref. 8, states that the distribution function is Gaussian, independently of the particular function $\tau_j(\vec{x}_j)$ where $\tau_j(\vec{x}_j)\, d^3x$ represents the probability that the displacement in the jth step lies between x_j and $x_j + dx_j$. We can therefore write for the probability density which is a Dirac δ function centered on the point $\vec{x}_0$ at time t (Green's initial condition)

$$P_{v0}(\vec{x}, t+\Delta t\,|\,\vec{x}_0, t) = (2\pi\Delta x^2)^{-3/2}\exp[-(\vec{x}-\vec{x}_0-\vec{v}_0\Delta t)^2/2\Delta x^2], \quad (5)$$

where the velocity $\vec{v}_0$ of the center of mass of the probability cloud remains constant since the stochastic motion is frictionless (i.e., the average braking force is zero) because of the isotropy of the zero-point radiation in every inertial frame.

The mean-square fluctuation Δx^2 depends on the function $\tau_j(\vec{x}_j)$ characterizing the various flights. One could use the Markoff method (Sec. IV of Ref. 8) if $\tau_j(x_j)$ were known. In our case $\tau_j(x_j)$ is very complicated since the effect of the zero-point em field is similar to vibrations due to trains of oscillations, each of different frequency. Fortunately, we know the spectrum of the zero-point em field and we can obtain the velocity power spectrum $G_{\dot{x}}(\omega)$ of a particle having charge e and mass m, subject to the random radiation whose spectral density is given by (3). As shown in Appendix A (see also an earlier paper[14]), we obtain

$$G_{\dot{x}}(\omega)=(\hbar\omega\tau/\pi m)/(1+\tau^2\omega^2), \tag{6}$$

where $\tau=2e^2/3mc^3$. Then the mean-square fluctuation along the x axis of the particle position is rigorously given by the Wiener-Khintchin inverse relationship (see Appendix A)

$$\begin{aligned}\Delta x^2&=\langle[x(t+\Delta t)-x(t)]^2\rangle\\&=2\int_0^\infty d\omega\, G_{\dot{x}}(\omega)[1-\cos(\omega\Delta t)]\omega^{-2}\\&=(\hbar\tau/\pi m)B(\Delta t/\tau),\end{aligned} \tag{7}$$

where the result has been obtained with the use of (6) and B turns out to be given by

$$\begin{aligned}B(\Delta t/\tau)=1.1544&+2\ln(\Delta t/\tau)\\&-\exp(-\Delta t/\tau)\mathrm{Ei}(\Delta t/\tau)\\&-\exp(\Delta t/\tau)\mathrm{Ei}(-\Delta t/\tau),\end{aligned} \tag{8}$$

Ei being the exponential integral.[16]

By (12) and (14) we obtain for the transition probability

$$P_{v0}(\vec{x},t+\Delta t\,|\,\vec{x}_0,t)=[2\hbar\tau m^{-1}B(\Delta t/\tau)]^{-3/2}\exp[-(\vec{x}-\vec{x}_0-\vec{v}_0\Delta t)^2/2\hbar\tau\pi^{-1}m^{-1}B(\Delta t/\tau)]. \tag{9}$$

We denote (9) as elementary propagator since it is relevant to no dispersion at time t. For $\Delta t\ll\tau$ the mean-square dispersion is proportional to Δt^2, as shown by Eq. (A17) of Appendix A and as is characteristic of any random walk in the Ornstein-Uhlenbeck theory.[8,9] For $\Delta t\gg\tau$ the mean-square dispersion increases as $\ln(\Delta t/\tau)$, as shown by Eq. (A16). These two approximate expressions (either for $\Delta t\ll\tau$ or $\Delta t\gg\tau$) have been already found by Santos.[15] The rigorous expression (8) gives all the intermediate evolution and shows the rapidity by which the asymptotic expression is reached. Practically (8) reduces to the first two terms of its right-hand side after a time Δt just larger than $\tau=2e^2/3mc^3$ [see Eq. (A2)] which is $\frac{2}{3}$ the time taken by light for crossing the classical electron radius.

The procedure of Appendix A also shows how it is possible to apply the Wiener-Khintchin inverse relationship to a nonstationary process, as the one for the position distribution which increases with time. Indeed, by a procedure parallel to that leading from (A10) to (A12) one relates the correlation function of the velocity to that relevant to acceleration, hence to the acceleration spectrum given by (A9). Now the stochastic process for the acceleration is stationary and the Wiener-Khintchin relationships can be applied.

