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Infinite-dimensional family of vacuum cosmological models with Taub-NUT (Newman-Unti-Tamburino)-type extensions

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We show that the Gowdy metrics on $T^3 \times R$ contain an infinite-dimensional subfamily of solutions which each admit a Taub-NUT (Newman-Unti-Tamburino)-type extension. However we also show that the generic (diagonal) Gowdy solution develops curvature singularities along the boundary of its maximal Cauchy development and thus is inextendible.

I. INTRODUCTION

In a recent paper[1] we studied the global Cauchy development problem for the vacuum metrics on $T^3 \times R$ which have two commuting, spacelike Killing fields. With a suitable choice of coordinates[2] these metrics take the form

$$ds^2 = \exp(2a)(-dt^2 + d\theta^2) + g_{ab}dx^a dx^b \tag{1.1}$$

with

$$\det(g_{ab}) = t^2 . \tag{1.2}$$

Here $x^1 \equiv \theta$ and $\{x^a\} = \{x^2, x^3\}$ are periodic coordinates on the three-torus and the function a and Riemannian two-metric g_{ab} are functions of t and θ alone. The Einstein equations for these metrics are the hyperbolic analogs of the elliptic equations studied in the stationary axisymmetric problem.

We showed that any choice of initial data in a suitable Sobolev space *determines* a C^2 solution of the vacuum field equations without singularity for all t in the interval $(0, +\infty)$. Thus no singularities arise until the time function t (which measures the area of the invariant two-tori tangent to $\partial/\partial x^a$) approaches its limiting values of 0 and $+\infty$. We also showed that each such solution defined on $T^3 \times (0, +\infty)$ is in fact the maximal Cauchy development of the given initial data. This followed from the observations that $\mathrm{tr}K(t)$ [the trace of second fundamental form $K(t)$ induced on the hypersurface of constant t] blows up uniformly as $t \to 0^+$ and that the curves of the normal congruence of the slicing tend to infinite proper length as $t \to +\infty$.

The uniform blowup of $\mathrm{tr}K(t)$ as $t \to 0^+$ means that each solution has a crushing singularity in the sense of Eardley and Smarr[3]. It is natural to ask whether these solutions all have curvature singularities at their crushing boundaries or whether, perhaps, some of the solutions are extendible across Cauchy horizons. The purpose of this paper is to show that there is an infinite-dimensional subfamily of these solutions which have no curvature singularities at their crushing boundaries and which are extendible across Cauchy horizons in much the same way that Taub space is extendible to Taub-NUT (Newman-Unti-Tamburino) space.

In spite of our result it remains quite plausible that the generic solution (of the symmetry type considered) does have curvature singularities at its crushing boundary. Indeed, we can demonstrate this explicitly for the special case of diagonal metrics of the type (1.1). These solutions, which correspond to Einstein-Rosen waves and admit only one polarization of gravitational radiation, are determined from the solutions of a linear wave equation on $S^1 \times (0, +\infty)$. We shall show that the "regular" solutions of this wave equation generate metrics which are extendible across Cauchy horizons as described above, whereas the "irregular" solutions (and hence the generic, diagonal solution) generate metrics with curvature singularities at their crushing boundaries.

To show that each of our regular solutions is in fact extendible we apply a technique recently developed by Schmidt[4]. Schmidt considered another variant of Einstein-Rosen waves which allows solutions compatible with (future null) asymptotic flatness. He constructs solutions in the interior of a certain null cone which intersects $\mathcal{I}^+$, establishes the regularity of $\mathcal{I}^+$, and then shows that a certain subfamily of his solutions are extendible across this null cone to the past.

The solutions we consider were first given by Gowdy[5] who generalized the classical Einstein-Rosen waves to allow the topologies $T^3 \times R$, $S^3 \times R$, and $S^2 \times S^1 \times R$. A proof that the generic, diagonal Gowdy solution develops curvature singularities was first given by Berger,[6] who showed that the square of the Riemann tensor typically blows up as $t \to 0^+$. As we shall show here, one can suitably restrict the solutions to a class (involving infinitely many parameters) in which the full Riemann tensor is well behaved at the crushing boundary. We then apply Schmidt's technique to show that every member of this class is extendible (in two different ways) through a Cauchy horizon. For simplicity we shall restrict our attention to analytic solutions and require analyticity of the extensions. A larger class of extendible solutions could presumably be obtained by considering all the solutions in a suitable Sobolev space as discussed in Ref. 1.

