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Gravitational/electromagnetic conversion scattering on fixed charges in the Born approximation

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In the presence of nonzero background electromagnetic and gravitational fields, perturbations in these fields are coupled, via the Einstein-Maxwell equations. We investigate the differential cross section for the conversion scattering of electromagnetic to gravitational radiation (and vice versa) on fixed charged masses. Our procedure uses a Born approximation at the level of accuracy in which the horizon structure of putative black holes is lost; our fixed charges therefore represent charged black holes at this order. We find conversion differential cross sections of the order of Ge^2 (G is Newton's gravitational constant, e is the value of the charge), with angular polarization factors, and with a forward-divergent behavior $\propto \sin^{-2}(\theta/2)$. The divergence is shown to arise not only in the Born approximation, but to be implicit in the known behavior of the large-angular-momentum phase shifts for the partial-wave spherical modes.

I. INTRODUCTION

It was pointed out by Choquet-Bruhat,[1] Gerlach,[2] and Sibgatullin[3] that in electrovac solutions (solutions to the coupled Maxwell-Einsten equations), gravitational and electromagnetic perturbations are necessarily coupled. This idea was subsequently developed by Moncrief[4] in the investigation of perturbations of Reissner-Nordström black holes. Chitre, Price, and Sandberg[5] developed a corresponding Newman-Penrose[6] formalism and investigated perturbations due to sources in orbit around charged black holes.

An important aspect to consider in such situations is the existence of a conversion cross section. That is, suppose an electromagnetic plane wave is incident on a charged mass. How much of that flux is converted to outgoing gravitational radiation? The inverse process can also be calculated. Olson and Unruh[7] used Moncrief's decomposition into modes labeled by angular momentum l, in the JWKB approximation to calculate conversion rates for high-frequency ($\omega \gg 1/m$) waves scattering in the Reissner-Nordström geometry (mass m, charge e). Matzner[8] numerically obtained the $\omega \to 0$, $l=2$ conversion cross section, and Fabbri[9] obtained the conversion phase shifts $(\Delta\delta)_l$ for $\omega \to 0$. Recently, Gunter[10] has done a numerical study of the conversion process for a range of ω, l, and e/m.

In this paper we consider the Reissner-Nordström conversion-scattering problem from a different viewpoint. Rather than investigating the partial-wave formulation, as has been done in previous discussions, we consider the full, summed conversion differential cross section, but only at the lowest nonvanishing order in the gravitational constant. This approach, applied here to the classical electromagnetic and gravitational wave fields, is the exact computational and conceptual analog of the Born approximation[11] in the quantum-mechanical scattering problem.

In addition to the conversion scattering, there is also ordinary, nonconversion scattering, in which, for instance, an incident electromagnetic wave is scattered (as an electromagnetic wave) by the black hole. Such scattering has been discussed for various wave fields on uncharged black holes in an angular decomposition by Sanchez,[12] Matzner,[13] Chrzanowski *et al.*,[14] Matzner and Ryan,[15] Handler and Matzner,[16] and in the Reissner-Nordström case by Gunter.[10] Furthermore, Peters,[17] Westervelt,[18] and Sanchez[19] have calculated Born-approximation nonconversion scattering from "Schwarzschild black holes."

The remarkably wide range of applicability of the Born approximation in such problems as spinless nonrelativistic electron scattering, where it reproduces the Rutherford and exact quantum-mechanical Coulomb cross sections, has been the subject of much discussion.[11] In general situations, however, it will not give the exact cross section, although the resulting calculation will still reflect some of the structure of the exact result.

In fact, we shall see that a consistent Born approximation to scattering from a black hole loses the horizon structure of the black hole, so it is more accurate to refer to the scattering center as a "fixed mass" m. Similarly, rather than referring to Born-approximation conversion on a "charged black hole" we use the term "fixed charge" e.

