PHYSICAL REVIEW D VOLUME 23, NUMBER 12 15 JUNE 1981

Z_N topology and charge confinement in SU(N) Higgs models

Z. F. Ezawa and A. Iwazaki
Department of Physics, Tohoku University, Sendai 980, Japan
(Received 17 December 1980)

We analyze topological effects in frozen SU(N) Higgs models in continuous space-time, where topological excitations are Z_N vortices together with associated Z_N monopoles. The space dimension is either two or three. We show that vortex condensation generates magnetic gauge symmetry and that monopole condensation leads to a spontaneous breakdown of this symmetry. By summing up all possible excitation modes of Z_N vortices and Z_N monopoles, we derive an effective Lagrangian in the strong-coupling regime. We obtain the following conclusions: (i) if external charges are introduced in the fundamental representation, they are confined by electric vortex strings, and (ii) if external charges are introduced in the adjoint representation, they are screened completely.

I. INTRODUCTION

It has been suggested[1] that field configurations important to charge confinement in gauge theories are distributions of quantized magnetic vortices. These vortices are defined with respect to the center of the group, that is, Z_N for SU(N). When we adopt a lattice formulation, it is possible to extract only the Z_N degrees of freedom and to investigate a possibility of charge confinement in a condensed phase of Z_N vortices.[2] This simplification is relevant because there is a theorem[3] which relates the problem of confinement in pure Yang-Mills theories to the same problem in Z_N gauge theories. However, situations are much more complicated in continuous space-time. Firstly, in pure Yang-Mills theories there are no stable field configurations to be identified with magnetic vortices. Secondly, it is impossible to freeze all the gauge degrees of freedom except for Z_N.

We recall that SU(N) Higgs models contain magnetic Z_N vortices as topological excitations.[4-6] Furthermore, we may consider that the models interpolate pure Yang-Mills theories to Z_N gauge theories, where interpolating parameters are masses of gauge bosons. It is our ultimate purpose to study the problem of charge confinement in pure Yang-Mills theories as a limiting case of SU(N) Higgs models. This paper is the first one in this program. Here, we analyze a special class of SU(N) Higgs models, where we freeze all the gauge degrees of freedom except for the minimal components that are necessary to define Z_N vortices in continuous space-time. In this way we are able to understand the importance of Z_N vortices in the problem of confinement at its simplest form.

It is well known[5] that the SU(N) Higgs model confines magnetic monopoles by quantized magnetic vortices. These vortices are only well defined in the weak-coupling regime. As the coupling constant increases, the vacuum fluctuation is gradually dominated by virtual creations of topological excitations carrying the vacuum quantum number. They are magnetic Z_N vortices. Eventually in the strong-coupling regime, we expect that the vacuum is a condensed phase of these vortices.[7] When we place a fundamental representation Wilson loop in this phase, the well-known area law is expected to follow due to the linking of magnetic vortices with the Wilson loop.[8] Alternatively, when we introduce fundamental representation charges into the system, electric vortices are expected to emerge and confine these charges.

The formation of such electric vortices is most easily seen by deriving an effective Lagrangian in the strong-coupling regime. We shall prove this explicitly in this paper. For this purpose we use a method proposed in Refs. 9–11. In so doing, we integrate over all possible excitations of magnetic Z_N vortices. We need to emphasize that SU(N) Higgs models contain open vortices as topological excitations.[12] Namely, the models contain effectively magnetic monopoles to terminate the Abelian flux at the end points of the vortices, which we call Z_N monopoles. Topologically, Z_N monopoles are the same objects as non-Abelian Dirac monopoles carrying multiple values of N Dirac units.[6] Thus, relevant topological excitations in SU(N) Higgs models are magnetic Z_N vortices together with associated magnetic Z_N monopoles.

In summing up all these topological excitations, we argue that vortex condensation generates a magnetic gauge symmetry and that monopole condensation leads to a spontaneous breakdown of this symmetry. The magnetic gauge symmetry is accompanied by a new field which couples with the external charge introduced in the fundamental representation. We show that this new field is the agent which leads to electric vortices to confine electric charges in the strong-coupling regime.

We derive the following conclusions: (i) If external charges are introduced in the fundamental representation, they are confined by way of electric vortex strings, and (ii) if external charges are introduced in the adjoint representation, they are screened completely. Therefore, SU(N) Higgs models present ideal examples of N-ality confinement of magnetic monopoles in the weak-coupling regime as well as N-ality confinement of electric charges in the strong-coupling regime.