Finally, (A15) can also give the mean-square dispersion of the position if an upper cutoff is introduced in the spectral density (3). The absence of a cutoff in (3) is one of the causes of the divergences in quantum electrodynamics. The mean-square dispersion Δx^2 in the position calculated by stochastic electrodynamics converges when the upper cutoff tends to infinity, as shown by (A15), and that is why we have not considered it so far. However, the mean-square dispersion of the particle velocity $\vec{v}_p$ diverges if no cutoff is introduced. Indeed, by the same procedure followed from (A10) to (A15), but introducing an upper cutoff ω_M, we find

$$\begin{aligned}\Delta v_{px}{}^2&=\Delta\dot{x}_p{}^2\\&=2\int_0^{\omega_M}d\omega\, G_{\dot{x}}\omega^{-2}[1-\cos(\omega\Delta t)]\\&=\frac{2\hbar\tau}{\pi m}\int_0^{\omega_M}d\omega\frac{\omega[1-\cos(\omega\Delta t)]}{1+\omega^2\tau^2}\\&=\frac{\hbar}{\pi m\tau}\left[\ln(1+\omega_M{}^2\tau^2)-2\int_0^{x_M}dx\frac{x\cos(\kappa\Delta t/\tau)}{1+x^2}\right],\end{aligned} \tag{10}$$

which diverges for $x_M=\omega_M\tau\to\infty$. A reasonable cutoff is that corresponding to pair production, i.e., $\omega_M=c/R_C$, where $R_C=\hbar/mc$ is the electron's Compton radius. Consequently, $x_M=2R_L/3R_C=2\alpha/3$, where $R_L=e^2/mc^2$ is the Lorentz radius and α the fine-structure constant. By this value we can neglect x^2 in the denominator of the integrand in (10) and we obtain, by also expanding the logarithm,

$$\begin{aligned}\Delta v_p{}^2\simeq\frac{\alpha}{\pi}c^2\Bigg[\frac{2}{3}&-2\frac{\tau}{\alpha\Delta t}\sin\frac{2\alpha\Delta t}{3\tau}\\&+\frac{3\tau^2}{\alpha^2\Delta t^2}\left(1-\cos\frac{2\alpha\Delta t}{3\tau}\right)\Bigg],\end{aligned} \tag{11}$$

which for $\alpha\Delta t\gg\tau$ tends to $\Delta v_p{}^2\simeq 2\alpha c^2/3\pi$. In quantum electrodynamics, this value corresponds to the transverse self-energy of the electron under the em fluctuations of the vacuum.

By the same cutoff, the mean-square fluctuation of the position results in

$$\Delta x^2=(\hbar\tau/\pi m)B_c(\Delta t/\tau), \tag{12}$$

where, with an error less than 10^{-4},

$$B_c(\Delta t/\tau) \simeq 2\gamma + \ln\frac{(2\alpha\,\Delta t/3\tau)^2}{1+(2\alpha/3)^2} - 2\mathrm{Ci}\left(\frac{2\alpha\,\Delta t}{3\tau}\right) + \frac{4\alpha\tau}{3\Delta t}\sin\frac{2\alpha\Delta t}{3\tau} + 2\frac{\tau^2}{\Delta t^2}\left(\cos\frac{2\alpha\Delta t}{3\tau} - 1\right). \tag{13}$$

For $\Delta t \gg \tau/\alpha$, since $(2\alpha/3)^2 \simeq 2.368\times10^{-5}$, (12) tends to

$$\Delta_x{}^2 \simeq \frac{\hbar\tau}{\pi m}\left(1.1544 + 2\ln\frac{2\alpha\,\Delta t}{3\tau}\right), \tag{14}$$

compared with (A16), which corresponds to $\omega_M \to \infty$. The difference is only in an attenuation by a factor $2\alpha/3 \simeq 4.8\times10^{-3}$ in the argument of the logarithm, or alternatively, in a constant $2\ln 2\alpha/3$ added inside the square brackets of (A16). For extremely large $\alpha\,\Delta t/\lambda$ values, the two expressions have the same dependence.

Because of the logarithmic dependence in (14), Δx^2 ranges from about $0.25\alpha R_C{}^2$ when $\Delta t \simeq R_C/c$ to $15\alpha R_C{}^2$ when $\Delta t \simeq 10^{-7}$ sec, which is the typical time taken by an electron for reaching the screen in a two-slit interference experiment.

When Δt is the average period of revolution of an electron in a hydrogen atom, i.e., $\Delta t \simeq 2\pi\alpha^{-2}R_C/c$, then $\Delta x^2 \simeq 2.7 R_C{}^2$. Let us compare this value with that of Welton[17] who assumes the same upper cutoff of the present paper and starts from his Eq. (2) in which the radiation reaction is neglected. The time independence of the spreading $\langle(\Delta q)^2\rangle = \Delta_x{}^2$ found by Welton is due to the fact that he, instead of using the Wiener-Khintchin inverse relationship (7) which gives $\Delta_x{}^2$, actually calculates $\langle q^2\rangle = \langle x^2\rangle = \int_0^\infty d\omega G_x(\omega)$, which gives half the mean-square dispersion after an infinite time. Indeed one gets from (7)

$$\Delta_x{}^2 = \langle x^2(t+\Delta t)\rangle + \langle x^2(t)\rangle - 2\langle x(t+\Delta t)x(t)\rangle. \tag{15}$$

