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Uniqueness of SU(5) and SO(10) grand unified theories

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We will prove that SU(5) and SO(10) grand unified theories are almost uniquely chosen under a few reasonable assumptions. The main ansatz is that there exist only left-handed particles in an SU(2) doublet and their antiparticles in an SU(2) singlet. For the case of a single-multiplet unification, SO(10) is unique, using the facts that the charge of the d quark is not neutral and the theory should be anomaly free. For the case of two-multiplet unification, SU(5) is unique among SU(N) ($N \leq 500$). The color group is effectively determined as the standard SU(3) with triplets of colored quarks. Quark charges must be the standard 2/3 and $-1/3$. No unifying simple group exists for G(horizontal) $\times G$(GUT) under the assumptions above. For the case of SU(5), the cancellation of the triangle anomaly emerges as a consequence rather than a hypothesis.

I. INTRODUCTION AND SUMMARY OF MAIN RESULTS

Among many models of grand unified theories (GUT's) of strong, weak, and electromagnetic interactions,[1] the original SU(5) (Ref. 2) and its extension SO(10) (Ref. 3) are the most popular ones. However, some of the consequences of grand unification have common features. For example, the baryon decay has almost always the property $\Delta(B-L)=0$, where B (L) denotes the baryon (lepton) number. This is because any local operator with its dimension 6 has this property.[4] This implies that it is a very difficult task to experimentally discriminate among various models of grand unification. On the theoretical side, the choice of a particular model appears to be a matter of personal taste.

Recently, Georgi[5] has provided an interesting way of looking at unification. He has posed a question of how unique the SU(5) group could be. Of course, we know that SU(5) is uniquely chosen among simple groups of rank 4.[2] It turns out that the color group plays an important role: If the color group is not SU(3) but SU(n) ($n \neq 3$), then we cannot unify the electron family. Note that in his argument the electric charges of quarks cannot assume values $\frac{2}{3}$ and $-\frac{1}{3}$, except for the case of $n=3$.

The purpose of this paper is to generalize this result by Georgi. Under some general conditions, we will show first that SO(10) [SU(5)] must be the essentially unique choice for the grand unification group for a single- [two] irreducible-multiplet unification. Secondly, the color group must be SU(3), and the quarks must have the standard fractional charges, $\frac{2}{3}$ and $-\frac{1}{3}$. Third, no simple group can unify the family structures, i.e., no simple group can accommodate G(horizontal)$\otimes G$(GUT). For SU(5), the absence of triangle anomaly emerges as a consequence.

In order to facilitate our discussions in the following sections, we will now state the following assumptions:

(1) The grand unified group G is a simple compact Lie group, and contains $G(\text{color})\otimes SU(2) \otimes U(1)$ as its subgroup. Here, the color group is not assumed to be SU(3), while $SU(2)\otimes U(1)$ refers to the standard Glashow-Weinberg-Salam group for unified electroweak interaction.

(2) The third component of the weak i-spin operator I_3 of the weak group SU(2) and the weak hypercharge operator Y of the weak U(1) group are elements of a Cartan subalgebra H of the Lie algebra L of G. Moreover, the electric charge operator Q is specified by[6]

$$Q=I_3+Y. \tag{1.1}$$

Here, a Cartan subalgebra implies a set consisting of maximal number of mutually commuting elements in L.

(3) Leptons and quarks denoted as N, E, U, and D have the following standard assignment of I_3 and Y as in Table I. Leptons (N_L, E_L) and antileptons (N_L^c, E_L^c) belong to color singlet representations, while quarks (U_L, D_L) and antiquarks (U_L^c, D_L^c) belong to an m-dimensional representation and its conjugate representation of the color group, respectively. We do not assume that the quark rep-

TABLE I. Quantum number assignments for leptons and quarks.

	N_L	E_L	E_L^C	N_L^C	U_L	D_L	U_L^C	D_L^C
I	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
Y	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	y	y	$-(y+\frac{1}{2})$	$\frac{1}{2}-y$
G_C	1	1	1	1	m	m	$\overline{m}$	$\overline{m}$
n	n_l	n_l	n_l	n_0	mn_q	mn_q	mn_q	mn_q

resentation of the color group belongs to the basic (i.e., lowest dimensional) representation. In Table I, y is an unknown number to be determined.[7] Also, n in the last column indicates the number of leptons and quarks of a given type contained in a given representation of G. It is assumed that only these particles listed in Table I form a basis of a representation of G, which we call $\{\rho\}$. It is not yet assumed that the representation is irreducible. Then, the dimension $d(\rho)$ of the representation is given by

$$d(\rho)=3n_l+n_0+4mn_q\,. \tag{1.2}$$

It is also assumed that $n_q\neq 0$, $m\neq 0$, and $n_l\neq 0$, although n_0 could be zero. Since we want to discuss the case for family unification, we never specify the relations among quarks and leptons, i.e., we never assume that $n_q=n_l$.

(4) The theory has no triangle anomaly[8] so as to make it renormalizable. This assumption will *not* be made, however, in Sec. IV for the uniqueness of the SU(5) group.

(5) No quark has zero electric charge. Leptons N and E have electric charges of 0 and -1, respectively.

Alternatively, we discuss the case of the following stronger ansatz:

(5′) Quarks cannot have integral electric charges. Leptons N and E have electric charges of 0 and -1, respectively.

Before going into detail, we note the following: The ansatz (5) and (5′) may create an asymmetry between leptons and quarks, since a neutrino N can be a Majorana particle but a quark cannot. Especially, there is no reason why we have to assume $n_l=n_0$. (Indeed, n_0 could be zero.) If we have $n_l=n_0$, then we have a certain symmetry between leptons and quarks as has been postulated earlier[9] for the Sakata model.

As a result of the five assumptions (1)–(5) listed above, we can prove the following facts:

(a) If all particles listed in Table I form a single irreducible representation $\{\rho\}$, then the only possible candidate for grand unification is SO(10) [or more accurately spin(10)]. Moreover, the dimension $d(\rho)$ of the representation is uniquely determined to be 16, corresponding to the fundamental spinor (or its complex conjugate) representation. We have either

(i) $n_l=n_0=n_q=1,\ m=3,\ Q_U=\frac{2}{3},\ Q_D=-\frac{1}{3}$,

(ii) $n_l=n_0=1,\ n_q=3,\ m=1,\ Q_U=\frac{2}{3},\ Q_D=-\frac{1}{3}$,

(iii) $n_l=n_0=3,\ n_q=1,\ m=1,\ Q_U=2,\ Q_D=1$.

The stronger ansatz (5′) yields that the case (iii) is not allowed. The additional requirement that the representation of the color group cannot contain trivial representations will yield that the case (ii) and (iii) are forbidden. Note that the color group must be either SU(3) or SU(2) for (i). The examination of the decomposition of SO(10) into SU(2)⊗SU(2)⊗U(1) shows the incompatibility of the SU(2) color group. Of course, the SU(2) color group is also experimentally forbidden in view of the quark-line rule as we noted elsewhere.[10] Then, only the case (i) with the SU(3) color group is allowed. In this sense, the fact that quarks are color SU(3) triplets with the standard electric charges can be said to be consequences of grand unification.

(b) If we assume that all particles in Table I now form a reducible representation of G, which is a direct sum of two irreducible multiplets, then we can conclude the following: First, quarks have the standard fractional electric charges $\frac{2}{3}$ and $-\frac{1}{3}$. Second, G must be one of the SU(N) ($N\geq 3$) group. If we restrict ourselves to cases of $N\leq 500$, then G is uniquely determined to be SU(5) with the representation $\{\rho\}=\{\bar{5}\}\oplus\{10\}$ (or its complex conjugate representation). Alternatively, if we assume $n_0=0$ from the beginning, then we can prove the same without restriction on the value of N. Third, the color group must be SU(3) with quarks in color-triplet states. In reaching these conclusions, we need not assume the ansatz (4). The cancellation of the triangle anomaly is a consequence of other hypotheses.

(c) Consequently, no simple group can accommodate the family structure in its multiplet. If a simple group is used, the family structure is a simple repetition. Similarly, if $\{\rho\}$ consists of more than two irreducible representations, then G is either SO(10) or SU(5) and that the representations are simple repetitions of those stated in (a) and (b).

We prove these facts in the following sections. Section II is devoted to general consequences of the assumptions (1)–(5). We will prove the result (a) in Sec. III, while Sec. IV is devoted to the discussion of (b).