II. EINSTEIN-ROSEN-GOWDY METRICS ON $T^3 \times R$

Specializing (1.1) to the diagonal case we write the metric as

$$ds^2=\exp(2a)(-dt^2+d\theta^2)+te^{2W}(dx^2)^2 + te^{-2W}(dx^3)^2\,. \tag{2.1}$$

The function W satisfies

$$\frac{\partial^2 W}{\partial t^2}+\frac{1}{t}\frac{\partial W}{\partial t}-\frac{\partial^2 W}{\partial \theta^2}=0\,, \tag{2.2}$$

whereas the function a is determined from

$$\frac{\partial a}{\partial \theta}=2t\,\frac{\partial W}{\partial t}\,\frac{\partial W}{\partial \theta} \tag{2.3}$$

and

$$\frac{\partial a}{\partial t}=-\frac{1}{4t}+t\left[\left(\frac{\partial W}{\partial t}\right)^2+\left(\frac{\partial W}{\partial \theta}\right)^2\right]. \tag{2.4}$$

The general solution to Eq. (2.2), with period 2π in the θ variable, is given by

$$W=\alpha+\beta\ln t+\sum_{n=1}^{\infty}\left[a_n J_0(nt)\sin(n\theta+\gamma_n) + b_n N_0(nt)\sin(n\theta+\delta_n)\right]\,, \tag{2.5}$$

where α, β, a_n, b_n, γ_n, and δ_n, are real constants and J_0 and N_0 are the regular and irregular Bessel functions of zeroth order. From Eq. (2.3) and the required continuity of a, it follows that one must impose the condition

$$\int_{-\pi}^{\pi} d\theta\left(t\,\frac{\partial W}{\partial t}\,\frac{\partial W}{\partial \theta}\right)=0\,. \tag{2.6}$$

This integral is a conserved quantity so the condition need only be imposed on an initial surface. The particular (nonsingular) solutions we consider below all satisfy this condition automatically.

To avoid questions of the convergence of series, etc., we shall restrict the solutions to have only finitely many of the coefficients $\{a_n\}$ and $\{b_n\}$ nonzero. A more general alternative would be to restrict the coefficients so that W and $\partial W/\partial t$ are initially in the Sobolev space $H_3\times H_2$ as discussed in Ref. 1.

The curvature tensor for the metrics (1.1) was computed by Gowdy in Ref. 2 and expressed in the orthonormal frame defined by

$$\hat e_0=e^{-a}\frac{\partial}{\partial t}\,,\quad \hat e_1=e^{-a}\frac{\partial}{\partial \theta}\,,$$
$$\hat e_2=t^{-1/2}e^{-W}\frac{\partial}{\partial x^2}\,,\quad \hat e_3=t^{-1/2}e^{W}\frac{\partial}{\partial x^3}\,. \tag{2.7}$$

In particular, for the diagonal metrics,

$$R_{\hat2\hat3\hat2\hat3}=e^{-2a}\left[\frac{1}{4t^2}-\left(\frac{\partial W}{\partial t}\right)^2+\left(\frac{\partial W}{\partial \theta}\right)^2\right], \tag{2.8}$$

and we showed in Ref. 1 that, for all t such that $0<t\leq t_0$,

$$e^{-2a(t,\theta)}\geq\left(\frac{t}{t_0}\right)^{1/2}e^{-2a(t_0,\theta)}\,. \tag{2.9}$$

By considering the asymptotic forms of $J_0(nt)$ and $N_0(nt)$ as $t\to 0^+$ we can easily show that $R_{\hat2\hat3\hat2\hat3}$ diverges as

$$R_{\hat2\hat3\hat2\hat3}\sim e^{-2a}\left[\frac{1}{4t^2}-\frac{1}{t^2}\left(\beta+\sum_n\frac{2b_n}{\pi}\sin(n\theta+\delta_n)\right)^2\right], \tag{2.10}$$

unless

$$b_n=0,\quad \beta=\tfrac{1}{2}\,, \tag{2.11}$$

in which case

$$R_{\hat2\hat3\hat2\hat3}\sim e^{-2a}\left[\sum_n \tfrac{1}{2}n^2 a_n\sin(n\theta+\gamma_n) +\left(\sum_n a_n n\cos(n\theta+\gamma_n)\right)^2\right]. \tag{2.12}$$

For these special solutions one can sharpen (2.9) to show that $e^{-2a(t,\theta)}$ approaches a finite nonzero limit as $t\to 0^+$.