Now in any stochastic process the correlation $\langle x(t+\Delta t)x(t)\rangle$ vanishes for $\Delta t \to \infty$ and the same fact occurs for the term containing $\cos(\omega\Delta t)$ in (7). Moreover, in a stationary process, $\langle x^2(t+\Delta t)\rangle = \langle x^2(t)\rangle$ so that (7) becomes

$$\lim_{\Delta t\to\infty} \Delta x^2_{\text{stationary}} = 2\langle x^2\rangle = 2\int_0^\infty d\omega\, G_x(\omega), \tag{16}$$

where we have denoted $G_x(\omega) = G_{\dot{x}}(\omega)/\omega^2$. Welton has used (16) which is valid for a stationary process. In our case of a nonstationary process, namely, of a particle subject to the fluctuations of the zero-point field (but otherwise free) the spreading increases with time and diverges, although logarithmically, for $\Delta t \to \infty$. That is why Welton is forced to introduce a (otherwise unjustified) lower cutoff k_0 in the wave-number spectrum and finds a logarithmic expression for $\langle x^2\rangle$. In our notation his Eq. (59) reads $\langle x^2\rangle = 2\pi^{-1}\alpha R_c \ln(2/R_C k_0)$. Only in a bound state the stochastic process becomes stationary for large Δt values and therefore $\Delta_x{}^2$ tends to (23). This is the case of an electron in the 1S state in a hydrogen atom. This case has been studied by Bethe[18] for finding the relevant effect of the vacuum fluctuations. If one substitutes the Bethe cutoff in the Welton expression, he finds $\langle x^2\rangle \simeq 3.85\alpha R_C$ to be compared with our value $\Delta\kappa^2 \simeq 2.7\alpha R_C$. I do not know more recent work on this subject in quantum electrodynamics.

III. EVOLUTION OF THE PROBABILITY DENSITY STARTING FROM AN INITIAL SMALL VOLUME: THE EFFECTIVE PROPAGATOR

As in the kinetic theory of the gases, it is very difficult to follow a single particle undergoing a stochastic motion. What is feasible is to consider a small volume $\Delta x_0\Delta y_0\Delta z_0$ in similarly prepared systems, theoretically constituting an ensemble. The present purpose is to calculate the evolution of the position probability density starting from $\Delta x_0\Delta y_0\Delta z_0$ and with an isotropic mean-square dispersion $\Delta v_0{}^2$ in the velocity. It is assumed independence between the distributions of the velocities and those of the positions. This can be due either to independent sources which produced the given distribution or even to a single source but after a long time. Indeed the cross correlation between position and velocity vanishes after a long time in any stochastic process. For instance in a Brownian motion, the joint probability distribution, which is a bivariate Gaussian distribution given by Eq. (178) of Ref. 8, tends to

$$\rho(r,v) = 2^{-3/2}q^{-3}t^{-3/2}\beta^{9/2}\exp[-\beta v^2(2q)^{-1} - \beta^2 r^2(4qt)^{-1}]$$

after a long time.

In order to have the same initial conditions considered in the Kennard wave packet, the position's initial distribution is assumed to be Gaussian. The velocity's initial distribution is also taken to be Gaussian. For this it is sufficient that the particle was free during a time $\Delta t > \tau/\alpha$ before the considered initial time. Indeed, if we join successive points of the stochastic process in the velocity space, we have an equivalent random walk to which we can apply the result of Ref. 8, i.e., that the distribution becomes Gaussian after many free flights. As we have seen in Sec. II, the zero-point field is so intense that the asymptotic condition for the velocity dispersion (11) is reached after a very short time $\Delta t > \tau/\alpha$. Consequently, even the most intense macroscopic fields met in practice produce negligible modifications as far as the velocity fluctuations are concerned. We can therefore assume for the joint, initial distribution function

$$[\rho_0(\vec{x}_0, \vec{v}_0) = (4\pi^2 \Delta v_0^2)^{-3/2}(\Delta x_0^2 \Delta y_0^2 \Delta z_0^2)^{-1/2} \times \exp[-(\vec{v}_0 - \vec{v})^2/(2\Delta v_0^2) - (x_0^2/2\Delta x_0^2) - (y_0^2/2\Delta y_0^2) - (z_0^2/2\Delta z_0^2)]. \quad (17)$$

The time evolution of (17) is obtained by a superposition of the results of Green's type, by dividing the distribution (in the phase space) in infinitesimal parts, each one of the Green kind, i.e., having single velocity $\vec{v}_{0k}$ (with an infinitesimal spread around it). Each part (of the initial probability cloud) which has a velocity $\vec{v}_{0k}$, diffuses with a diffusive radius slowly increasing as given by (7) and (8) [see also (A16) of Appendix A] around a point A_k (with $k = 1, 2, 3, \ldots$), which moves along a straight line with constant velocity $\vec{v}_{0k}$ (see Fig. 1). Since the distance of A_k from the initial cloud is increasing proportionally to Δt, the "envelope", i.e., the total effect due to all the parts (with the different initial velocities) is spreading (for large Δt values) proportionally to Δt.