In the following, we use the same lexicographical ordering of a simple root system for the Lie algebra L of G as in Ref. 11. Similarly, $\{\Lambda_1,\Lambda_2,\ldots,\Lambda_n\}$ designate the corresponding fundamental weight system, where n is now the rank of the Lie algebra L. Then, any irreducible representation with its highest weight Λ will be characterized by n nonnegative integers, $(m_1,m_2,\ldots,m_n)$ with

$$\Lambda=m_1\Lambda_1+m_2\Lambda_2+\cdots+m_n\Lambda_n\,. \tag{1.3}$$

II. GENERAL CONSIDERATION

Let x be a generic element of the Lie algebra L of the simple group G. Let X be its representation matrix in a representation $\{\rho\}$, which may be reducible. Then, since G is simple, we must have

$$\mathrm{Tr}X=0\,. \tag{2.1}$$

We identify the representation $\{\rho\}$ with the lepton-quark multiplet specified in the previous section. Then, the condition (2.1) is automatically satisfied for $X=I_3$ or Y. Next, let us consider

$$\mathrm{Tr}X^3=0\,. \tag{2.2}$$

This relation holds automatically for any simple group except SU(N) ($N\geq 3$) (Ref. 12). (See Appendix A.) For SU(N) ($N\geq 3$), the validity of Eq. (2.2) is equivalent to the absence of triangle anomaly.[12,13] Therefore, Eq. (2.2) must hold for any grand unification group G. Since X is an arbitrary element of the Lie algebra L, we may choose

$$X=Y+tI_3 \tag{2.3}$$

for arbitrary real or complex number t. Then, Eq. (2.2) gives us

$$\mathrm{Tr}Y^3=\mathrm{Tr}(I_3)^3=\mathrm{Tr}(YI_3{}^2)=\mathrm{Tr}(Y^2I_3)=0\,. \tag{2.4}$$

We can compute $\mathrm{Tr}Y^3$, for example, from Table I to find that y must satisfy

$$y=\alpha/2m\,. \tag{2.5}$$

Here, we have defined α by

$$\alpha=n_l/n_q\,. \tag{2.6}$$

Then the rest of Eq. (2.4) are easily shown to be identically satisfied.[14] This, of course, reproduces the result by Georgi for $n_l=n_q$.[5] We note that n_0 is undertermined.

In view of Eqs. (2.5) and (2.6), we rewrite the content of Table I in the following standard notation:

$$\begin{aligned}
&n_l(N_L,E_L):\ (1,2,-\tfrac{1}{2})\,,\\
&n_lE_L^c:\ (1,1,1)\,,\\
&n_0N_L^c:\ (1,1,0)\,,\\
&n_q(U_L,D_L):\ (m,2,\alpha/2m)\,,\\
&n_qU_L^c:\ (\overline{m},1,-(m+\alpha)/2m)\,,\\
&n_qD_L^c:\ (\overline{m},1,(m-\alpha)/2m)\,,
\end{aligned} \tag{2.7}$$

where a,b,c in the parentheses (a,b,c) denote the representations for G(color), SU(2), and U(1), respectively.

Next, following the method utilized elsewhere[15,16] on a similar problem, let us suppose we have the validity of the quintuple trace identity

$$\mathrm{Tr}X^5=0\,. \tag{2.8}$$

Using Eq. (2.3) and the arbitrariness of t, we have

$$\mathrm{Tr}Y^5=\mathrm{Tr}(Y^3I_3^2)=0 \tag{2.9}$$

since other conditions

$$\mathrm{Tr}(I_3)^5=\mathrm{Tr}(I_3)^4Y=\mathrm{Tr}(I_3)^3Y^2=\mathrm{Tr}I_3Y^4=0$$

are automatically satisfied. From Table I or Eqs. (2.7), we calculate

$$\begin{aligned}
&\mathrm{Tr}Y^5=5n_l(m^2-\alpha^2)/8m^2\,,\\
&\mathrm{Tr}Y^3I_3^2=n_l(\alpha^2-m^2)/16m^2\,.
\end{aligned} \tag{2.10}$$

Therefore, if we have Eq. (2.8), it is required to give $m=\alpha$. Then the charge eigenvalues of U_L and D_L are 1 and 0, respectively. However, this contradicts the fifth assumption of the previous section. In this way we conclude that we cannot have the validity of Eq. (2.8).

The first consequence of the impossibility of Eq. (2.8) is that the representation $\{\rho\}$ under consideration cannot be self-contragredient. Let X^T be the transpose matrix of X, and suppose that there exists a nonsingular matrix S such that

$$X^T=-S^{-1}XS\,. \tag{2.11}$$

Then, we have

$$\mathrm{Tr}X^{2l+1}=0\quad(l=\text{non-negative integer}) \tag{2.12}$$

for any odd-power trace, and especially the validity of Eq. (2.8). If the representation is irreducible, then S must obey furthermore a condition

$$S^T=S\ \text{or}\ S^T=-S$$

in view of Schur's lemma. Any irreducible representation satisfying these conditions is called real (or orthogonal) or pseudoreal (or symplectic), depending on the two signs in $S^T=\pm S$. Whether a given irreducible representation is real, pseudoreal, or neither (the case which is often called complex) is known and listed by various authors.[17] Especially, all irreducible representations of any simple Lie groups except SU(N) ($N\geq 3$), E_6, and SO($4N+2$) ($N\geq 1$) are known to be self-contragredient. Since any representation of a simple Lie algebra is fully reducible by the Weyl theorem, this implies that any representation of these algebras is also self-contragredient, leading to the validity of Eq. (2.12) and hence of Eq. (2.8). Therefore, we conclude that the possible group G is one of SU(N) ($N\geq 3$), E_6, and SO($4N+2$) ($N\geq 1$) and that we must use a non-self-contragredient representation(s) for our multiplet(s). Since the SO(6) group is locally isomorphic to SU(4), we consider hereafter only cases of SO($4N+2$) ($N\geq 2$) for orthogonal groups.

Actually, in this discussion, we only need the validity of Eq. (2.8), but not the stronger Eq. (2.12). Then we can eliminate all orthogonal groups

$SO(4N+2)$ with $N \geq 3$, leading only to SO(10) as follows. We recall the fact that the validity of $\mathrm{Tr}X^3 = 0$ is intimately connected with the absence of the third-order Casimir invariant $I_3(\rho)$.[12,18] Since the $SO(4N+2)$ group has only one fundamental odd-order Casimir invariant of order $2N+1$,[19,20] we conclude that only SO(10) among $SO(4N+2)$ $(N \geq 2)$ can have a fifth-order Casimir invariant. This implies the validity of $\mathrm{Tr}X^5 = 0$ for all $SO(4N+2)$ except SO(10) (Ref. 21) as has already been noted in Ref. 15. For more details, see Appendix B. Physically, the use of $SO(4N+2)$ $(N \geq 3)$ would include particles with opposite chirality which are not present in Table I. This is the only way to satisfy $\mathrm{Tr}X^5 = 0$. These particles, if we include them, have to be assigned large masses.

Summarizing, we have found that only E_6, SO(10), and $SU(N)$ $(N \geq 3)$ are acceptable as candidates for grand unification. Furthermore, the representation to be used must be non-self-contragredient.

We may remark that in deriving the conclusion above, we did not really utilize the part of the assumption (2) in the previous section that I_3 and Y are elements of a Cartan subalgebra of the Lie algebra of G. When we assume this, we can say more on the representation to be used. We note from Table I that the third component I_3 of the weak i spin can assume only three eigenvalues $\frac{1}{2}, -\frac{1}{2}, 0$ in $\{\rho\}$. Fully reducing $\{\rho\}$ into its irreducible components, we see that each irreducible representation must have this property. Then, generalizing the result stated in Ref. 11, we will prove in Appendix C that the highest weight of each irreducible component must be of the form, either $\Lambda = \Lambda_j$ or $\Lambda = \Lambda_j + \Lambda_k$ $(1 \leq j, k \leq n)$ in terms of n fundamental weights Λ_j for the Lie algebra L of rank n.

We remark that the condition $\mathrm{Tr}X^5 \neq 0$ together with $\mathrm{Tr}X^3 = 0$ is equivalent to the nonvanishing of pentagonal fermion loop diagrams, although its physical meaning is unknown to us.

III. UNIQUENESS OF THE SO(10) GROUP

As has been demonstrated in Sec. II, one need consider only SO(10), E_6, and $SU(N)$ $(N \geq 3)$. Demanding further that the multiplet must belong to *a single irreducible representation*, it can be proven that only SO(10) is tenable under assumptions stated in Sec. I. This section is aimed at proving this statement, as well as the dimension of the representation.