The significance of this result may be seen by evaluating the curvature invariant

$$C\equiv R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}R^{\hat\alpha\hat\beta\hat\gamma\hat\delta} =4[(R_{\hat2\hat3\hat2\hat3})^2-2(R_{\hat1\hat2\hat2\hat0})^2-2(R_{\hat1\hat3\hat3\hat0})^2+\cdots]\,, \tag{2.13}$$

where $+\cdots$ signifies a sum of additional nonnegative terms. If condition (2.11) is not satisfied, the components $R_{\hat1\hat2\hat2\hat0}$ and $R_{\hat1\hat3\hat3\hat0}$ behave as

$$R_{\hat1\hat2\hat2\hat0}=-R_{\hat1\hat3\hat3\hat0}\sim 3e^{-2a}\left(\frac{\ln t}{t}\right)\sum_n\frac{2b_n n}{\pi}\cos(n\theta+\delta_n)\left[+\tfrac{1}{4}-\left(\beta+\sum_m\frac{2b_m}{\pi}\sin(m\theta+\delta_m)\right)^2\right], \tag{2.14}$$

and thus cannot cancel the divergence of $R_{\hat{2}\hat{3}\hat{2}\hat{3}}$ in the curvature invariant. Thus if condition (2.11) is not satisfied, C blows up almost everywhere as $t \to 0^+$. This result was given earlier by Berger.[6]

If, however, condition (2.11) is satisfied then one can show by straightforward evaluation that all the components of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ have finite limits as $t \to 0^+$ along the curves of constant $\{x^i\}$. Furthermore, one can show directly that each of the curves of this normal congruence has bounded acceleration as $t \to 0^+$ and that the basis fields $\hat{e}_{(\mu)}$ define a Fermi-Walker transported frame along each of these curves. Thus one may regard the normal congruence as a preferred family of spacetime filling "observers" who fall into the crushing singularity with bounded acceleration without experiencing infinite tidal forces. One can also show that the (time-dependent) Lorentz transformation which carries $\hat{e}_{(\mu)}$ in a parallel-propagated basis along each world line of the normal congruence is well behaved as $t \to 0^+$. It follows that the components of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ in a parallel-propagated basis also have finite limits along each of the normal trajectories.

It is natural to ask whether these nonsingular solutions are extendible. The special, homogeneous case with $\beta = \frac{1}{2}$, $a_n = b_n = 0$ is the well-known flat Kasner solution

$$ds^2 = -dt^2 + t^2 d\psi^2 + d\theta^2 + d\phi^2 , \tag{2.15}$$

where we have put $x^2 = \psi$ and $x^3 = \phi$. This space is the product of a flat two-torus (with coordinates θ and ϕ) and Misner's flat, two-dimensional model for Taub space. The extension of Misner's model to give a two-dimensional analog of Taub-NUT space is discussed by Misner[7] and by Hawking and Ellis.[8] Evidently the flat Kasner model admits a corresponding extension—the extension being simply the product of Misner's model for Taub-NUT space with a flat two-torus. We shall now show that each of the nonsingular solutions discussed above has an analogous extension.

For the regular solutions W [i.e., those satisfying Eq. (2.11)] we define $b(t, \theta)$ by

$$W = \tfrac{1}{2} \ln t + b , \tag{2.16}$$

so that

$$\begin{aligned} ds^2 = {} & \exp(2a)(-dt^2 + d\theta^2) \\ & + t^2 \exp(2b) d\psi^2 + \exp(-2b) d\phi^2 , \end{aligned} \tag{2.17}$$

and b has the form

$$b(t, \theta) = \sum_n a_n J_0(nt) \sin(n\theta + \gamma_n) . \tag{2.18}$$

We may solve the remaining Einstein equations (2.3) and (2.4) by setting, for $t > 0$,

$$a(t, \theta) = b(t, \theta) + \int_0^t ds \left[s\left(\frac{\partial b(s, \theta)}{\partial s}\right)^2 + s\left(\frac{\partial b(s, \theta)}{\partial \theta}\right)^2 \right]. \tag{2.19}$$