Let us give a mathematical form to the above considerations. Each domain $\rho_0\, d^3x_0\, d^3v_0$ maintains its velocity $\vec{v}_0$ and diffuses as given by (9). The time evolution of (17) is obtained as the superposition of the single parts, i.e., as the convolution of (17) with (9):

$$P_{\text{eff}}(\vec{x}, \Delta t) = \int_{-\infty}^{\infty} d^3x_0 \times \int_{-\infty}^{\infty} d^3v_0 \rho_0(\vec{x}_0, \vec{v}_0) P_{v0}(\vec{x}, t + \Delta t \,|\, \vec{x}_0, t). \quad (18)$$

Taking into account that

$$\int_{-\infty}^{\infty} dx \exp(-a^2x^2 \pm bx) = \frac{\pi^{1/2}}{a} \exp\left(\frac{b^2}{4a^2}\right), \quad (19)$$

(18) becomes

$$P_{\text{eff}}(\vec{x}, \Delta t) = P_{\text{eff}x}(x, \Delta t) P_{\text{eff}y}(y, \Delta t) P_{\text{eff}z}(z, \Delta t), \quad (20)$$

where

$$P_{\text{eff}y}(y, \Delta t) = [2\pi\Delta y_0^2 + 2\pi\Delta v_0^2 \Delta t^2 + 2\hbar\tau m^{-1} B(\Delta t/\tau)]^{-1/2} \times \exp\{-(y - v\Delta t)^2/[2\Delta x^2 + 2\Delta v_0^2 \Delta t^2 + 2\hbar\tau\pi^{-1} m^1 B(\Delta t/\tau)]\}, \quad (21)$$

and similarly for $P_{\text{eff}x}$ and $P_{\text{eff}z}$ by substituting y and z, respectively.

If $\Delta y_0^2 \ll \Delta v_0^2 \Delta t^2$ (and similarly for Δx_0^2 and Δz_0^2), (21) is a "quasipropagator" and we denote

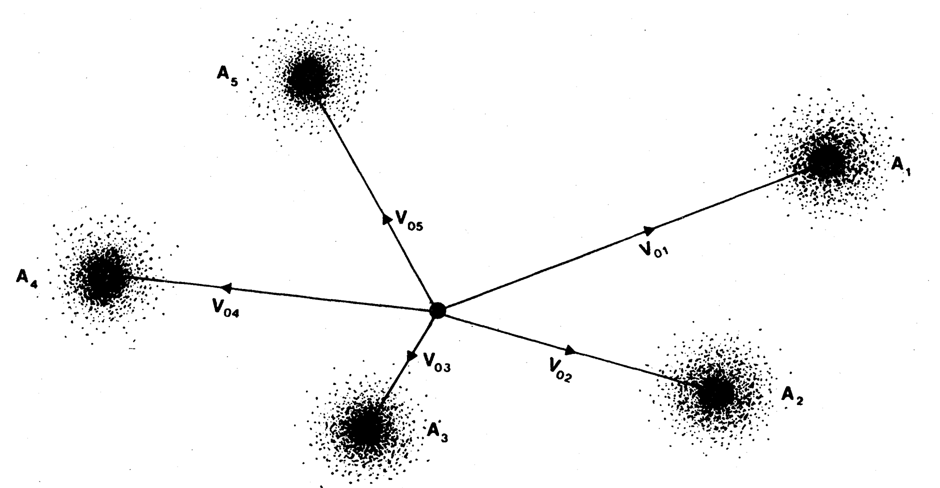

FIG. 1. Frictionless diffusion of an initially concentrated probability cloud having an initial spreading in the velocity. As an example, five different initial velocities $\vec{v}_{ok}$ are shown. The center of mass of each domain (of the initial probability cloud) having velocity v_{ok} has an uniform motion in absence of external forces since the zero point em field does not produce a friction, on average. Around each center of mass (having velocity $\vec{v}_{ok}$) there is a spreading which slowly increases as given by Eqs. (14) and (15) due to the zero-point em field. The resultant diffusion is the convolution of the spatial spreading due to the initial dispersion in the velocity and the diffusion (further due to the zero-point em field) around each point moving with uniform motion.

it as effective propagator since it takes into account the initial distribution of the velocities which cannot be disregarded if an initial (small) volume is considered. The elementary propagator found in Sec. II is useful from a theoretical point of view only. For instance, it has been used to obtain (21) and will be exploited in a future paper to deduce the fine-structure constant.

If we let $\tau \to 0$, (21) becomes equal to $\psi\psi^*$ with ψ given by the Kennard wave packet for a particle having an initial Gaussian distribution.[19] We therefore find for a free particle what we found for a harmonic oscillator (see Refs. 13 and 15), i.e., that the result given by *quantum mechanics* is the approximation obtainable by stochastic electrodynamics for $\tau \to 0$. If we keep τ with its true value, stochastic electrodynamics gives for the harmonic oscillator the same result of *quantum electrodynamics*. For a free particle it should be so but we do not have, at present, the corresponding result found by quantum electrodynamics. It is this the first case in which stochastic electrodynamics precedes quantum electrodynamics. There are only the old, partial results of Welton[17] and Bethe[18] discussed in Sec. II, but (21) (or a corresponding expression with radiative corrections) has not yet been found.

Finally, stochastic electrodynamics eliminates any problem of renormalization.

Note added in proof. Another achievement of stochastic electrodynamics has been obtained by A. Rueda [Nuovo Cimento 48A, 155 (1978)] who explained the origin of the cosmic rays as due to the continuous action of the zero-point, random em field.