First, we dispose of the case of $SU(N)$ $(N \geq 3)$. As we noted in Sec. II, the irreducible representation $\{\rho\}$ must have the highest weight, either Λ_j or $\Lambda_j + \Lambda_k$ for some $1 \leq j,\ k \leq N-1$. However, for $SU(N)$, only the case of $\Lambda = \Lambda_j$ is possible because of the following reason: From Table I or Eq. (2.7), our representation contains only doublets and singlets with respect to the weak i-spin group SU(2). Then, by the same reasoning given by Gell-Mann, Ramond, and Slansky[22] on a similar problem, the representation must be completely antisymmetric.

Indeed, consider the basic (i.e., the lowest dimensional) representation of $SU(N)$. Decomposing the representation content as to its SU(2) subgroup, it must be a direct sum of various i-spin states. Since any irreducible representation of $SU(N)$ can be obtained from products of basic representations by means of a Young tableau, the basic representation cannot contain any i-spin state higher than $\frac{1}{2}$. Otherwise, any irreducible representation and hence our $\{\rho\}$ would contain such higher i-spin states, which is in contradiction to our ansatz. Similarly, the basic representation cannot contain either zero i-spin only or i-spin-$\frac{1}{2}$ states only. It must contain both i-spin-0 and i-spin-$\frac{1}{2}$ states.[23] Then, by the same reasoning, we can be assured of the fact that only completely antisymmetric states contain i-spin states 0 and i-spin-states $\frac{1}{2}$, but nothing else. This fact can also be proved as follows: Let B^μ_ν $(\mu, \nu = 1, 2, \ldots, N)$ be generators of the $SU(N)$ group, so that they satisfy

$$[B^\mu_\nu, B^\alpha_\beta] = \delta^\mu_\beta B^\alpha_\nu - \delta^\alpha_\nu B^\mu_\beta \quad (\mu, \nu, \alpha, \beta = 1, 2, \ldots, N), \qquad \sum_{\mu=1}^{N} B^\mu_\mu = 0 . \tag{3.1}$$

Then, I_3 may be identified as

$$I_3 = ({B_1}^1 - {B_2}^2)/2 \tag{3.2}$$

after a suitable reordering of indices. Applying the action of B^μ_ν to tensor representations as in Ref. 15, it can readily be seen that eigenvalues of I_3 would involve values larger than $\frac{1}{2}$, if the representation is not completely antisymmetric. This proves the desired result again.

Now, the eigenvalue of the third-order Casimir invariant $I_3(\Lambda)$ for the $SU(N)$ group is well known.[12] For convenience, we reproduce it in Appendix A. Especially, for the fundamental (i.e., antisymmetric) representation $\Lambda = \Lambda_j$, we find

$$I_3(\Lambda_j) = (N+1)(N+2)j(N-j)(N-2j)/4N^2 \quad (1 \leq j \leq N-1) . \tag{3.3}$$

Since the anomaly coefficient is proportional to $I_3(\Lambda)$, the absence of the triangle anomaly requires $I_3(\Lambda) = 0$ so that it is necessary to have $j = N/2$ for N being an even integer. However, $\Lambda = \Lambda_j$ for such j corresponds to the fact that the representation is self-contragredient, since the j-th simple root α_j is invariant under the left-right symmetric interchange of the Dynkin diagram for $SU(N)$. Thus, $SU(N)$ is excluded from the candidates.

Next, let us investigate the case of E_6. It has been noted elsewhere[24] that all exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8 as well as A_1 and A_2 satisfy the quartic trace identity

$$\mathrm{Tr}X^4 = K(\rho)(\mathrm{Tr}X^2)^2 \tag{3.4}$$

for any irreducible representation $\{\rho\}$ and for any generic element X of these Lie algebras. Here, $K(\rho)$ is defined by

$$K(\rho) = \frac{d(\rho_0)}{2[2+d(\rho_0)]d(\rho)}\left[6 - \frac{I_2(\rho_0)}{I_2(\rho)}\right], \tag{3.5}$$

where $\{\rho_0\}$ designates the adjoint representation, and $I_2(\rho)$ and $d(\rho)$ are the eigenvalues of the second-order Casimir invariant and the dimension of the representation $\{\rho\}$, respectively. As we noted in Ref. 24, the validity of Eq. (3.4) is intimately related to the fact that none of these Lie algebras has a genuine fundamental fourth-order Casimir invariant. Since X is arbitrary, we set $X = Y + tI_3$ for an arbitrary constant t as in Eq. (2.3). Then, Eq. (3.4) implies the validity of

$$\frac{\mathrm{Tr}I_3^{\,4}}{(\mathrm{Tr}I_3^{\,2})^2} = \frac{3\mathrm{Tr}I_3^{\,2}Y^2}{(\mathrm{Tr}I_3^{\,2})(\mathrm{Tr}Y^2)} = \frac{\mathrm{Tr}Y^4}{(\mathrm{Tr}Y^2)^2} = K(\rho) \tag{3.6}$$

when we note

$$\mathrm{Tr}(I_3Y^3) = \mathrm{Tr}(I_3^{\,3}Y) = \mathrm{Tr}(I_3Y) = 0$$

for the representation $\{\rho\}$. From Table I or Eqs. (2.7), we calculate traces in Eq. (3.6) to find

$$\begin{aligned}\frac{1}{m+\alpha} &= \frac{3\alpha}{(m+\alpha)(m+2\alpha)}\\ &= \frac{m^4+9\alpha m^3+6\alpha^2m^2+2\alpha^4}{m(m+\alpha)^2(m+2\alpha)^2} = 2n_q K(\rho)\,. \end{aligned}\tag{3.7}$$

The first two equations demand

$$m = \alpha\,,$$

which requires the zero electric charge for the D quark. This is contrary to our assumption. Hence, E_6 is also ruled out. We have finally found that SO(10) is the only surviving candidate.

The remaining task is to determine which representation of SO(10) should be realizable as the multiplet $\{\rho\}$. As we shall see in Appendix C, the highest weight Λ of any realizable irreducible representation must be one of the following types,

$$\Lambda_4,\ \Lambda_5,\ 2\Lambda_4,\ 2\Lambda_5,\ \Lambda_1+\Lambda_4,\ \Lambda_1+\Lambda_5\,.$$

However, pairs of irreducible representations (Λ_4, Λ_5), $(2\Lambda_4, 2\Lambda_5)$, and $(\Lambda_1+\Lambda_4,\ \Lambda_1+\Lambda_5)$ are complex conjugates of each other, because of the corresponding Dynkin-diagrammatic symmetry for exchange of two simple roots α_4 and α_5. Therefore, we discuss, for a while, only Λ_5, $2\Lambda_5$, and $\Lambda_1+\Lambda_5$. The dimensions of these representations are given by

$$d(\Lambda_5) = 16,\quad d(2\Lambda_5) = 126,\quad d(\Lambda_1+\Lambda_5) = 144\,.$$

Now, we prove the impossibility of the last two cases. Both SO(10) and E_6 do not possess fundamental third- and seventh-order Casimir invariants. Because of this, any seventh-order Casimir invariant must be a product of the second- and fifth-order Casimir invariants. In this way, we can prove the identity[24,25] (see also Appendix B)

$$\mathrm{Tr}X^7 = D(\rho)\mathrm{Tr}X^2\mathrm{Tr}X^5\,, \tag{3.8}$$

where

$$D(\rho) = \frac{35d(\rho_0)}{4[10+d(\rho_0)]d(\rho)}\left[\frac{12}{5} - \frac{I_2(\rho_0)}{I_2(\rho)}\right]. \tag{3.8$'$}$$

Setting $X = I_3 + tY$, this leads to

$$\begin{aligned}\frac{\mathrm{Tr}Y^2\mathrm{Tr}Y^5}{\mathrm{Tr}Y^7} &= \frac{10\mathrm{Tr}Y^2\mathrm{Tr}Y^3I_3^{\,2} + \mathrm{Tr}I_3^{\,2}\mathrm{Tr}Y^5}{21\mathrm{Tr}Y^5I_3^{\,2}}\\ &= \frac{2}{7}\,\frac{\mathrm{Tr}I_3^{\,2}\mathrm{Tr}Y^3I_3^{\,2}}{\mathrm{Tr}Y^3I_3^{\,4}} = \frac{1}{D(\rho)}\end{aligned}\tag{3.9}$$

when we note

$\mathrm{Tr}YI_3^{2l} = \mathrm{Tr}Y^lI_3^{2k+1} = 0$ (l, k: non-negative integers).