In Sec. II, we define frozen SU(N) Higgs models and discuss how to parametrize all possible excitation modes of Z_N vortices together with Z_N monopoles. In Secs. III and IV, we analyze these models in 3+1 dimensions and 2+1 dimensions, respectively, and show charge confinement in the strong-coupling regime. Finally, in Sec. V we summarize our conclusions.

II. Z_N VORTICES AND Z_N MONOPOLES

The Lagrangian density of a SU(N) Higgs model is defined by

$$L=-\frac{1}{4g^2}\vec{\mathrm{F}}_{\mu\nu}\cdot\vec{\mathrm{F}}_{\mu\nu}+\sum_{i=1}^{k}(D_\mu\vec{\phi}_i)^2+V(\vec{\phi}_1,\ldots,\vec{\phi}_k)\,,$$
$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu\,, \tag{2.1}$$

where k Higgs fields have been introduced in the adjoint representation to implement an appropriate symmetry breakdown. Topologically there are two simple but important ways of breaking symmetry. They are as follows.

(i) Gauge symmetry SU(N) is completely broken except for one continuous Abelian symmetry. This symmetry is chosen to include the center Z_N as a subgroup. The essence of the model is that it contains magnetic monopoles as topological solitons. These monopoles have finite masses and their charges assume multiple values of N Dirac units.[6] (We shall later give the definition of one Dirac unit in non-Abelian gauge theories.) An expedient way of achieving such a symmetry breakdown is obtained by first breaking SU(N) to its maximal Abelian subgroup and then breaking it successively up to the desired Abelian subgroup. Here, we freeze all the massive fields by increasing all their masses infinitely large. We end up with

$$L=-\frac{1}{4g^2}\sum_{n=2}^{N}F^{n^2-1}_{\mu\nu}F^{n^2-1}_{\mu\nu} \tag{2.2}$$

in the first step of breakdown, and finally with

$$L=-\frac{1}{4g^2}F^{N^2-1}_{\mu\nu}F^{N^2-1}_{\mu\nu}\,. \tag{2.3}$$

We call (2.3) the frozen SU(N) Georgi-Glashow model, where the monopole mass is given by $\sim m_S/g^2$ with m_S being a typical mass of frozen fields. Because g^2 and m_S are independent parameters, the monopole mass is arbitrary even in the frozen limit $m_S\to\infty$. The frozen SU(N) Georgi-Glashow model is simple enough to analyze,[9,10] and yet it has the same topological structure as the unfrozen model.

(ii) Gauge symmetry SU(N) is completely broken. The essence of the model is that it contains magnetic Z_N vortices together with associated Z_N monopoles as topological excitations.[5,6] An expedient way of achieving this model is obtained by introducing one more Higgs field into the SU(N) Georgi-Glashow model to break the remaining Abelian symmetry spontaneously. It is obvious by construction that the mass of vector field A^{N^2-1} can be assigned independently. Thus, corresponding to frozen Georgi-Glashow model (2.3), we obtain

$$L=-\frac{1}{4g^2}(F^{N^2-1}_{\mu\nu}F^{N^2-1}_{\mu\nu}-2m_V{}^2A^{N^2-1}_\mu A^{N^2-1}_\mu)\,, \tag{2.4}$$

which we call the frozen SU(N) Higgs model. This Lagrangian is simple enough to analyze, and yet it has the same topological structure as the unfrozen model.

In the succeeding sections, we analyze (2.4) in detail. Here we recall some properties of Z_N vortices and associated Z_N monopoles.[5,6] The essential feature of Z_N vortices is that the flux is only defined modulo N. From this we may argue that the model contains open vortices[12] as topological excitations when the associated Abelian flux assumes multiple values of N Dirac units. They are unstable topologically but have finite self-energy and well-defined size. Note that such an open vortex must have a source or sink for the Abelian flux at the end points, which we call Z_N monopoles. Topologically the Z_N monopole is the same object as the non-Abelian Dirac monopole whose charge is a multiple of N Dirac units.[6] There is also a one-to-one correspondence between Z_N monopoles in the Higgs model and monopole solitons in the Georgi-Glashow model.[6] Indeed, the Georgi-Glashow model contains a monopole soliton carrying a multiple of N Dirac units. Let us take a system of two monopole solitons with the opposite charges, and attempt to break the Abelian gauge symmetry spontaneously. Then, the Abelian flux would be squeezed into an open vortex. This open vortex is topologically unstable and it is identified to be an open Z_N vortex in the Higgs model. An explicit construction of such an open vortex has been made in an instance of gauge group SU(2).[12] In general, we may consider arbitrary configurations of Z_N monopoles together with Z_N vortices bridging them as topological excitations in the SU(N) Higgs model.