APPENDIX A: DEDUCTION OF THE POSITION POWER SPECTRUM IN STOCHASTIC ELECTRODYNAMICS

The object of this appendix is to derive Eqs. (13), (14), and (15) from the zero-point, electromagnetic power spectrum given by Eq. (3). Let us consider a particle having mass m and charge e, subject to an external force $\vec{f}$ and to a stochastic field $\vec{E}$. Its nonrelativistic, Lorentz-Dirac equation of motion, projected on the x axis is

$$m\ddot{x} = eE_x + f_x + m\tau\dddot{x}, \tag{A1}$$

where

$$\tau = 2e^2/3mc^3, \tag{A2}$$

with c being the light speed. As known, (A1) has exponentially increasing solutions. These "runaway solutions" have been thought of as the cause of the electron *Zitterbewegung*.[20] However, in order to evaluate the spreading only due to the zero-point radiation field E_x, the *Zitterbewegung* is considered as an ordered motion inside a sphere having the Compton radius. Hence, in order to calculate the spread due to E_x and the consequent power spectrum of the position x, the runaway solutions must be eliminated. For this purpose we use, instead of (A1), the following equivalent integro-differential equation:

$$\begin{aligned} m\ddot{x} &= \int_0^\infty ds[\exp(-s)][eE_x(t+\tau s) + f_x(t+\tau s)] \\ &= F_x(t) + f_x^*(t). \end{aligned} \tag{A3}$$

Indeed, if we differentiate (A3) with respect to t and take into account that inside the integral sign, $\partial/\partial t = \tau^{-1}\partial/\partial s$ we get, integrating by parts,

$$\begin{aligned} m\dddot{x} &= \frac{1}{\tau}\int_0^\infty ds[\exp(-s)]\frac{\partial}{\partial s}[eE_x(t+\tau s) + f_x(t+\tau s)] \\ &= \frac{1}{\tau}\Big|[\exp(-s)][eE_x(t+\tau s) + f_x(t+\tau s)]\Big|_0^\infty + \frac{1}{\tau}\int_0^\infty ds[\exp(-s)][eE_x(t+\tau s) + f_x(t+\tau s)]. \end{aligned} \tag{A4}$$

The first term in the last side of (A4) is equal to $-\tau^{-1}[eE_x(t) + f_x(t)]$, while the second term, because of (A3), is equal to $m\ddot{x}$. Consequently, (A4) is equivalent to (A1). It is easy to show that (A3) has the same solutions of (A1) except the runaway solutions.

Note that (A3) has the formal appearance of the Newton law for a particle which is subject to a random force F_x besides the external force $f_x^*(t)$. The acceleration $\ddot{x} = F_x(t)/m$, caused by the equivalent random force $F_x(t)$ [i.e., when the equivalent external force $f_x^*(t)$ is zero] has a power spectrum $G_{\ddot{x}}(\omega)$ which can be derived from that of a component of $\vec{E}$. Indeed, the energy density of a random, isotropic, electromagnetic, pure radiation field (for which $E = H$) can be written in one of the following expressions:

$$u = \int_0^\infty G_E(\omega)d\omega = \frac{1}{8\pi}\langle E^2 + H^2\rangle = \frac{1}{4\pi}\langle E^2\rangle = \frac{3}{4\pi}\langle E_x{}^2\rangle = \frac{3}{4\pi}\int_0^\infty G_{Ex}(\omega)d\omega. \tag{A5}$$

Identifying the second and last step of (A5) and using (3) of the main text gives

$$G_{Ex}(\omega) = 4\pi G_E(\omega)/3 = 2\hbar\omega^3/3\pi c^3. \tag{A6}$$

Now $G_{\ddot{x}}(\omega)$ can be obtained by the usual expression[21]

$$G_x(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}dT\,\langle\ddot{x}(0)\ddot{x}(T)\rangle\exp(-i\omega T). \tag{A7}$$

In the case $f_x=0$, let us substitute (A3) in (A7), exchange the orders of integration, and set $T=y+\tau\times(s-s')$ We obtain

$$G_{\ddot{x}}(\omega)=\frac{e^2}{m^2}\int_0^{\infty}ds\int_0^{\infty}ds'\exp[-s(1-i\omega\tau)-s'(1+i\omega\tau)] \times\frac{1}{\pi}\int_{-\infty}^{\infty}dy\langle E_x(t+\tau s)E_x(t+\tau s+y)\rangle\exp(-i\omega y)=\frac{e^2G_{Ex}(\omega)}{m^2(1+\omega^2\tau^2)}. \tag{A8}$$

Finally, with the use of (A6) and the properties[22] of the spectrum of differentiated quantities we get

$$G_{\dot{x}}=\left(\frac{1}{\omega^2}G_{\ddot{x}}\right)=\frac{2\hbar e^2\omega}{3\pi c^3m^2(1+\omega^2\tau^2)}, \tag{A9}$$

which, with the use of (A2), is equal to (13) of the main text. For finding (15) we start from

$$\Delta x^2=\langle[x(t+\Delta t)-x(t)]^2\rangle =\left\langle\int_t^{t+\Delta t}dt_1v_x(t_1)\int_t^{t+\Delta t}dt_2\,v_x(t_2)\right\rangle. \tag{A10}$$

If we set $\tau=t_2-t_1$ so that $dt_1dt_2=dt_1d\tau$ (the Jacobian is unity), we have

$$\Delta x^2=\int_t^{t+\Delta t}dt_1\int_{t-t_1}^{t-t_1+\Delta t}d\tau\langle v_x(t_1)v_x(t_1+\tau)\rangle. \tag{A11}$$