We evaluate

$$\mathrm{Tr}I_3^{\,2} = \tfrac{1}{2}n_q(m+\alpha)\,,$$

$$\mathrm{Tr}Y^2 = \frac{1}{2m}n_q(m+\alpha)(m+2\alpha)\,,$$

$$\mathrm{Tr}Y^5 = \frac{5}{8}\,\frac{\alpha n_q}{m^2}(m^2-\alpha^2)\,,$$

$$\mathrm{Tr}Y^7 = \frac{7}{64}\,\frac{\alpha n_q}{m^4}(m^2-\alpha^2)(8m^2+3\alpha^2)\,,$$

$$\mathrm{Tr}Y^3I_3^{\,2} = \frac{1}{16}\,\frac{\alpha n_q}{m^2}(\alpha^2-m^2)\,,$$

$$\mathrm{Tr}Y^5I_3^{\,2} = \frac{1}{64}\,\frac{\alpha n_q}{m^4}(\alpha^2-m^2)(\alpha^2+m^2)\,,$$

$$\mathrm{Tr}Y^3I_3^{\,4} = \frac{1}{64}\,\frac{\alpha n_q}{m^2}(\alpha^2-m^2)\,.$$

Then, the first two equations in (3.9) give the same equation:

$$(m^2-\alpha^2)(3\alpha-m)(\alpha-3m) = 0\,.$$

Since $m^2 \neq \alpha^2$, we must have either

$$\text{(I)}\quad m = 3\alpha \tag{3.10}$$

or

$$\text{(II)}\quad \alpha = 3m\,. \tag{3.11}$$

Inserting this into Eq. (3.9), we find

$$\frac{1}{D(\rho)} = \tfrac{4}{7}n_q(m+\alpha) = \begin{cases}\tfrac{16}{21}mn_q & \text{for (I)},\\[4pt] \tfrac{16}{7}mn_q & \text{for (II)}.\end{cases} \tag{3.12}$$

Now the dimension of the representation is given by

$$d(\rho)=(3\alpha+4m)n_q+n_0=\begin{cases}5mn_q+n_0 & \text{for (I)}\\ 13mn_q+n_0 & \text{for (II)}.\end{cases} \tag{3.13}$$

Since the value of $D(\rho)$ is determined by Eq. (3.8)′ from properties of the representation $\{\rho\}$ alone, we can test whether Eqs. (3.12) and (3.13) are consistent with integer character of m, n_0, and n_q. The answer is negative for $\Lambda=2\Lambda_5$ and $\Lambda=\Lambda_1+\Lambda_5$. However, for $\Lambda=\Lambda_5$ (and hence also for $\Lambda=\Lambda_4$), we find the following three sets of solutions:

I(a): $m=3,\quad \alpha=1,\quad n_l=n_0=n_q=1,$

I(b): $m=1,\quad \alpha=\frac{1}{3},\quad n_l=n_0=1,\quad n_q=3,$

II: $m=1,\quad \alpha=3,\quad n_l=n_0=3,\quad n_q=1.$

As we noted in Sec. I, the solutions I(a) and I(b) give the standard electric charge assignment of $Q=\frac{2}{3}$ and $-\frac{1}{3}$ for U_l and D_L, respectively. Also, only the case I(a) corresponds to nontrivial color realization for the color group. Therefore, only the solution I(a) is physically viable. Moreover, if we insist that the color representation for each quark is irreducible, then only SU(2) and SU(3) are admissible as the color group, since only these groups can have an irreducible representation with its dimension $m=3$. However, we see that there exist *no* decomposition of SO(10) into SU(2) $\otimes$SU(2)$\otimes$U(1) which has only SU(2) color triplets and singlets.[26] Another way to see the incompatibility of SU(2) as the color group is to use Eq. (3.8) for $X=X_C+ty$, where $X_C\in G(\text{color})=\text{SU}(2)$. Therefore, the color group must be SU(3) and quarks must belong to its three-dimensional basic representation. Of course, this assignment is the standard one. [The use of a reducible representation, $2\oplus 1$, for the SU(2) color group leads to no inconsistency, however.]

Alternatively, if we use the information on the quark-line rule,[10,27] then it is easy to show that the viable color group is only SU(3) by examining subgroups of SO(10). Then, using the decomposition of SO(10) into SU(3)$\otimes$SU(2)$\otimes$U(1), the impossibility of representations, $\Lambda_1+\Lambda_5$ or $2\Lambda_5$, are manifested again by the appearance of SU(2) triplets, which are not in our multiplet structure.

IV. UNIQUENESS OF SU(5)

We will now consider the case where our multiplet forms a direct sum of two irreducible representations $\{\rho_1\}$ and $\{\rho_2\}$, i.e.,

$$\{\rho\}=\{\rho_1\}\oplus\{\rho_2\}. \tag{4.1}$$

Following Georgi,[5] we discuss only cases where given leptons or quarks with the same Y and the same i spin will not split into two or more different representations, except possibly N_L^C. Then we have the six possibilities shown in Table II. However, as we shall see shortly, all cases except for case 1 in Table II will be ruled out.

As we emphasized in Sec. I, we do not assume the absence of the triangle anomaly in this section. Actually, it will emerge as a consequence of other postulates.

Since $\{\rho_1\}$ and $\{\rho_2\}$ are irreducible representations of L, we must have

$$\text{Tr}^{(1)}X=\text{Tr}^{(2)}X=0, \tag{4.2}$$

where $\text{Tr}^{(i)}$ designates the trace over the representation space $\{\rho_i\}$.

Next, let X_μ $(\mu=1,2,\ldots,p)$ be a basis of the simple Lie algebra L in an irreducible representation $\{\lambda\}$. Then, it is well known (e.g., see Ref. 12) that we have

$$\text{Tr}X_\mu X_\nu=\frac{d(\lambda)}{d(\rho_0)}I_2(\lambda)g_{\mu\nu},$$

where $g_{\mu\nu}$ is the Killing form. Since any element X can be expressed as

$$X=\sum_{\mu=1}^{p}\xi^\mu X_\mu \tag{4.3}$$

for some real or complex numbers ξ^μ, we find

$$\text{Tr}X^2=\frac{d(\lambda)}{d(\rho_0)}I_2(\lambda)\sum_{\mu,\nu=1}^{p}g_{\mu\nu}\xi^\mu\xi^\nu. \tag{4.4}$$

TABLE II. Possible two-multiplet structures.

	$\{\rho_1\}$	$\{\rho_2\}$
1.	$n_l(N_L,E_L),n_0^{(1)}N_L^C,n_qD_L^C$	$n_lE_L^C,n_0^{(2)}N_L^C,n_q(U_L,D_L),n_qU_L^C$
2.	$n_l(N_L,E_L),n_0^{(1)}N_L^C,n_qU_L^C$	$n_lE_L^C,n_0^{(2)}N_L^C,n_q(U_L,D_L),n_qD_L^C$
3.	$n_lE_L^C,n_0^{(1)}N_L^C,n_q(U_L,D_L)$	$n_l(N_L,E_L),n_0^{(2)}N_L^C,n_qU_L^C,n_qD_L^C$
4.	$n_lE_L^C,n_0^{(1)}N_L^C,n_qU_L^C,n_qD_L^C$	$n_l(N_L,E_L),n_0^{(2)}N_L^C,n_q(U_L,D_L)$
5.	$n_lE_L^C,n_0^{(1)}N_L^C,n_qD_L^C$	$n_l(N_L,E_L),n_0^{(2)}N_L^C,n_q(U_L,D_L),n_qU_L^C$
6.	$n_lE_L^C,n_0^{(1)}N_L^C,n_qU_L^C$	$n_l(N_L,E_L),n_0^{(2)}N_L^C,n_q(U_L,D_L),n_qD_L^C$

If X is chosen to be Hermitian, then $\mathrm{Tr}X^2$ can never vanish. Since Eq. (4.4) holds for any irreducible representation, we especially have

$$\frac{\mathrm{Tr}^{(2)}X^2}{\mathrm{Tr}^{(1)}X^2}=\frac{d(\rho_2)I_2(\rho_2)}{d(\rho_1)I_2(\rho_1)}. \tag{4.5}$$

Letting $X=I_3+tY$ for an arbitrary parameter t, this leads to

$$\frac{\mathrm{Tr}^{(2)}Y^2}{\mathrm{Tr}^{(1)}Y^2}=\frac{\mathrm{Tr}^{(2)}I_3{}^2}{\mathrm{Tr}^{(1)}I_3{}^2}=\frac{d(\rho_2)I_2(\rho_2)}{d(\rho_1)I_2(\rho_1)}. \tag{4.6}$$

After this preparation, let us first prove the impossibility of cases 2–6 in Table II. For cases 4, 5, and 6 we evidently have $\mathrm{Tr}^{(1)}(I_3)^2=0$, but $\mathrm{Tr}^{(2)}(I_3)^2\neq 0$. Therefore, Eq. (4.6) requires $\mathrm{Tr}^{(1)}Y^2=0$ which is easily seen to be impossible. For case 2, Eq. (4.2) with $X=Y$ gives

$$y=-(m+2\alpha)/2m$$

if we use the assignment of Y as in Table I. Then, Eq. (4.6) is rewritten as

$$\frac{3m^2+10m\alpha+6\alpha^2}{\alpha(m+2\alpha)}=\frac{m}{\alpha}=\frac{d(\rho_2)I_2(\rho_2)}{d(\rho_1)I_2(\rho_1)},$$

which leads to a contradictory equation

$$(m+3\alpha)(m+\alpha)=0.$$

For case 3, $\mathrm{Tr}^{(1)}Y=0$ gives

$$y=-\alpha/2m.$$

Then, from an analogous consideration of Eq. (4.6), we find

$$(m^2-\alpha^2)(m+\alpha)=0,$$

so that we conclude $m=\alpha$ and $y=-\frac{1}{2}$. But the electric charge of the U quark would then be

$$Q(U_L)=I_3+Y=\tfrac{1}{2}+(-\tfrac{1}{2})=0,$$

which contradicts our ansatz 5 in Sec. I. In this way, we have ruled out all cases 2-6, except case 1.