II. CONVERSION SCATTERING: ELECTROMAGNETIC TO GRAVITATIONAL

In order to display some of the features of the approximation, consider the equation for a metric perturbation propagating on a Reissner-Nordström background[20]:

$$\bar{h}_{\alpha\beta|\sigma}{}^{|\sigma} = -16\pi G\,\delta T_{\alpha\beta}\,, \tag{2.1}$$

where $g_{\alpha\beta} = {}^0g_{\alpha\beta} + h_{\alpha\beta}$ is the perturbed metric (${}^0g_{\alpha\beta}$ is the background metric, all indices are raised and lowered using this metric) and

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\,{}^0g_{\alpha\beta}h_\sigma{}^\sigma\,. \tag{2.2}$$

A slash denotes a covariant derivative in the background, and $\delta T_{\alpha\beta}$ is typically nonzero because there is a concomitant electromagnetic perturbation [Eq. (3.1) below]. In Eq. (2.1) the gauge $\bar{h}^{\alpha\beta}{}_{|\beta} = 0$, which is consistent because of the conservation of the stress tensor, has been assumed.

In the spirit of the Born approximation, we consider the lowest nonvanishing interaction in calculating conversion scattering. If we momentarily ignore the source in Eq. (2.1), we see that the left-hand side refers only to the metric perturbation $h_{\alpha\beta}$, and is therefore a description of metric waves moving in a fixed Reissner-Nordström background. The Born approximation for *nonconversion* scattering of gravitational waves would be obtained by linearizing this background to $\eta_{\mu\nu} + O(G)(m/r + e^2/r^2) + \cdots$ (keeping first-order terms in G). This is the technique used by Peters[17] and Westervelt[18] to calculate the gravitational-wave-scattering cross section of an uncharged mass.

We, however, are interested in conversion scattering, which is due only to the perturbed source in Eq. (2.1). This source contains terms such as ${}^1F_{\alpha\beta}{}^0F^{\alpha\rho}$, where ${}^1F_{\alpha\beta}$ is the electromagnetic field tensor perturbation while ${}^0F_{\mu\nu}$ is the (Coulomb) background electric field of the Reissner-Nordström solution. Conversion scattering occurs when an electromagnetic wave is incident upon the fixed charge. The metric perturbation which is generated is proportional to Ge. For this reason, the deviation of the background metric from flatness can be ignored in the subsequent propagation of the gravitational wave generated. (These metric deviations introduce corrections of next order in G to a term already of first order.) Hence, in the Born approximation one solves the flat-space propagation of a tensor field $\bar{h}_{\alpha\beta}$ with source $\delta T_{\alpha\beta}$,

$$\Box\bar{h}_{\alpha\beta} = -16\pi G\,\delta T_{\alpha\beta}\,, \tag{2.3}$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ and $\delta T_{\alpha\beta}$ is the linearized electromagnetic stress-energy tensor, and the horizon structure completely disappears from the problem.

The calculation then proceeds in a straightforward manner. The background electric field is (using rectangular coordinates in Minkowski space)

$${}^0F_{0k} = ex^k/r^3\,, \quad r^2 = x^2 + y^2 + z^2\,. \tag{2.4}$$

We choose an incident plane electromagnetic wave of specific polarization moving in the positive $\hat{z}$ direction $[\hat{e}_i = (\hat{x}, \hat{y}, \hat{z})]$:

$$\delta\vec{E} = E\hat{x}e^{i\omega(t-z)}\,,\qquad \delta E_i \equiv {}^1F_{0i} \tag{2.5}$$

(there is also a complementary magnetic field). The energy flux for this wave is

$$T^{EM}_{0i}\hat{e}_i = T^{EM}_{00}\hat{z} = \frac{E^2}{8\pi}\hat{z}. \tag{2.6}$$

The conversion cross section is given as the energy flux of the outgoing gravitational waves per steradian, divided by the energy flux of the incident electromagnetic wave [Eq. (2.5)].