Since the correlation is the antitransform of the spectrum (Wiener-Khintchin inverse relationship), (A11) becomes

$$\Delta x^2=\int_t^{t+\Delta t}dt_1\int_{t-t_1}^{t-t_1+\Delta t}d\tau\int_0^{\infty}d\omega\,G_{\dot{x}}(\omega)\cos\omega\tau =2\int_0^{\infty}d\omega\,G_{\dot{x}}(\omega)\omega^{-2}[1-\cos(\omega\Delta t)]. \tag{A12}$$

Substituting (A9) in (A12), with the use of (A2), and setting $x=\omega\tau$, gives

$$\Delta x^2=\frac{\hbar\tau}{\pi m}2\int_0^{\infty}dx\frac{1-\cos(x\Delta t/\tau)}{x(1+x^2)}=\frac{\hbar t}{\pi m}B\left(\frac{\Delta t}{\tau}\right). \tag{A13}$$

Multiplying the numerator of the integrand of (A13) by $(1+x^2-x^2)$, gives

$$\frac{B}{2}=\lim_{x_M\to\infty}\left\{\int_0^{x_M\Delta t/\tau}dy\frac{1-\cos y}{y}-\int_0^{x_M}dx\frac{x}{1+x^2}\right\} +\int_0^{\infty}dx\frac{x\cos(x\Delta t/\tau)}{1+x^2}, \tag{A14}$$

where we have set $x\Delta t/\tau=y$ in the first integral.

By performing the integrations we obtain[23]

$$B=\lim_{x_M\to\infty}2\left\{\gamma+\ln\left(\frac{\Delta t}{\tau}x_M\right)-\mathrm{Ci}\left(\frac{\Delta t}{\tau}x_M\right)-\ln(1+x_M^2)^{1/2}\right\} -\left\{\exp(-\Delta t/\tau)\,\mathrm{Ei}(\Delta t/\tau)+\exp(\Delta t/\tau)\,\mathrm{Ei}(-\Delta t/\tau)\right\}$$
$$=1.1544+2\ln(\Delta t/\tau)-\exp(-\Delta t/\tau)\,\mathrm{Ei}(\Delta t/\tau) -\exp(\Delta t/\tau)\,\mathrm{Ei}(-\Delta t/\tau), \tag{A15}$$

where $\gamma=0.5772$ is the Euler constant,

$$\mathrm{Ci}(x)=\int_{\infty}^{x}dy\,y^{-1}\cos y =\gamma+\ln x+\int_0^x dy\,y^{-1}(\cos y-1)$$

the cosine integral, and $\mathrm{Ei}(x)=⨍_{-\infty}^{x}dy\,y^{-1}\exp(y)$ the exponential integral. For $\Delta t>\tau$ the last two terms of (A15) give a very small contribution to the first term so that

$$\Delta x^2\simeq(\hbar\tau/\pi m)[1.1544+2\ln(\Delta t/\tau)],\quad \Delta t>\tau. \tag{A16}$$

For $\Delta t\ll\tau$ a second-order expansion of (A15) in $\Delta t/\tau$ gives

$$\Delta x^2\simeq(\hbar\Delta t^2/\pi m\tau)\ln(\tau/\Delta t),\quad \Delta t\ll\tau. \tag{A17}$$

APPENDIX B: STOCHASTIC MECHANICS WITH A BROWNIAN MOTION CANNOT BE EQUIVALENT TO QUANTUM MECHANICS

The power spectrum of the random *impulses* causing a Brownian motion is white[9,15] and therefore, because of (3), the motion of a charged harmonic oscillator cannot be frictionless. This already excludes the possibility of describing quantum mechanics by a Brownian process. However, other drawbacks of the approach to quantum mechanics by a Brownian process will be emphasized, also for showing that all the authors who claimed to have obtained the Schrödinger equation by a stochastic approach, have assumed a Brownian motion so that their procedure cannot be correct.

The power spectrum of the x component of the *position* in a Brownian motion is given by[9,14]

$$G_x(\omega) = (2D/\pi\omega^2)/(1+\tau^{*2}\omega^2)\,, \tag{B1}$$

where D is the diffusion coefficient and the relaxation-time constant τ^* is zero in the Einstein-Smoluchowski[8] approximation and $\tau^* \neq 0$ in the more refined Ornstein-Uhlenbeck theory.[9] By the Wiener-Khintchin inverse relationship [given by (7) of the main text] one obtains

$$\Delta x^2 = 2D\Delta t \quad \text{for } \Delta t \gg \tau^*\,, \tag{B2}$$

$$\Delta x^2 = D\Delta t^2/\tau^* \quad \text{for } \Delta t \ll \tau^* \tag{B3}$$

[a factor 2 has been neglected in Eq. (4.17) by Santos[15]]. One immediately sees that (B2) represents an irreversible process which cannot describe the reversible quantum processes governed by the Schrödinger equation.