We devote the rest of this section to a summary of Z_N vortices and Z_N monopoles in the frozen Higgs model.[5,6] When we assume the unitary gauge, the boundary condition on vortex solutions is given at the vortex center. It reads

$$A_\mu^{N^2-1} \to n a_N \partial_\mu \theta\ , \quad (n = \text{integer})\ , \tag{2.5}$$

where θ is the azimuthal angle around the vortex center, and the constant a_N will be determined just below. The topological charge Q of a vortex is defined by

$$\exp(iQ/N) = \exp\left(i\tfrac{1}{2} \oint dx_\mu A_\mu^{N^2-1}\lambda\right)\ , \tag{2.6}$$

provided that phase factor (2.6) belongs to center Z_N, where λ is a diagonal Gell-Mann matrix given by

$$\lambda = \left[\frac{2}{N(N-1)}\right]^{1/2} \mathrm{diag}(1,1,\ldots,1-N)\ . \tag{2.7}$$

Therefore, we obtain $Q = 2\pi n$ (modulo N), which fixes a_N to be

$$a_N = \left[\frac{2(N-1)}{N}\right]^{1/2}\ . \tag{2.8}$$

We now define the Abelian flux Φ by

$$\Phi = \frac{1}{g} \oint dx_\mu A_\mu^{N^2-1}\ , \tag{2.9}$$

which implies $\Phi = a_N g^{-1} Q$. Thus, the fundamental Abelian magnetic charge is given by $2\pi a_N/g$, which we call one Dirac unit in SU(N) gauge theories in the standard normalization. On the other hand, the Abelian electric charge of the Higgs field is given by g/a_N in the same normalization. Note that the Dirac quantization condition is valid between the magnetic unit and the electric unit of the Abelian charges.

Singular behavior (2.5) of A_μ produces a Dirac string to field tensor $F_{\mu\nu}$ along the vortex center. Instead of giving the boundary condition as in (2.5), we may as well give it in terms of a Dirac string.[5,13] Thus,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \rho_{\mu\nu}\ , \tag{2.10}$$

where $\rho_{\mu\nu} = n a_N [\partial_\mu, \partial_\nu]\theta$ in an instance of (2.5). The advantage of doing so is that we are able to incorporate arbitrary configurations of Z_N vortices as well as associated Z_N monopoles.[5,13] They are characterized by an ensemble of Dirac strings

$$\rho_{\mu\nu}^*(x) = 2\pi a_N \sum_q n_q \int d\sigma\, d\tau \frac{\partial(y_\mu^q, y_\nu^q)}{\partial(\sigma,\tau)} \delta^{(4)}(x - y^q)\ , \tag{2.11a}$$

which satisfies

$$\partial_\mu \rho_{\mu\nu}^*(x) = 2\pi a_N N \sum_p \hat{n}_p \int d\sigma\, \dot{y}_\nu^p \delta^{(4)}(x - y^p)\ , \tag{2.11b}$$

where y^q (y^p) stands for space-time positions of Z_N vortices (monopoles), and n_q ($N\hat{n}_p$) for charges of each of the vortices (monopoles).

III. CHARGE CONFINEMENT IN 3 + 1 DIMENSIONS

The Lagrangian density of the frozen SU(N) Higgs model is given by (2.4), or

$$L = -\frac{1}{4g^2}(F_{\mu\nu}{}^2 - 2m_V{}^2 A_\mu{}^2) + eA_\mu j_\mu\ , \tag{3.1}$$

where an external source ej_μ has been introduced explicitly and the superscript N^2-1 has been suppressed. The charge e is measured in units of g and takes the following typical values: (i) $e = 1/N$ if the external charge is in the fundamental representation, and (ii) $e = 1$ if it is in the adjoint representation. For the purpose of our analysis, it is more convenient to write the source term as

$$A_\mu j_\mu \to \tfrac{1}{2} F_{\mu\nu} J_{\mu\nu}\ , \tag{3.2}$$

where

$$\partial_\nu J_{\mu\nu} = j_\mu\ . \tag{3.3}$$

The external current is represented in general as

$$j_\mu(x) = a_N{}^{-1} \sum_p \int d\sigma\, \dot{z}_\mu^p \delta^{(4)}(x - z^p)\ , \tag{3.4}$$

where the integrations are to be performed along world lines of the external charges. We may solve (3.3) with (3.4) as

$$J_{\mu\nu} = a_N^{-1} \sum_p \int d\sigma\, d\tau \frac{\partial(z_\mu^p, z_\nu^p)}{\partial(\sigma,\tau)} \delta^{(4)}(x - z^p)\ , \tag{3.5}$$

which describes world sheets swept by electric Dirac strings attached to the external charges. These strings are unphysical.