Equation (2.3) gives

$$\Box\bar{h}_{zz} = -16\pi G\,\delta T_{zz} = -4GExr^{-3}e^{i\omega(t-z)}. \tag{2.7}$$

Furthermore, it is straightforward to show that

$$\delta T_{yy} = \delta T_{zz}\,,\quad \delta T_{xx} = -\delta T_{zz}\,,\quad \delta T_{xy} = -(y/x)\delta T_{zz}\,,\quad \delta T_{xz} = -(z/x)\delta T_{zz}\,,\quad \delta T_{yz} = 0\,. \tag{2.8}$$

We need not evaluate $\delta T_{0\alpha}$ because only spatial components h_{ij} are needed to evaluate the transverse-traceless outgoing gravitational wave. The Green's function G for Eq. (2.7) satisfies

$$(\nabla^2 + \omega^2)G(\vec{r}, \vec{r}\,') = -4\pi\,\delta(\vec{r} - \vec{r}\,')\,, \tag{2.9}$$

from which we get

$$\bar{h}_{zz} \cong G\frac{e^{i\omega(t-r)}}{r}\frac{eE}{\pi}\int (x'/r'^3)e^{-i\vec{\Delta\omega}\cdot\vec{r}\,'}d^3r'\,, \tag{2.10}$$

where

$$\vec{\Delta\omega} = \omega(\hat{z} - \hat{r}_{\rm out})\,,$$

with $\hat{r}_{\rm out}$ the direction to the observation point, and where we impose outgoing boundary conditions on the scattered field. The $\cong$ sign in Eq. (2.10) means that we have only taken the lowest-order term of the Green's function. Equation (2.10) is easily evaluated by aligning the pole with the direction of

$\overrightarrow{\Delta\omega}$; one introduces new coordinates x'', y'', z'' so aligned and finds that

$$x' = \cos\tilde{\phi}\,\sin\tilde{\theta} z'' + \cdots ,\quad y' = \sin\tilde{\phi}\,\sin\tilde{\theta} z'' + \cdots ,\quad z' = \cos\tilde{\theta} z'' + \cdots , \tag{2.11}$$

where the ellipses involve quantities independent of z''. By symmetry about $\overrightarrow{\Delta\omega}$, only terms in x' proportional to z'' contribute to the integral (2.10). In Eq. (2.11), $(\tilde{\theta}, \tilde{\phi})$ are the angles coordinatizing $\overrightarrow{\Delta\omega}$ in the original frame; they are related by

$$\tilde{\phi} = \varphi + \pi ,\quad \tilde{\theta} = \tfrac{1}{2}(\theta + \pi) \tag{2.12}$$

to the angles θ, φ giving the outgoing direction $\hat{r}$. We obtain

$$\bar{h}_{ij} = \frac{2ieE}{\omega r} e^{i\omega(t-r)} \times \begin{bmatrix} -\cos\varphi\cot\frac{\theta}{2} & -\sin\varphi\cot\frac{\theta}{2} & 1 \\ -\sin\varphi\cot\frac{\theta}{2} & \cos\varphi\cot\frac{\theta}{2} & 0 \\ 1 & 0 & \cos\varphi\cot\frac{\theta}{2} \end{bmatrix} . \tag{2.13}$$

An interesting alternative evaluation of the integrals is given in the Appendix.

For the outgoing flux of gravitational radiation we use Isaacson's[21] prescription:

$$T^{\mathrm{GW}}_{oi}\hat{e}_i = T^{GW}_{00}\hat{r}_{\mathrm{out}} = \frac{\hat{r}_{\mathrm{out}}}{32\pi G}\langle h^{TT}_{ij,0}h^{TT}_{ij,0}\rangle \text{summed on } i,j , \tag{2.14}$$

where h^{TT}_{ij} is the transverse-traceless part of h_{ij}, and the average is over several cycles. In our case one has

$$\langle h^{TT}_{ij,0}h^{TT}_{ij,0}\rangle = \frac{\omega^2}{2}\,|h^{TT}_{ij}h^{TT}_{ij}| .$$

Because gauge transformations

$$h'_{ij} = h_{ij} + 2\xi_{(i,j)} \tag{2.15}$$

can never introduce nor modify transverse-traceless terms, a simple algebraic projection operator based on the direction $\hat{r}_{\mathrm{out}}$ allows extraction of the transverse-traceless part. Furthermore, because of the definition $\bar{h}_{ij}$ in terms of h_{ij} [Eq. (2.2)], we see $h_{ij} = \bar{h}_{ij}$ when both are traceless. The symmetric projector is