The transition probability for a Brownian motion (starting from an initial probability density given by a Dirac δ function centered on point $\vec{x} - \vec{\xi}$ at time t) is[4,7]

$$P_B(\vec{x}, t+\Delta t\,|\,\vec{x}-\vec{\xi}, t) = (4\pi D\Delta t)^{-3/2}\exp[-(\vec{\xi} - \vec{V}\Delta t)^2/4D\Delta t]\,, \tag{B4}$$

where $\vec{V}$ is given by

$$\vec{V} = \int_{-\infty}^{\infty} d^3 v_p \int_{-\infty}^{\infty} d^3\xi\, \rho(\vec{V}_p, \xi)\vec{V}_p\,, \tag{B5}$$

with $\vec{V}_p$ being the particle velocity and $\rho(\vec{V}_p, \vec{\xi})$ the joint probability for position and velocity. Indeed $\vec{V}$ is denoted as mean particle velocity by Kershaw[4] [after his Eq. (15)] and drift velocity by Bess[7] [just before his Eq. (1)].

From (B4) one can obtain a generalization of (B2), i.e., for $\vec{V} = 0$,

$$\langle \xi_i \xi_j \rangle = 2D\Delta t\delta_{ij} \tag{B6}$$

(where δ_{ij} is the Kronecker symbol), which is Eq. (12) of Nelson,[5] used in his Eqs. (22) and (23). Even Surdin[6] implicitly uses (B1)–(B6) since it starts from the Fokker-Planck equation as found in the classical paper by Chandrasekhar,[8] where Brownian motion only is treated.

Kershaw[4] and Bess[7] use (B4) in the Fokker-Planck method applied to the probability balance. This use is inconsistent since the velocity appearing in (B4) is the mean velocity of the probability cloud representing the entire particle (or all the particles). The above authors expand the propagator P_B given by (B4) obtaining

$$\begin{aligned} P_B(x_i, t+\Delta t\,|\,x_i - \xi_i, t) &= P_B(\xi)\exp\left\{\frac{1}{2D}\left[V_i\xi_i - \xi_i\xi_j\partial_i V_j - \tfrac{1}{2}V_iV_i\Delta t\right]\right\} \\ &\simeq P_B(\xi)\left[1 + \frac{1}{2D}\left(V_i\xi_i - \xi_i\xi_j\partial_i V_j - \tfrac{1}{2}V_iV_i\Delta t + \frac{1}{4D}\xi_i\xi_jV_iV_j\right)\right]\,, \end{aligned} \tag{B7}$$

where

$$P_B(\xi) = (4\pi D\Delta T)^{-3/2}\exp[-(\xi_i\xi_i)/(4D\Delta t)]\,. \tag{B8}$$

Substituting (B7) and the expansion of V in the Fokker-Planck equation and truncating after the terms in Δt and $\xi_i\xi_j$ gives

$$\begin{aligned} \rho + \Delta t\partial_t\rho = \int_{-\infty}^{\infty} d^3\xi\, P_B(\xi)\Big(\rho &+ \frac{\rho}{2D}\xi_iV_i - \frac{\rho}{4D}V^2\Delta t - \frac{\rho}{2D}\xi_i\xi_j\partial_iV_j + \frac{\rho}{8D^2}\xi_i\xi_jV_iV_j - \xi_i\partial_i\rho + \tfrac{1}{2}\xi_i\xi_j\partial_{ij}{}^2\rho \\ &- \frac{1}{2D}\xi_iV_i\xi_j\partial_j\rho + \frac{\Delta t}{2D}V^2\xi_i\partial_i\rho\Big)\,. \end{aligned} \tag{B9}$$

Since

$$\int_{-\infty}^{\infty} d^3\xi\, P_B(\xi) = 1\,, \quad \int_{-\infty}^{\infty} d^3\xi\, P_B(\xi)\xi_i = 0\,,$$

and

$$\int_{-\infty}^{\infty} d^3\xi\, P_B(\xi)\xi_i\xi_j = 2D\Delta t\delta_{ij}\,, \tag{B10}$$

where δ_{ij} is the Kronecker δ, one gets by (B9) and (B10) after some simplifications

$$\partial_t\rho = -\partial_i(\rho V_i - D\partial_i\rho)\,. \tag{B11}$$

At this point the above authors[4,7] introduce the "transport" velocity v_i defined as

$$\rho v_i = \rho V_i - D\partial_i\rho\,, \tag{B12}$$

so that (B11) becomes

$$\partial_t\rho = -\partial_i(\rho v_i)\,. \tag{B13}$$

Indeed (B13) is obtainable from the Schrödinger equation provided v_i be the mean, local velocity, given in quantum physics, by

$$\vec{v} = \frac{\hbar}{2im\psi\psi^*}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)\,. \tag{B14}$$

But (B12) is meaningless in a frictionless motion since the relationship between $\vec{v}$ and $\vec{V}$ depends on all the preceding history of the particle. That is why a double integration appears in Eqs. (11) and (13) of Gilson[24] [see the comments of his Eq. (13), where Gilson emphasizes the second criticism against the Brownian stochastic picture; his first criticism is that the same $\vec{v}$ must appear in the propagation kernel and in the continuity equation]. In a frictionless motion, any velocity depends on its initial value and on an integral over time of the acceleration undergone starting from the initial condition. This is valid even for the diffusion velocity, which therefore loses its meaning. What are meaningful are the differences of velocities relevant to a very small time interval Δt and to two near points. This is not the case of $\vec{v}-\vec{V}$, the first velocity being pertinent to a given point which may be very far from the center of mass (of the probability cloud) which has velocity $\vec{V}$. Consequently, what is always meaningful in a frictionless motion are the accelerations, like the "diffusion acceleration".