The frozen Higgs model (3.1) describes only a massive vector boson as a physical particle. It is easy to see that the external charge e is always screened. However, this argument neglects vacuum fluctuations due to topological excitations. Indeed, the model contains magnetic Z_N vortices as well as Z_N monopoles as topological excitations. Although these vortices do not appear in the physical spectrum, their existence in the vacuum fluctuation could significantly modify the structure of the Lagrangian system. This must be especially so in the strong-coupling limit $g \to \infty$ with m_V fixed, where the vortex mass density $\sim m_V{}^2/g^2$ becomes arbitrarily small and hence the vacuum would be dominated by these excitations. We now analyze this problem by deriving an effective Lagrangian in the strong-coupling regime.

We start with the generating functional

$$Z=\int [dF_{\mu\nu}][dA_\mu]\delta(F_{\mu\nu}-\partial_\mu A_\nu+\partial_\nu A_\mu)\times\exp\left(-\int L\right) \quad (3.6)$$

in the Euclidean metric, where field variables are to be integrated over all possible configurations of Z_N vortices together with Z_N monopoles. As we have noticed in Sec. II, we may generate arbitrary configurations of these excitations by introducing the corresponding set of Dirac strings and monopoles. Thus, we may rewrite (3.6) as

$$Z=\int [d(\text{string})][dF_{\mu\nu}][dA_\mu]\delta(F_{\mu\nu}-\partial_\mu A_\nu+\partial_\nu A_\mu-\rho_{\mu\nu})\times\exp\left(-\int L\right) \quad (3.7)$$

with (2.11), where $\int[d(\text{string})]$ stands for an integration over all possible configurations of Dirac strings and monopoles described by $\rho_{\mu\nu}$.

Here, we make a remark about the chemical potential of a topological excitation. When we integrate over field variables $F_{\mu\nu}$ and A_μ in (3.7), we obtain an interacting system of Z_N vortices and monopoles. We can extract the self-energy density of each excitation from the interacting system, and summarize them as follows:

$$\mathfrak{M}_1(x)=\alpha_1\frac{m_V{}^2}{g^2}\sum_q n_q{}^2\int d\sigma\, d\tau\left|\det\left(\frac{\partial(y^q_\mu,y^q_\nu)}{\partial(\sigma,\tau)}\right)\right|^{1/2}\times\delta^{(4)}(x-y^q)\,, \quad (3.8a)$$

and

$$\mathfrak{M}_2(x)=\alpha_2\frac{m_S}{g^2}N^2\sum_p \hat{n}_p{}^2\int d\sigma(\dot{y}^p_\mu\dot{y}^p_\mu)^{1/2}\delta^{(4)}(x-y^p)\,, \quad (3.8b)$$

where m_S denotes a typical mass of frozen fields, while α_κ are dimensionless positive constants whose precise values are not important, and the integrations are to be performed over world sheets (lines) swept by Dirac strings (monopoles). After separating the self-energy density terms, we recover field variables $F_{\mu\nu}$ and A_μ. The result is

$$Z=\int [d(\text{string})][dF_{\mu\nu}][dA_\mu]\delta(F_{\mu\nu}-\partial_\mu A_\nu+\partial_\nu A_\mu-\rho_{\mu\nu})\times\exp\left[-\int\left(\frac{1}{4g^2}F_{\mu\nu}{}^2+\frac{m_V{}^2}{2g^2}A_\mu{}^2+\mathfrak{M}_1+\mathfrak{M}_2-\frac{i}{2}eJ_{\mu\nu}F_{\mu\nu}\right)\right], \quad (3.9)$$

where $F_{\mu\nu}$ and A_μ now describe interactions between different points on Z_N vortices and monopoles exclusively. Namely, the Lagrangian has been normal ordered.