$$P = \begin{bmatrix} 1-\sin^2\theta\cos^2\varphi & -\sin^2\theta\sin\varphi\cos\varphi & -\sin\theta\cos\theta\cos\varphi \\ -\sin^2\theta\sin\varphi\cos\varphi & 1-\sin^2\theta\sin^2\varphi & -\sin\theta\cos\theta\sin\varphi \\ -\sin\theta\cos\theta\cos\varphi & -\sin\theta\cos\theta\sin\varphi & \sin^2\theta \end{bmatrix} \tag{2.16}$$

and

$$h^{TT}_{ij} = \bar{h}^{TT}_{ij} = P_{il}\bar{h}_{lm}P_{mj} - \tfrac{1}{2}P_{ij}\,\mathrm{tr}(P\bar{h}P) . \tag{2.17}$$

We calculate the quantity

$$h^{TT}_{ij}h^{TT}_{ij} = \mathrm{tr}(\bar{h}^{TT}\bar{h}^{TT}) = \mathrm{tr}(P\bar{h}PP\bar{h}P) - \tfrac{1}{2}[\mathrm{tr}(P\bar{h}P)]^2 = \mathrm{tr}(P\bar{h}P\bar{h}) - \tfrac{1}{2}[\mathrm{tr}(P\bar{h})]^2 \tag{2.18}$$

(using the cyclic property of the trace and the idempotent property of projectors). The cross section is defined as

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\mathrm{EM\ to\ GW}\\ \mathrm{Born}}} = \frac{r^2 T^{\mathrm{GW}}_{00}}{T^{\mathrm{EM}}_{00}} . \tag{2.19}$$

Multiplying $\bar{h}_{ij}$ as given by Eq. (2.13) by the projection matrix (P_{ki}) from Eq. (2.16) and calculating $\frac{1}{2}\omega^2|h^{TT}_{ij}h^{TT}_{ij}|$ by means of (2.18), we obtain [with T^{EM}_{00} given by Eq. (2.6)]

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\mathrm{EM\ to\ GW}\\ \mathrm{Born}}} = Ge^2\,\frac{\cos^2(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta)}(1-\sin^2\theta\cos^2\varphi) . \tag{2.20}$$

The expression calculated from Eq. (2.18) appears much more complicated than (2.20) at first sight. Only repeated use of trigonometric identities gives the final expression.

III. CONVERSION SCATTERING: GRAVITATIONAL TO ELECTROMAGNETIC

The production of outgoing electromagnetic radiation when gravitational radiation is incident on a fixed charge is governed by the Maxwell equation[20]

$$g^{\alpha\beta}A_{\mu;\alpha;\beta} - R_{\mu}{}^{\alpha}A_{\alpha} = 0 , \tag{3.1}$$

where $R_{\mu}{}^{\alpha}$ is the Ricci tensor, A_{α} is the total electromagnetic potential, and the semicolon means covariant differentiation using the full metric. In Eq. (3.1) we assume the Lorentz guage $A^{\mu}{}_{;\mu} = 0$. We will see below that it is consistent to make this choice for the perturbed potential.

The approach here is, as in the preceding section, to take only the lowest-order terms that enter conversion scattering and to ignore all others. Expand

$$A_\mu = {}^0A_\mu + {}^1A_\mu \,, \tag{3.2}$$

where ${}^0A_\mu$ is the background potential and ${}^1A_\mu$ the perturbation. The metric and covariant derivatives in (3.1) must also be expanded in terms of the metric perturbation $h_{\alpha\beta}$. We will need to examine each term in the expansion of (3.1) to check the order at which it affects the conversion scattering. This is easily carried out, even though the perturbed Eq. (3.1) does not fall as neatly into conversion and nonconversion parts as the equation for metric perturbations did. The exercise affords an opportunity to apply physical reasoning to the estimates.