The functions a and b satisfy

$$\lim_{t \to 0^+} (a(t, \theta) - b(t, \theta)) = 0 \tag{2.20}$$

and

$$\lim_{t \to 0^+} \left(\frac{\partial a(t, \theta)}{\partial t} - \frac{\partial b(t, \theta)}{\partial t} \right) = 0 . \tag{2.21}$$

Let us temporarily relax the (toroidal) coordinate identification in ψ and introduce new coordinates

$$\begin{aligned} & t' = t^2, \quad \psi' = 2\psi - 2 \ln t , \\ & \theta' = \theta, \quad \phi' = \phi \end{aligned} \tag{2.22}$$

in terms of which the line element becomes

$$\begin{aligned} ds^2 = {} & -\frac{1}{4t'} [\exp(2a) - \exp(2b)](dt')^2 \\ & + \tfrac{1}{2} \exp(2b) d\psi' dt' + \frac{t'}{4} \exp(2b)(d\psi')^2 \\ & + \exp(2a)(d\theta')^2 + \exp(-2b)(d\phi')^2 . \end{aligned} \tag{2.23}$$

Here, with a slight ambiguity of notation, we let a and b signify the original functions a and b reexpressed in the new coordinates so that

$$b(t', \theta') = \sum_n a_n J_0(nt'^{1/2}) \sin(n\theta' + \gamma_n) \tag{2.24}$$

and

$$\begin{aligned} a(t', \theta') = b(t', \theta') + \tfrac{1}{2} \int_0^{t'} ds' \Bigg[& 4s' \left(\frac{\partial b(s', \theta')}{\partial s'}\right)^2 \\ & + \left(\frac{\partial b(s', \theta')}{\partial \theta'}\right)^2 \Bigg]. \end{aligned} \tag{2.25}$$

From the power-series expansion of J_0,

$$J_0(nt'^{1/2}) = \sum_{k=0}^{\infty} \frac{[(-\frac{1}{4}n^2)t']^k}{(k!)^2} , \tag{2.26}$$

we see that $b(t', \theta')$ may be analytically extended to negative values of t' and is in fact defined by Eq. (2.24) for all $(t', \theta') \in (-\infty, +\infty) \times S^1$. The function $a(t', \theta')$ may also be analytically extended to $(-\infty, +\infty) \times S^1$ by defining it everywhere on this domain by Eq. (2.25).

Having extended the definitions of a and b we need only show that Eq. (2.23) defines a Lorentzian metric everywhere on the extended manifold. In view of Eqs. (2.20) and (2.21) and the analytic character of a and b, it is clear that the function $h \equiv [\exp(2a) - \exp(2b)]$ admits an expansion of the form

$$h(t', \theta') = \sum_{k=2}^{\infty} \rho_k(\theta') t'^k , \tag{2.27}$$

where the $\rho_k(\theta')$ are smooth functions of period 2π. It follows that

$$g_{t't'} = -\frac{1}{4t'} h(t', \theta') = -\tfrac{1}{4} \sum_{k=1}^{\infty} \rho_{k+1}(\theta') t'^k \tag{2.28}$$

is smooth on the extended manifold. The other metric components are obviously smooth since a and b are. Finally, one can easily show that (2.23) has Lorentzian signature everywhere on the extended manifold.

We now identify points with fixed t', θ', and ϕ' with $\psi' = \pm\pi$. The resulting models have the topology $T^3 \times R$ and each contains a region (defined by $t' > 0$) isometric to its unextended counterpart in the original Gowdy family. In each of the extended models the curves of the original normal congruence wrap infinitely many times around the torus as $t' \to 0^+$ and do not extend through the Cauchy horizon at $t' = 0$. The extended models all have closed timelike lines in the region $t' < 0$. Since we have extended the metrics analytically, it is clear that the vacuum Einstein equations are satisfied throughout the extensions. A second, inequivalent family of extensions could be defined by taking $\psi' = 2\psi + 2 \ln t$ and proceeding as above.

One of the aims underlying Ref. 1 was the hope to prove the inextendibility of the generic (in general, nondiagonal) Gowdy metric beyond its maximal Cauchy development. The present examples show, however, that one cannot simply apply estimates to show that the curvature must blow up at the crushing boundary. A more subtle argument seems to be needed to cover the general case. Perhaps one can argue that the linear perturbations of any particular solution with a crushing singularity always admit some irregular solutions (indicating a curvature blowup) and then appeal to linearization stability arguments for the conclusion in the nonlinear case.

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