Therefore, we consider case 1 in Table II. The condition $\mathrm{Tr}^{(1)}Y=0$ now requires

$$y=(m-2\alpha)/2m \tag{4.7}$$

so that Eq. (4.6) becomes

$$\frac{3(m^2-2m\alpha+2\alpha^2)}{\alpha(m+2\alpha)}=\frac{m}{\alpha}=\frac{d(\rho_2)I_2(\rho_2)}{d(\rho_1)I_1(\rho_1)}.$$

This is possible only if we have

$$m=3\alpha,\quad y=\tfrac{1}{6} \tag{4.8}$$

as well as

$$d(\rho_2)I_2(\rho_2)=3d(\rho_1)I_2(\rho_1). \tag{4.9}$$

We note that the validity of Eqs. (4.8) implies fractional charges $\frac{2}{3}$ and $-\frac{1}{3}$ for U and D quarks, respectively. Here, we discarded an alternative solution, $m=\alpha$ and $y=-\frac{1}{2}$, since then $Q(U)=0$.

Next, we calculate

$$\mathrm{Tr}^{(1)}Y^3=2n_l(-\tfrac{1}{2})^3+mn_q(\tfrac{1}{2}-y)^3$$
$$=-5n_l/36\neq 0, \tag{4.10}$$

where we have used Eqs. (4.8). The fact $\mathrm{Tr}^{(1)}Y^3\neq 0$ demands that the grand-unification group G must be now one of SU(N) ($N\geq 3$), since otherwise we would have $\mathrm{Tr}^{(1)}Y^3=0$.

As we shall see in Appendix A, we must have

$$\frac{\mathrm{Tr}^{(2)}X^3}{\mathrm{Tr}^{(1)}X^3}=\frac{d(\rho_2)I_3(\rho_2)}{d(\rho_1)I_3(\rho_1)} \tag{4.11}$$

for SU(N) ($N\geq 3$), where $I_3(\rho)$ is the third-order Casimir invariant. Setting $X=I_3+tY$ again, we have

$$\frac{\mathrm{Tr}^{(2)}Y^3}{\mathrm{Tr}^{(1)}Y^3}=\frac{\mathrm{Tr}^{(2)}YI_3{}^2}{\mathrm{Tr}^{(1)}YI_3{}^2}=\frac{d(\rho_2)I_3(\rho_2)}{d(\rho_1)I_3(\rho_1)}.$$

By calculating traces, this is rewritten as

$$d(\rho_1)I_3(\rho_1)+d(\rho_2)I_3(\rho_2)=0, \tag{4.12}$$

which is equivalent to $\mathrm{Tr}X^3=0$ for any X in L in our reducible representations, $\{\rho\}=\{\rho_1\}\oplus\{\rho_2\}$. This relation is precisely the condition for cancellation of the triangle anomaly between two irreducible representations $\{\rho_1\}$ and $\{\rho_2\}$. Note that the absence of the triangle anomaly is a consequence but *not* a requirement. In this connection, it may be instructive to inquire what would have happened if we allowed integral quark charges, corresponding to case 3 as well as the discarded solution $m=\alpha$ of case 1. In that case, a similar calculation would have given an equation

$$d(\rho_1)I_3(\rho_1)-d(\rho_2)I_3(\rho_2)=0,$$

which leads to the presence of anomaly. Hence, we see that the absence of the triangle anomaly is intimately related to the question of quark charges.

Now, by the same reasoning as was given in the previous section, both irreducible representations $\{\rho_1\}$ and $\{\rho_2\}$ must be completely antisymmetric representations so as to ensure the fact that they contain only isosinglets and isodoublets. We label their highest weights by Λ_j and Λ_k ($1\leq j,k\leq N-1$). Here $\Lambda_1,\Lambda_2,\ldots,\Lambda_{N-1}$ are the fundamental weight system of the Lie algebra A_{N-1} in the Cartan notation. Then, we rewrite Eqs. (4.9) and (4.12) as

$$d(\Lambda_k)I_2(\Lambda_k)=3d(\Lambda_j)I_2(\Lambda_j), \tag{4.13}$$

$$d(\Lambda_k)I_3(\Lambda_k)+d(\Lambda_j)I_3(\Lambda_j)=0. \tag{4.14}$$

The dimensions of representations are given by

$$d(\Lambda_j) = 5\alpha n_q + n_0^{(1)} = 5n_l + n_0^{(1)},$$
$$d(\Lambda_k) = 10\alpha n_q + n_0^{(2)} = 10n_l + n_0^{(2)}, \tag{4.15}$$

where we have used Eqs. (4.8) and $n_l = \alpha n_q$ [see Eq. (2.6)]. These are essentially all the constraints.

For the fundamental representation $\Lambda = \Lambda_j$ we know

$$d(\Lambda_j) = \frac{N!}{j!\,(N-j)!},$$
$$I_2(\Lambda_j) = \frac{N+1}{2N}\, j(N-j), \tag{4.16}$$
$$I_3(\Lambda_j) = \frac{(N+1)(N+2)}{4N^2}\, j(N-j)(N-2j).$$

These relations are invariant under the interchange $j \to N-j$ and $k \to N-k$, which corresponds to its complex conjugate representation. Hence, we assume hereafter $j \leq k$ without loss of generality. Now, Eqs. (4.13) and (4.14) imply

$$2N = j + 3k. \tag{4.17}$$

First, let us investigate consequences of the absence of the triangle anomaly without imposing the constraint Eq. (4.17). We have found a class of solutions for Eq. (4.14) (see Appendix A), assuming $j+k \neq N$, since the case $j+k=N$ implies that two irreducible representations, $\{\Lambda_j\}$ and $\{\Lambda_k\}$, are contragredient to each other. We have checked up to $N = 500$ that the solutions we have found are the only solutions for Eq. (4.14) (see Appendix A and Table III). Out of these, only two satisfy the relation Eq. (4.17) [or equivalently Eq. (4.13)]. They are

$$\text{(I)}\ N=5,\ j=1,\ k=3,\ d(\Lambda_1)=5,\ d(\Lambda_3)=10,$$
$$\text{(II)}\ N=16,\ j=5,\ k=9,\ d(\Lambda_5)=4368,\ d(\Lambda_9)=11\,440. \tag{4.18}$$

Of course, their complex conjugate representations also furnish other solutions. The first solution corresponds to the standard SU(5) model[2] with the unique possibility of

$$\alpha=1,\quad m=3,\quad n_l=n_q=1,\quad n_0^{(1)}=n_0^{(2)}=0. \tag{4.19}$$

The possibility of SU(2) as the color group is excluded, because the decomposition of $\{5\}$ into SU(2)$\otimes$SU(2)$\otimes$U(1) (Ref. 26) yields no SU(2) color triplet. Thus, the color group must be identified with SU(3).

For solution II, we cannot have $n_0^{(1)} = 0$. To determine $n_0^{(1)}$, we proceed as follows. We express the third component of the weak i spin as in Eq. (3.2),

$$I_3 = (B_1{}^1 - B_2{}^2)/2,$$

and note that a basis of the irreducible representation $\{\rho_1\} = \{\Lambda_j\}$ is spanned by the completely antisymmetric tensor $\varphi_{\mu_1\mu_2\cdots\mu_j}$ $(1 \leq \mu_k \leq N)$, on which B^μ_ν operates as

$$B^\mu_\nu \varphi_{\mu_1\mu_2\cdots\mu_j} = \sum_{p=1}^{j} \delta^\mu_{\mu_p} \varphi_{\mu_1\cdots\hat{\nu}\cdots\mu_j} - \frac{j}{N}\,\delta^\mu_\nu \varphi_{\mu_1\mu_2\cdots\mu_j}. \tag{4.20}$$

Here, the symbol $\hat{\nu}$ in Eq. (4.20) indicates to delete μ_p and replace it by ν. By Eq. (4.20), we can compute the number of i spin doublets as

$$n(I=\tfrac{1}{2}) = \binom{N-2}{j-1} = \frac{(N-2)!}{(j-1)!\,(N-j-1)!}$$

by counting the number of states with the eigenvalue $I_3 = \frac{1}{2}$.