The 0th-order version of (3.1) is of course satisfied. Furthermore, we drop terms of the form ${}^0\Gamma^\alpha_{\beta\gamma}$ (background connection coefficients) multiplied by $h_{\mu\nu}$ or by ${}^1A_\sigma$. The reason is that ${}^0\Gamma^\alpha_{\beta\gamma} \sim Gm/r^2$, and $h_{\mu\nu}$ and ${}^1A_\sigma$ are already small quantities. (Some such terms contribute to *nonconversion* scattering.) On the other hand, products such as ${}^1\Gamma^\alpha_{\mu\nu}\,{}^0A_\lambda$ do survive, since they are of only first order. Making simplifications of this type, we arrive at an intermediate reduced form for the perturbed Maxwell equations:

$$\begin{aligned}0 = -h^{\sigma\rho}\,{}^0A_{\mu,\sigma,\rho} + {}^0g^{\sigma\rho}[{}^1A_{\mu,\sigma,\rho} - ({}^1\Gamma^\lambda_{\mu\sigma}\,{}^0A_\lambda)_{,\rho} \\ - {}^1\Gamma^\alpha_{\mu\rho}\,{}^0A_{\alpha,\sigma} - {}^1\Gamma^\alpha_{\sigma\rho}\,{}^0A_{\mu,\alpha}] \\ - {}^0R_\mu^\alpha\,{}^1A_\alpha + {}^0g^{\alpha\delta}\,{}^1R_{\mu\delta}\,{}^0A_\alpha - h^{\alpha\delta}\,{}^0R_{\mu\delta}\,{}^0A^\delta \,.\end{aligned} \tag{3.3}$$

The quantity ${}^0R_\mu{}^\alpha\,{}^1A_\alpha$ is a nonconversion-scattering term, and will be dropped. On the other hand, the last term, $h^{\alpha\delta}\,{}^0R_{\mu\delta}\,{}^0A^\delta$, is a conversion-scattering term, but is of higher order in G because ${}^0R_{\mu\delta} = O(Ge^2)$, $h_{\alpha\delta} = O(G)$, ${}^0A^\delta = O(e)$ so this term is a factor Ge smaller than other terms in (3.3) and we will drop it. The remaining Ricci tensor term ${}^0g^{\alpha\delta}\,{}^1R_{\alpha\beta}\,{}^0A_\mu$ will be dropped as well. In flat space the vacuum equation for a metric perturbation is ${}^1R_{\alpha\beta}=0$. Here there is a source, ${}^1R \sim O(Ge\,{}^1A)$. Hence this term also has an explicit additional power of Ge and will be dropped.

Notice also that the combination

$${}^0g^{\sigma\rho}\,{}^1\Gamma^\alpha_{\sigma\rho} = \tfrac{1}{2}\,{}^0g^{\sigma\rho}(h_{\alpha\sigma,\rho} + h_{\alpha\rho,\sigma} - h_{\sigma\rho,\alpha}) \,, \tag{3.4}$$

which appears in (3.3), can be eliminated if a transverse-traceless gauge is chosen for the metric perturbation. Thus, by choosing as an incident plane gravitational wave

$$h_{xx} = -h_{yy} = he^{i\omega(t-z)} \tag{3.5}$$

(all other components are zero)—which is transverse and traceless—(3.4) vanishes.

The wave equation for ${}^1A_\mu$ is then given by

$$\begin{aligned}\Box A = h^{\sigma\rho\,0}A_{\mu,\sigma,\rho} + \tfrac{1}{2}\,{}^0A_\lambda\,\eta^{\sigma\rho}\eta^{\tau\lambda}(h_{\tau\mu,\sigma} + h_{\tau\sigma,\mu} - h_{\sigma\mu,\tau})_{,\rho} \\ + \eta^{\sigma\rho}\eta^{\alpha\tau}(h_{\tau\mu,\sigma} + h_{\tau\sigma,\mu} - h_{\sigma\mu,\tau})\,{}^0A_{\alpha,\rho} \,.\end{aligned} \tag{3.6}$$

The background has only a static electric field so we can take it in a Lorentz gauge where in addition only ${}^0A_0$ is nonzero. Then we derive from (3.5) and (3.6) the equations

$$\begin{aligned}&\Box\,{}^1A_0 = h({}^0A_{0,xx} - {}^0A_{0,yy})e^{i\omega(t-z)} \,, \\ &\Box\,{}^1A_x = i\omega h\,{}^0A_{0,x}e^{i\omega(t-z)} \,, \\ &\Box\,{}^1A_y = -i\omega h\,{}^0A_{0,y}e^{i\omega(t-z)} \,, \\ &\Box\,{}^1A_z = 0 \,.\end{aligned} \tag{3.7}$$

It is straightforward to show that the terms on the right-hand side of (3.7) are the components of a conserved current; hence the assumed Lorentz gauge is consistent for the perturbation ${}^1A_\mu$. As in Sec. II, the background electric field is taken as ${}^0A_0 = -e/r$; then $E_i = ex^i/r^3$.