As said, Gilson[24] has shown that a necessary condition for having equivalence between quantum mechanics and a stochastic approach is that the velocity v appearing in (B13) must be equal to the velocity appearing in the propagator. This is not the case for the Brownian motion. In other words, the diffusion coefficient D appearing in (B12) should be zero.

It is surprising that, despite the above three inconsistencies, Kershaw[4] (who does not use tricks) find the correct *time-independent* Schrödinger equation. However, in order to find the *time-dependent* Schrödinger equation by the same procedure, Bess[7] had to introduce *ad hoc* quantum-mechanical forces.

Let us summarize the four drawbacks of the above approach.

(i) A Brownian motion cannot be frictionless.

(ii) The diffusion coefficient D cannot vanish for $\Delta t \to 0$ (see Gilson[24]) or, equivalently, the velocity appearing in the continuity equation cannot be equal to the velocity appearing in the propagator (B4).

(iii) It relates the local velocity with the mean velocity of the complete probability cloud. This is wrong since such relation must depend, in a frictionless motion, by all the preceding history of the probability cloud.

(iv) Either the time-independent Schrödinger equation is not obtained[4] or quantum-mechanical forces have to be introduced.[7]

Nelson[5] cannot avoid drawback (ii) and replace (iii) by another drawback. Indeed he introduces two different local velocities $\vec{v}=\vec{b}+\vec{b}_*$ and $\vec{u}=\vec{b}-\vec{b}_*$ related to his forward velocity $\vec{b}$ and backward velocity b_*. By contrast, in any macroscopic stochastic process one has a single *local* velocity and this occurs even for quantum mechanics where $\vec{v}$ is given by (B14). Nelson[5] apparently avoids drawbacks (i) and (iv) by defining the acceleration as half the sum of the forward derivative of the backward velocity and the backward derivative of the forward velocity

$$a=\tfrac{1}{2}(Db_*+D_*b)\,. \tag{B15}$$

This strange definition (different and irreducible to the macroscopic acceleration) is justified by Nelson[5] by his restriction to the Einstein-Smoluchosky theory[8] where $dx=b\{x(t),t\}\,dt+dw$ is not differentiable since his Eq. (12) implies

$$\langle dx_i dx_j\rangle = 2D\delta_{ij}\,dt + b_i b_j\,dt^2\,. \tag{B16}$$

But the final result of Nelson, i.e., the Schrödinger equation, is such that with the use of the propagator (21) of the main text when both $\Delta y_0{}^2$ and τ vanish,

$$\langle dx_i dx_j\rangle = \langle v_0{}^2\rangle\delta_{ij}\,dt^2\,. \tag{B17}$$

Consequently, $x(t)$ is differentiable and Nelson's pretext is no longer valid. What is still worse is that (B17) is different from the initial assumption (B16), which is also used in Nelson's[5] Eqs. (22) and (23).

For a similar and more detailed criticism against Nelson's paper[5] see Kracklauer.[25] However, Nelson's paper[5] can be saved if not intended as based on classical physics, as shown by de La Peña-Auerbach and Cetto,[26] contrary to what is asserted by Nelson himself in his introduction.[5] In this interpretation, Nelson's paper[5] is a new recipe for first quantization and has nothing to do with classical physics, as emphasized by his definition (B15) for the acceleration. If strictly connected to (B15), even Nelson's Brownian motion is something different from classical Brownian motion and therefore his first assumption (B16) is uncorrelated (in the classical sense) with the result (B17) obtainable by Nelson's procedure, thus eliminating the contradictions.

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[16]The expression (8) may seem very unusual to the people used to the Brownian motion only. Moreover, in Sec. IVD of Ref. 8 only such motion is considered since the probability $\tau_j(\bar{x}_j)$ of having the free path x_j is assumed to be independent of j. By contrast, a charged particle subject to the wave trains of the zero-point em field undergoes vibrations so that its displacement at the beginning of the wave train is comparable to those near the end of the same wave train. The random walk aspect is due to the fact that the elongations in one direction may differ by unity with respect to those in the opposite direction. This is equivalent to having a τ_j decreasing with the increasing number N of the free flights. For taking into account the N dependence of $\tau_j=\tau_j(x_j,N)$ one should multiply the second side of Eqs. (53) and (94) of Ref. 8 by a convenient function $f(N)$. The same $f(N)$ would appear in the numerator of (193) of Ref. 8. Our result (9) would be obtained if $f(N)\propto\{NB(N)\}^{-1}$ since $N\propto\Delta t/\tau$. Anyway, as said in the main text, Δx^2 only and not the x dependence of the distribution function (i.e., its Gaussian nature), depends on $\tau_j(x_j,N)$. Moreover, Δx^2 is easily obtained by the rigorous Eq. (7) via the velocity power spectrum $G_x(\omega)$ given by (6) which is a consequence of (3).

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