Since this must be equal to n_l, we find

$$\binom{N-2}{j-1} = n_e = \alpha n_q. \tag{4.21}$$

Similarly, counting the number of i spin doublets in the representation $\{\rho_2\}$, we must have

$$\binom{N-2}{k-1} = m n_q = 3n_l. \tag{4.22}$$

For solution I, these, of course, give $m = 3$, $\alpha = 1$, and $n_l = n_q = 1$. For solution II with $N = 16$, we must have

$$n_l = 1001.$$

But then, $d(\rho_1) \geq 5n_l = 5005$, which contradicts the dimension 4368. Therefore, we conclude that the case II is forbidden.

TABLE III. Anomaly-free combinations for a reducible representation, $\{\rho\} = \{\Lambda_j\} \oplus \{\Lambda_k\}$ $(j+k \neq N)$.

N	j	k	$d(\Lambda_j)$	$d(\Lambda_k)$
5	1	3	5	10
9	2	5	36	126
10	3	6	120	210
16	5	9	4368	11 440
17	6	10	$\sim 1.2\times10^4$	$\sim 1.9\times10^4$
25	9	14	$\sim 2.0\times10^6$	$\sim 4.5\times10^6$
26	10	15	$\sim 5.3\times10^6$	$\sim 7.7\times10^6$
36	14	20	$\sim 3.8\times10^9$	$\sim 7.3\times10^9$
37	15	21	$\sim 9.4\times10^9$	$\sim 1.3\times10^{10}$
49	20	27	$\sim 2.8\times10^{13}$	$\sim 5.0\times10^{13}$
50	21	28	$\sim 6.7\times10^{13}$	$\sim 8.9\times10^{13}$
64	27	35	$\sim 8.5\times10^{17}$	$\sim 1.4\times10^{18}$
65	28	36	$\sim 2.0\times10^{18}$	$\sim 2.5\times10^{18}$
81	35	44	$\sim 1.0\times10^{23}$	$\sim 1.6\times10^{23}$
82	36	45	$\sim 2.3\times10^{23}$	$\sim 2.9\times10^{23}$
100	44	54	$\sim 4.9\times10^{28}$	$\sim 7.3\times10^{28}$

In summary, we have proved the uniqueness of SU(5), provided that we restrict ourselves to SU(N) with $N \leq 500$. We conjecture that this result will hold for arbitrary N.

Actually, if we assume $n_0^{(1)} = 0$ from the beginning, then we can prove the uniqueness of solution I with $N = 5$ without any restriction on the range of N as follows. We define a new operator $\tilde{Y}$ and $Z_\pm$ by

$$\tilde{Y} = 2mY/(2m - \alpha) = 6Y/5\,,$$
$$Z_\pm = I_3 \pm mY/(2m - \alpha) = I_3 \pm 3Y/5\,. \tag{4.23}$$

Then, the eigenvalues of these operators in the irreducible representation $\{\rho_1\}$ are given in Table IV. We see from Table IV that all the operators $\tilde{Y}$, Z_+, and Z_- can assume only two eigenvalues if there is no N_L^C in the representation $\{\rho_1\}$, i.e., if $n_0^{(1)} = 0$. Moreover, denoting two eigenvalues as x and y, we normalized them as $x - y = 1$. As shown in Ref. 11, any irreducible representation satisfying this condition must be one of the fundamental representations. In our case, this requires[11] further

$$N = 5l \tag{4.24}$$

for some integer l. Let X represent any one of $\tilde{Y}$, Z_+, Z_- and express it as a linear combination of the Cartan subalgebra elements $H_1, H_2, \ldots, H_{N-1}$ as

$$X = \sum_{\mu=1}^{N-1} \xi^\mu H_\mu\,. \tag{4.25}$$

Then, the explicit form of $\vec{\xi}$ regarded as a vector in the $(N-1)$ dimensional root space must possess the following form:

$$\xi = \pm \frac{2}{(\alpha,\alpha)} \Lambda_p \quad \text{for } \Lambda = \Lambda_1 (j=1)\,,$$

$$\xi = \pm \frac{2}{(\alpha,\alpha)} \Lambda_1 \quad \text{for } \Lambda = \Lambda_j (2 \leq j \leq N-2)\,,$$

$$\xi = \pm \frac{2}{(\alpha,\alpha)} \Lambda_{N-p} \quad \text{for } \Lambda = \Lambda_{N-1} (j = N-1)\,,$$

where Λ is the highest weight of the representation $\{\rho_1\}$ and p is an integer, satisfying $1 \leq p \leq N-1$. Especially, if $\Lambda = \Lambda_j$ $(2 \leq j \leq N-2)$, then ξ is effectively determined apart from a constant. This implies that eigenvalues of X are uniquely determined, apart from an overall sign, irrespective of whether X is either $\tilde{Y}$, Z_+, or Z_-. However, this contradicts the content of Table III. Therefore, we conclude that Λ must be either Λ_1 or Λ_{N-1}. Since $\{\Lambda_{N-1}\}$ is complex conjugate to $\{\Lambda_1\}$ we may assume that $\Lambda = \Lambda_1$, i.e., $j = 1$ without loss of generality. For $j = 1$, Eqs. (4.13) and (4.17) are rewritten as

$$3(k-1)!\,(N-k-1)! = (N-2)!\,,$$
$$2N = 1 + 3k\,,$$

which admit only one solution,

$$N = 5,\ k = 3\,,$$

reproducing solution I in Eqs. (4.18). It is amusing to note that we have $\mathrm{Tr}Y^5 \neq 0$ for the present solution of SU(5), just as in the case of SO(10).

TABLE IV. Eigenvalues of $\tilde{Y}$, Z_+, and Z_-.

	N_L	E_L	N_L^C	D_L^C
$\tilde{Y}$	$-\frac{3}{5}$	$-\frac{3}{5}$	0	$\frac{2}{5}$
Z_+	$\frac{1}{5}$	$-\frac{4}{5}$	0	$\frac{1}{5}$
Z_-	$\frac{4}{5}$	$-\frac{1}{5}$	0	$-\frac{1}{5}$

V. FINAL COMMENT

In the previous sections, we have considered cases where $\{\rho\}$ consists of at most two irreducible representations. However, this is not a serious restriction for the following reason: Suppose that $\{\rho\}$ consists of more than two irreducible representations. If any one of the irreducible components, say $\{\rho_1\}$, has the multiplet structure specified as in Table I, then we can apply our discussion to the irreducible representation $\{\rho_1\}$ to find that the group G must be SO(10). Moreover, $\{\rho\}$ must be a direct sum of 16-dimensional irreducible representations, i.e., repetitive. Suppose that any irreducible component of $\{\rho\}$ does not have the structure in Table I. Then, we may have at least two irreducible components $\{\rho_1\}$ and $\{\rho_2\}$ which have the structure specified by Sec. VI. We restrict our discussion for these two multiplets. Thus, under the restriction of SU(N) ($N \leq 500$), we conclude again that G may be now SU(5) and that the representation $\{\rho\}$ must be repetitive, consisting of pairs, $\{\bar{5}\} + \{10\}$. For each case, the color group G_C must be SU(3) with quarks in color triplet. Moreover, U and D quarks must possess the standard electric charges, $\frac{2}{3}$ and $-\frac{1}{3}$, respectively. Summarizing, we have found that G must be either SO(10) or SU(5) and families must be repetitive.

Note that SO(10) can admit the lepton-baryon symmetry.[9] Rewriting I_3 and SU(2) as $(I_3)_L$ and $\mathrm{SU}(2)_L$, we may introduce another SU(2) group, which we write $\mathrm{SU}(2)_R$ with $(I_3)_R$. Assigning (D_L^c, U_L^c) and (E_L^c, N_L^c) as isodoublets with respect to $\mathrm{SU}(2)_R$, we see that the electric charge Q can be written as

$$Q = (I_3)_L + (I_3)_R + (B - L)/2\,, \tag{5.1}$$

where B and L are baryon and lepton numbers, respectively. Combining this with Eq. (1.1) we find

$$Y=(I_3)_R+(B-L)/2\,.$$

Possible consequences of the relation (5.1) have been discussed by Marshak and Mohapatra.[28] Note that SO(10) contains a subgroup $SU(3)_c\otimes SU(2)_L \otimes SU(2)_R$.