The solution of Eqs. (3.7)—again only the spatial components A_i are needed—is obtained by an identical procedure to the one used to solve Eq. (2.7). We find, with outgoing boundary conditions,

$$\begin{aligned}&{}^1A_x = \frac{he}{2r}e^{i\omega(t-r)}\cos\varphi\,\cot\frac{\theta}{2} \,, \\ &{}^1A_y = \frac{he}{2r}e^{i\omega(t-r)}\sin\varphi\,\cot\frac{\theta}{2} \,, \\ &{}^1A_z = 0 \,.\end{aligned} \tag{3.8}$$

The cross section is obtained by evaluating the outgoing (transverse) energy flux in the transverse waves:

$$\begin{aligned}T^{\rm EM}_{0i}\hat{e}_i = T^{\rm EM}_{00}\hat{r} &= \frac{1}{8\pi}\langle E^2 + B^2\rangle\hat{r} \\ &= \frac{\omega^2}{8\pi}\,|{}^1A_i^T\,{}^1A_i^T|\,\hat{r} \,,\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}{}^1A_i^T &= P_{ij}\,{}^1A_j \\ &= \frac{he}{2r}e^{i\omega(t-z)}\begin{pmatrix}\cos\varphi\cot\frac{\theta}{2}(1-\sin^2\theta\cos2\varphi) \\ -\sin\varphi\cot\frac{\theta}{2}(1+\sin^2\theta\cos2\varphi) \\ -\sin\theta\cos\theta\cot\frac{\theta}{2}\cos2\varphi\end{pmatrix}\end{aligned} \tag{3.10}$$

and, therefore,

$$\begin{aligned}\frac{\omega^2}{8\pi}\,|{}^1A_i^T\,{}^1A_i^T| &= \frac{\omega^2}{8\pi}\,|P_{ij}\,{}^1A_i\,{}^1A_j| \\ &= \frac{\omega^2}{32\pi r^2}\,\frac{h^2e^2}{\sin^2(\frac{1}{2}\theta)}\cos^2(\tfrac{1}{2}\theta) \\ &\quad\times(1-\sin^2\theta\cos^2 2\varphi) \,.\end{aligned} \tag{3.11}$$

The flux due to the incident gravitational plane wave of Eq. (3.5) is

$$T^{\rm GW}_{0i}\hat{e}_i = T^{\rm GW}_{00}\hat{z} = \hat{z}\frac{\omega^2}{32\pi G}h^2 . \qquad (3.12)$$

Hence, the Born-approximation cross section is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{GW to EM}\\ \text{Born}}} = r^2 T^{\rm EM}_{00}/T^{\rm GW}_{00} = Ge^2\frac{\cos^2(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta)}(1-\sin^2\theta\cos^2 2\varphi) . \qquad (3.13)$$

The similarity between the two cross sections (3.13) and (2.20) is due to the fact that the electromagnetic and gravitational potentials are almost the same within the approximation considered. The differences in the cross sections are due to the different tensorial nature (spin s) of the fields involved. They show up in the azimuthal dependence of the cross section, which is $\sim\cos^2(s\varphi)$, where s is the spin of the *incident* wave.

It is not obvious that this similarity would exist in the full relativistic problem because of the asymmetry introduced by ingoing-wave boundary conditions at the horizon. Perhaps a reciprocity theorem of the type given by Sanchez[19] could still be established for the full conversion-scattering problem.

IV. DISCUSSION: COMPARISON WITH OTHER CALCULATIONS

These cross sections were derived for plane incident polarization; φ is an angle from the direction of polarization. Cross sections for circular incident radiation are obtained by simply averaging over φ.

The appearance of the coefficient Ge^2 is consistent with our notation which has the electrostatic potential $A_0 \simeq e/r$ (as in electrostatic cgs units, for instance). Then, the potential energy is given by e^2/r. Hence e^2 has dimensions energy $\times$distance.

Ignoring factors of the speed of light, then, since $G\times$energy ~ length, $Ge^2\sim(\text{length})^2$. For the charge on an electron, $Ge^2\sim 10^{-68}$ cm^2.

Perhaps the most surprising feature of the results (2.20) and (3.13) is the appearance of the forward-divergent term $[\sin^2(\frac{1}{2}\theta)]^{-1}$ in the cross section. (This differs from the Coulomb forward divergence $[\sin^4(\frac{1}{2}\theta)]^{-1}$ because the integrand in Eq. (2.10) is not the same as the integrand in the Coulomb problem.) This forward divergence is one feature we definitely anticipate in any exact treatment (summed over l) of the conversion cross section.

The small-angle scattering arises from distant encounters, and it is for these that one expects the assumptions of the Born method (planeness of the incident wave, linearization of the background metric) will be most accurate. There is corroborating evidence for this result from the behavior of the phase shifts as determined by Fabbri.[9] In Moncrief's[4] angular decomposition, there are two independently propagating modes, each of which is a (different) linear combination of the gravitational and electromagnetic waves. As noted by Matzner[8] and Fabbri,[9] it is the *difference* $(\Delta\delta)_l$ between the phase shifts for these two modes which determines the conversion scattering. This means that the logarithmic phase shift typical of a long-range (e.g., Coulomb) scattering force, and typical of waves in the gravitational field of a central mass, cancel out in the calculation of the conversion scattering, a fact which lends credence to our calculation [cf. Eq. (2.5)] which ignores the logarithmic terms entirely.

More interesting is the phase-shift behavior discovered by Fabbri[9] for $l\to\infty$, $l/\omega\to\infty$. He found $(\Delta\delta)_l\sim 2e\omega G^{1/2}/l$. Although this quantity decreases with l, it does so sufficiently slowly that the cross section diverges as $\sim\sin^{-2}(\frac{1}{2}\theta)$ as we now show. [Matzner[8] noted the falloff with l of $(\Delta\delta)_l$, but incorrectly concluded that large-l terms could be ignored.].

Matzner's[8] results can be used to show that Fabbri's $(\Delta\delta)_l$ produces a conversion amplitude (gravitational radiation $\to$ electromagnetic radiation, here with circular incident polarization for simplicity):

$$f(\theta)\cong\frac{e(4\pi G)^{1/2}}{i}\sum_l(2l+1)^{1/2}\frac{1}{l}\,{}_{-1}S^2_l(\theta) , \qquad (4.1)$$

where

$${}_{-1}S^2_l(\theta)e^{2i\varphi} = {}_{-1}Y^2_l(\theta,\varphi) \qquad (4.2)$$

is a spin-weighted spherical harmonic. We now use the relation[22] (cf. Matzner and Ryan[23] for a similar calculation)

$$\left(\partial_\theta+\frac{2}{\sin\theta}-\cot\theta\right){}_{-1}S^2_l = [(l-1)(l+2)]^{1/2}\,{}_{-2}S^2_l \qquad (4.3)$$

to write the large-l relation

$$\left(\partial_\theta+\frac{2}{\sin\theta}-\cot\theta\right)f\cong\frac{4\pi e}{i}G^{1/2}\sum\left(\frac{2l+1}{4\pi}\right)^{1/2}{}_{-2}S^2_l(\theta) . \qquad (4.4)$$

Hence, if we concentrate on large-l terms which dominate the singular behavior of the cross section, we find that the conversion amplitude solves Eq. (4.4) whose source is proportional to[22]

$\delta(\cos\theta - 1)$. The solution of Eq. (4.4) is

$$f \cong \frac{k\cos^3(\frac{1}{2}\theta)}{\sin(\frac{1}{2}\theta)}. \tag{4.5}$$