The assignment of quantum numbers in the SU(2) $\otimes$ U(1) is crucial for our conclusion. Although the Glashow-Weinberg-Salam SU(2) $\otimes$ U(1) group is now experimentally well established as the electroweak group, it may be of some interest to see what will happen if we replace this group. In Ref. 11, it has been proven that the possible electroweak groups of rank 2 are only SU(2) $\otimes$ U(1) and SU(3), provided that quarks have the fractional electric charges of $\frac{2}{3}$ and $-\frac{1}{3}$. Theoretically, any model based on SU(3) instead of SU(2) $\otimes$ U(1) predicts[29] the pure axial-vector current for hadronic neutral current, which is in contradiction with the experimental facts available now. However, in order to see how this difference of the electroweak group can influence the grand-unified group, we assume that G contains $SU_c(3)\otimes SU(3)$ as its subgroup. The simplest model of the SU(3) electroweak group is one with two triplets of quarks and an octet of leptons: $(3,3)_{L,R}\oplus(\bar{3},\bar{3})_{L,R}\oplus(1,8)_{L,R}$, where the bracket shows the SU(3) (color) $\otimes$ SU(3) (electroweak) quantum numbers. Then assuming $\{\rho\}$ to be irreducible, we find that only F_4, SO(26), and Sp(26) are possible candidates. An interesting fact is that the 26-dimensional representation of F_4 is capable of unifying all particles with

$$26=(3,3)\oplus(\bar{3},\bar{3})\oplus(1,8)$$

for the decomposition $F_4 \to SU(3)\otimes SU(3)$. Therefore, if nature had chosen SU(3) as the electroweak group, we could have had F_4 as the smallest grand-unified group.

Concluding this paper, we note that Zee[30] has recently found the uniqueness of SU(5) or SO(10) by posing a question: What is the largest simple subgroup of $SU(N_f)\otimes U(1)$ which is free from anomaly and free from bare masses. The choice of $N_f=45$ gives SU(5) and SO(10) only. Note that our conclusion is stronger than those obtained by Georgi[5] and Zee.[30] However, our program still may receive criticism, as Zee has acknowledged, that it is an argument that the world is as it is because it is the way it is.

Note added in proof. We can now prove the uniqueness of SU(5) without restriction of $N\leq 500$. The absence of the triangle anomaly is found unnecessary also for the uniqueness of SO(10). Hence, the absence of the triangle anomaly is a consequence of grand unification. These facts will be discussed in a future publication.

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APPENDIX A

Let X_μ $(\mu=1,2,\ldots,p)$ be a basis of a simple Lie algebra. Then, we have proved the following facts in Ref. 12:

(i) $\mathrm{Tr}(\{X_\alpha,X_\beta\}X_\gamma)=0$ unless G is SU(N) ($N\geq 3$); (A1)

(ii) if G is SU(N) ($N\geq 3$), then

$$\mathrm{Tr}(\{X_\alpha,X_\beta\}X_\gamma)=d_{\alpha\beta\gamma}K(\rho)\,, \tag{A2}$$

$$K(\rho)=d(\rho)I_3(\rho)\frac{2N}{(N^2-1)(N^2-4)}\,, \tag{A3}$$

where $d_{\alpha\beta\gamma}$ is the d symbol of Gell-Mann and $I_3(\rho)$ is the third-order Casimir invariant of SU(N) for a irreducible representation $\{\rho\}$. Note that $K(\rho)$ designates the triangle-anomaly coefficient. Since any generic element X can be expressed as

$$X=\sum_{\mu=1}^{p}\xi^\mu X_\mu \tag{A4}$$

for some real or complex number ξ^μ, Eq. (A1) implies the validity of

$$\mathrm{Tr}X^3=0$$

for any group other than SU(N) ($N\geq 3$), while for SU(N) ($N\geq 3$) we have

$$2\,\mathrm{Tr}X^3=K(\rho)d_{\alpha\beta\gamma}\,\xi^\alpha\xi^\beta\xi^\gamma\,.$$

Thus, for any two irreducible representations, $\{\rho_1\}$ and $\{\rho_2\}$, we must have

$$\frac{\mathrm{Tr}^{(2)}X^3}{\mathrm{Tr}^{(1)}X^3}=\frac{d(\rho_2)I_3(\rho_2)}{d(\rho_1)I_3(\rho_1)}\,. \tag{A5}$$

The eigenvalue $I_3(\rho)$ has been calculated in Refs. 12 and 13 as follows: Let $\Lambda_1,\Lambda_2,\ldots,\Lambda_{N-1}$ be the fundamental weight system of the Lie algebra A_{N-1}. Then, any highest weight Λ may be expressed as

$$\Lambda=\sum_{j=1}^{N-1}m_j\Lambda_j \tag{A6}$$

in terms of non-negative integers m_j $(1\leq j\leq N-1)$. They are related to the Young tableau notation f_j $(1\leq j\leq N)$ by

$$m_j=f_j-f_{j+1}(1\leq j\leq N-1)\,. \tag{A7}$$

For our purpose, we use the expression of $K(\rho)$ given by Banks and Georgi,[13] which is rewritten as

$$I_3(\rho)=\frac{1}{2N^2}\sum_{i,j,k=1}^{N-1} b_{ijk}(m_i+1)(m_j+1)(m_k+1)$$
$$=\frac{1}{2N^2}\left\{\sum_{i,j,k=1}^{N-1} b_{ijk}m_i m_j m_k+\frac{3N}{2}\sum_{j,k=1}^{N-1} b_{jk}m_j m_k+\frac{N^2}{2}\sum_{j=1}^{N-1} j(N-j)(N-2j)m_j\right\}. \tag{A8}$$

Here b_{ijk} is completely symmetric in three indices i, j, and k with its value as

$$b_{ijk}=i(N-2j)(N-k) \quad \text{if } i\leq j\leq k \tag{A9}$$

while b_{jk} satisfies $b_{jk}=b_{kj}$ and

$$b_{jk}=j(N-k)(N-j-k) \quad \text{if } j\leq k. \tag{A10}$$

Especially, if we have

$$\Lambda=m_j\Lambda_j+m_k\Lambda_k \quad (j\leq k) \tag{A11}$$

then, $I_3(\rho)$ can be written as

$$\begin{aligned}2N^2I_3(\rho)=&\tfrac{1}{2}j(N-j)(N-2j)(2m_j+N)(m_j+N)m_j\\&+\tfrac{1}{2}k(N-k)(N-2k)(2m_k+N)(m_k+N)m_k\\&+3[j(N-2j)(N-k)m_j+j(N-2k)(N-k)m_k\\&+Nj(N-2k)(N-j-k)]m_jm_k.\end{aligned} \tag{A12}$$

For a special case of $m_k=0$ and $m_j=1$, we find

$$I_3(\Lambda_j)=\frac{1}{4N^2}(N+1)(N+2)j(N-2j)(N-j). \tag{A13}$$

For $m_j=1$ and $m_k=1$, we find

$$I_3(\Lambda_j+\Lambda_k)=\frac{N+2}{4N^2}(j+k-N)\{(N+1)[2(j^2+k^2-jk)-(j+k)N]-6j(N-k)\}. \tag{A14}$$

Note that $I_3(\Lambda_j+\Lambda_k)=0$ for $j+k=N$, which corresponds to a self-contragredient representation.

It is known[12] that there exists a non-self-contragredient irreducible representation Λ with $I_3(\Lambda)=0$. We have found one more example. Suppose that $\Lambda=\Lambda_j+\Lambda_k$, then $j=3$ and $k=21$ with $N=32$ (or its complex conjugate $j=11$ and $k=29$) leads to $I_3(\Lambda)=0$. As a matter of fact, this is the only one up to SU(300) for irreducible representations of the type $\Lambda=\Lambda_j+\Lambda_k(j+k\neq N)$. The dimension of this representation is extremely large ($\sim 6\times 10^{11}$).