The constant must be determined by demanding expansion coefficients with the form (4.1). Using the relation between ${}_sY_l^m$ and Jacobi polynomials given by Breuer *et al.*,[24] and evaluating integrals via Sec. 7.39 of the book by Gradshteyn and Ryzhik[25] one obtains $k = e\sqrt{G}$, and

$$\left.\frac{d\sigma}{d\Omega}\right|_{\substack{\text{GW to EM}\\ \text{large } l}} \cong Ge^2\frac{\cos^6(\frac{1}{2}\theta)}{\sin^2(\frac{1}{2}\theta)}. \tag{4.6}$$

Hence the small-angle cross section is proportional to $\sin^{-2}(\frac{1}{2}\theta)$, although of course Eq. (4.6), which was calculated using only the dominant large-l parts of the phase shift, is not exact. Nonetheless, there is agreement between this method and the Born approximation on the forward divergence.

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APPENDIX

The calculation of the integral in Eq. (2.10) may alternately be carried out as follows. If the outgoing direction $\hat{r}_{\text{out}}$ is given by $\hat{r}_{\text{out}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ then

$$-\vec{\Delta\omega}\cdot\vec{r}'_{\text{out}} = \omega r'[\cos\theta'(\cos\theta - 1) + \sin\theta'\sin\theta\cos(\varphi - \varphi')] \equiv \omega r'\alpha(\theta', \tilde{\varphi}'), \tag{A1}$$

where $\tilde{\varphi}' = \varphi - \varphi'$. Thus we have to solve

$$\bar{h}_{zz} = K\int_0^\infty dr'\int_0^\pi d\theta'\int_{\varphi-2\pi}^{\varphi} d\tilde{\varphi}'\exp[i\omega r'\alpha(\theta', \cos\tilde{\varphi}')]\sin^2\theta'\cos(\varphi - \tilde{\varphi}'), \tag{A2a}$$

$$\bar{h}_{xy} = K\int_0^\infty dr'\int_0^\pi d\theta'\int_{\varphi-2\pi}^{\varphi} d\tilde{\varphi}'\exp[i\omega r'\alpha(\theta', \cos\tilde{\varphi}')]\sin^2\theta'\sin(\varphi - \tilde{\varphi}'), \tag{A2b}$$

$$\bar{h}_{xz} = K\int_0^\infty dr'\int_0^\pi d\theta'\int_{\varphi-2\pi}^{\varphi} d\tilde{\varphi}'\exp[i\omega r'\alpha(\theta', \cos\tilde{\varphi}')]\sin\theta'\cos\theta', \tag{A2c}$$

where $K \equiv (eE/\pi r)e^{i\omega(t-r)}$.

If we expand $\cos(\varphi - \tilde{\varphi}')$ and $\sin(\varphi - \tilde{\varphi}')$ in (A2) we find that all the integrals involving $\sin\tilde{\varphi}'$ vanish because these integrands are odd in the regimes $[\varphi - 2\pi, \varphi - \pi]$ and $[\varphi - \pi, \varphi]$. For the remaining terms, the radial integration gives a δ_+ function[26]:

$$\bar{h}_{xy} = -\frac{\sin\varphi}{\cos\varphi}\bar{h}_{zz}, \tag{A3a}$$

$$\bar{h}_{zz} = K\cos\varphi\int_{\varphi-2\pi}^{\varphi} d\tilde{\varphi}'\int_0^\pi d\theta'\sin^2\theta'\cos\tilde{\varphi}'\left[\pi\delta(\alpha) + i\frac{P}{\alpha}\right], \tag{A3b}$$

$$\bar{h}_{xz} = -K\int_{\varphi-2\pi}^{\varphi} d\tilde{\varphi}'\int_0^\pi d\theta'\sin\theta'\cos\theta'\left[\pi\delta(\alpha) + i\frac{P}{\alpha}\right]. \tag{A3c}$$

The δ function part in (A3) can only contribute when $\alpha = 0$ which is the case for $\theta' \in [\theta/2, \pi - \theta/2]$. But in this range, the corresponding integrals vanish. Thus only the principal parts in (A3) contribute. The θ' integration of these also contributes nothing in the middle range.[27] Then, from integration over the outer intervals $[0, \theta/2]$ and $[\pi - \theta/2, \pi]$ we obtain (Ref. 25, formula 2.554) Eq. (2.13) in the text.

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