If the representation $\{\rho\}$ is a direct sum of two irreducible representations $\{\rho_1\}$ and $\{\rho_2\}$ with highest weights $\{\Lambda_j\}$ and $\{\Lambda_k\}$, we find that either

$$j=(N\mp 1-\sqrt{N-1})/2, \quad k=(N\mp 1+\sqrt{N-1})/2$$

or

$$j=(N\mp 2-\sqrt{N})/2, \quad k=(N\mp 2+\sqrt{N})/2 \tag{A15}$$

always satisfies the anomaly-free condition Eq. (4.14), provided that $\sqrt{N-1}$ or $\sqrt{N}$ is an integer. Among these solutions, Eq. (4.17) can be satisfied only for $N=5$ and $N=16$. We have checked that no other anomaly-free solutions exist up to $N=500$ and the results up to $N=100$ are presented in Table III, where we listed only the upper sign cases of Eq. (A15), since the lower cases correspond to complex conjugate representations.

APPENDIX B

Let x_μ $(\mu=1,2,\ldots,p)$ be a basis of the simple Lie algebra and let X_μ be its representation matrix in an irreducible representation $\{\rho\}$. Suppose that the Lie algebra possesses a fifth-order Casimir invariant:

$$I_5=\sum_{\mu_1,\ldots,\mu_5=1}^{p} g^{\mu_1\mu_2\cdots\mu_5}x_{\mu_1}x_{\mu_2}x_{\mu_3}x_{\mu_4}x_{\mu_5}, \tag{B1}$$

where $g^{\mu_1\mu_2\cdots\mu_5}$ are constants which are completely symmetric in any exchange of any two indices. Then, the eigenvalue $I_5(\rho)$ is calculated as

$$I_5(\rho)E=\sum_{\mu_1,\ldots,\mu_5=1}^{p} g^{\mu_1\mu_2\ldots\mu_5}X_{\mu_1}X_{\mu_2}\cdots X_{\mu_5}, \tag{B2}$$

where E is the unit matrix in the representation space $\{\rho\}$. Consider a quantity

$$\sum_P\frac{1}{5!}\operatorname{Tr}(X_{\mu_1}X_{\mu_2}X_{\mu_3}X_{\mu_4}X_{\mu_5})=K_{\mu_1\mu_2\cdots\mu_5}) \tag{B3}$$

where the summation in the left side is overall 5! permutations P of indices $\mu_1,\mu_2,\ldots,\mu_5$. Raising the indices by the raising operator $g^{\mu\nu}$ as usual, then we can verify, using the method of Ref. 19, that

$$J=\frac{1}{5!}\sum_{\mu_1\mu_2,\ldots,\mu_5=1}^{p} K^{\mu_1\mu_2\cdots\mu_5}x_{\mu_1}x_{\mu_2}\cdots x_{\mu_5} \tag{B4}$$

is a Casimir invariant, i.e., it commutes with any element x_λ; $[J,x_\lambda]=0$. Therefore, if the Lie algebra possesses no fifth-order Casimir invariant, then we must have

$$K_{\mu_1\mu_2\cdots\mu_5}=0.$$

Since any X is expressed as in Eq. (A4), this implies the validity of

$$\operatorname{Tr}X^5=0, \tag{B5}$$

which must hold for any simple Lie algebra except E_6, SO(10), and SU(N) ($N \geq 3$).

Since both SO(10) and E_6 possess precisely one fifth-order Casimir invariant, $K_{\mu_1\mu_2\cdots\mu_5}$ must be proportional to $g_{\mu_1\mu_2,\ldots,\mu_5}$ defined in Eq. (B1) and hence

$$K_{\mu_1\mu_2\cdots\mu_5} = c(\rho) g_{\mu_1\mu_2\cdots\mu_5}. \tag{B6}$$

The constant $c(\rho)$ can be computed as follows: Multiply $g^{\mu_1\mu_2\cdots\mu_5}$ to both sides of Eq. (B3) and use Eq. (B2). We find

$$(g^{\mu_1\mu_2\cdots\mu_5} g_{\mu_1\mu_2\cdots\mu_5}) c(\rho) = d(\rho) I_5(\rho). \tag{B7}$$

Then the analog of Eq. (B5) is given by

$$\mathrm{Tr} X^5 = c(\rho) \xi^{\mu_1} \xi^{\mu_2} \cdots \xi^{\mu_5} g_{\mu_1\mu_2\cdots\mu_5}. \tag{B8}$$

This proves that $\mathrm{Tr}X^5$ is proportional to $I_5(\rho)$.

Now, we restrict ourselves to the study of the SO(10) group, whose Lie algebra may be defined by J_{ab} ($a, b = 1, 2, \ldots, 10$) satisfying

$$\begin{aligned} &J_{ab} = -J_{ba}, \\ &[J_{ab}, J_{cd}] = \delta_{bc} J_{ad} + \delta_{ad} J_{bc} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac}. \end{aligned} \tag{B9}$$

Now, we identify

$$X_{\mu_j} = J_{a_j b_j} = -J_{b_j a_j} \ (j = 1, 2, 3, 4, 5), \quad \mu_j = (a_j, b_j) \tag{B10}$$

to find

$$g_{\mu_1\mu_2\cdots\mu_5} = \epsilon_{a_1 b_1 \, a_2 b_2 a_3 b_3 a_4 b_4 a_5 b_5}, \tag{B11}$$

where $\epsilon_{a_1 \cdots b_5}$ is the completely antisymmetric Levi-Civita symbol in the ten-dimensional space. The eigenvalue of the fifth-order Casimir invariant can be calculated as[19,31]

$$I_5(\rho) = l_1 l_2 l_3 l_4 l_5 \tag{B12}$$

apart from some normalization constant which does not concern us here. The value l_j ($j = 1, 2, 3, 4, 5$) is related to the usual symbol f_j ($j = 1, 2, 3, 4, 5$) by

$$l_j = f_j + 5 - j \ (1 \leq j \leq 5). \tag{B13}$$

In order not to have $\mathrm{Tr}X^5 = 0$, we must have $l_5 = f_5 \neq 0$, since $l_1 > l_2 > l_3 > l_4 > |l_5|$. In terms of m_j, we can express f_j as

$$\begin{aligned} f_1 &= m_1 + m_2 + m_3 + \tfrac{1}{2}(m_4 + m_5), \\ f_2 &= m_2 + m_3 + \tfrac{1}{2}(m_4 + m_5), \\ f_3 &= m_3 + \tfrac{1}{2}(m_4 + m_5), \\ f_4 &= \tfrac{1}{2}(m_4 + m_5), \\ f_5 &= \tfrac{1}{2}(m_4 - m_5). \end{aligned}$$

Therefore, we must have

$$m_4 \neq m_5$$

in order to have $I_5(\rho) \neq 0$.

The identity Eq. (3.8) involving $\mathrm{Tr}X^7$ for SO(10), can be obtained as follows: From the SO(10) rotational invariance, we see that the quantity

$$\frac{1}{7!} \sum_P \mathrm{Tr}(X_{\mu_1} X_{\mu_2} \cdots X_{\mu_7}),$$

$$\mu_j = (a_j, b_j) \quad (j = 1, 2, \ldots, 7),$$

must be proportional to a symmetric sum of

$$\epsilon_{(a_1 b_1) \cdots (a_5 b_5)} (\delta_{a_6 a_7} \delta_{b_6 b_7} - \delta_{a_6 b_7} \delta_{a_7 b_6}).$$

Then, just as we have obtained the formula (B8), this leads to the formula (3.8) after some calculations.

APPENDIX C

Suppose that an element of a Cartan subalgebra of a simple Lie algebra has the property of having only three distinct eigenvalues in an irreducible representation $\{\rho\}$. Then, following the method given in Ref. 11, we can verify the following facts.

(i) The highest weight Λ of $\{\rho\}$ must have one of the following;

$$\Lambda = \Lambda_j \text{ or } \Lambda_j + \Lambda_k.$$

(ii) For any weight M in the representation $\{\rho\}$ and for any nonzero root α, the value of

$$\frac{2(M, \alpha)}{(\alpha, \alpha)}$$

is limited to values $2, 1, 0, -1, -2$.

Now let us restrict ourselves to the Lie algebra D_n ($n \geq 4$) which corresponds to SO($2n$). Then, since Λ_2 is the highest root of D_n, we must have

$$\left| \frac{2(\Lambda_2, \Lambda)}{(\Lambda_2, \Lambda_2)} \right| \leq 2.$$

This requires that Λ must be one of

$$\Lambda_j (1 \leq j \leq n), 2\Lambda_1, 2\Lambda_{n-1}, 2\Lambda_n, \Lambda_1 + \Lambda_{n-1}, \Lambda_1 + \Lambda_n, \Lambda_{n-1} + \Lambda_n.$$

For SO(10) and $\mathrm{Tr}X^5 \neq 0$, only the following are allowed: Λ_4, Λ_5, $2\Lambda_4$, $2\Lambda_5$, $\Lambda_1 + \Lambda_4$, and $\Lambda_1 + \Lambda_5$, as was stated in Sec. III.

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