We note that the same procedure has been made to extract self-energies in analyzing compact QED in the continuum formulation[14] as well as in the lattice formulation.[15] The importance of the self-energy density term is clear since each component of $\mathfrak{M}_k$ acts as the chemical potential for the corresponding excitation. For instance, we observe from (3.8) and (3.9) that the vortex excitation with higher topological charge n_q is more difficult since $\mathfrak{M}_1\propto n_q{}^2$. We observe also that the vortex excitation is practically impossible in the weak-coupling limit $g\to 0$ where $\mathfrak{M}_1\to\infty$, while it is quite easy in the strong-coupling limit $g\to\infty$ where $\mathfrak{M}_1\to 0$.

Now, inserting the equation

$$\delta(F_{\mu\nu}-\partial_\mu A_\nu+\partial_\nu A_\mu-\rho_{\mu\nu})=\int [dC_{\mu\nu}]\exp[i\tfrac{1}{2}C_{\mu\nu}(F_{\mu\nu}-\partial_\mu A_\nu+\partial_\nu A_\mu-\rho_{\mu\nu})] \quad (3.10)$$

into (3.9), and integrating it over $F_{\mu\nu}$ and A_μ, we obtain

$$Z=\int [dC_{\mu\nu}]I(C_{\mu\nu})\exp\left\{-\int\left[\frac{g^2}{4}(C_{\mu\nu}+eJ_{\mu\nu})^2+\frac{g^2}{2m_V{}^2}(\partial_\mu C_{\mu\nu})^2\right]\right\}, \quad (3.11)$$

where

$$I(C_{\mu\nu})=\int [d(\text{string})]\exp\left[-\int\left(\mathfrak{M}_1+\mathfrak{M}_2+\frac{i}{2}C_{\mu\nu}\rho_{\mu\nu}\right)\right]. \quad (3.12)$$

We go on to integrate over all possible configurations of topological excitations. In order to fix the integration measure, we consider discrete space-time in the form of cubic lattices, where lattice spacing b is taken to be the width of vortices. From (2.11) and (3.8), we get

$$I(C_{\mu\nu})=\sum_{n_\mu}\sum_{n_{\mu\nu}}\sum_{x}\delta(b\partial_\mu n_{\mu\nu}-Nn_\nu)\exp\left[-b^2\left(\frac{\alpha_1 m_V{}^2}{g^2}n_{\mu\nu}{}^2+i\pi a_N C^*_{\mu\nu}n_{\mu\nu}\right)-b\frac{\alpha_2 N^2 m_s}{g^2}n_\mu{}^2\right], \tag{3.13}$$

where $C^*_{\mu\nu}$ is the dual of $C_{\mu\nu}$, and n_μ ($n_{\mu\nu}$) is an integer-valued field defined on the lattice link (plaquette). Then, inserting the equation

$$\delta(b\partial_\mu n_{\mu\nu}-Nn_\nu)=\int[dB_\nu]\exp[ibB_\nu(b\partial_\mu n_{\mu\nu}-Nn_\nu)] \tag{3.14}$$

into (3.13), and using the Poisson resummation formula, we obtain

$$I(C_{\mu\nu})=\sum_{m_\mu}\sum_{m_{\mu\nu}}\prod_{x}\int[dB_\mu]\exp\left[-\frac{b^2g^2}{16\alpha_1 m_V{}^2}\left(2\pi a_N C^*_{\mu\nu}+\partial_\mu B_\nu-\partial_\nu B_\mu+\frac{2\pi}{b^2}m_{\mu\nu}\right)^2 -\frac{bg^2}{4\alpha_2 N^2 m_s}\left(NB_\mu+\frac{2\pi}{b}m_\mu\right)^2\right], \tag{3.15}$$

where m_μ ($m_{\mu\nu}$) is an integer-valued field defined on the lattice link (plaquette). Then, making a change of variable

$$B_\mu\to B_\mu-\frac{2\pi}{bN}m_\mu\,,$$

we rewrite (3.15) as

$$I(C_{\mu\nu})=\sum_{m_\mu}\sum_{m_{\mu\nu}}\prod_{x}\int[dB_\mu]\exp\left\{-\frac{b^2g^2}{16\alpha_1 m_V{}^2}\left[2\pi a_N C^*_{\mu\nu}+\partial_\mu B_\nu-\partial_\nu B_\mu+\frac{2\pi}{b^2}m_{\mu\nu}-\frac{2\pi}{bN}(\partial_\mu m_\nu-\partial_\nu m_\mu)\right]^2 -\frac{bg^2}{4\alpha_2 m_s}B_\mu{}^2\right\}. \tag{3.16}$$

We now take the continuum limit of (3.16) and combine it with (3.11). We derive

$$Z=\int[d(\text{string})][dC_{\mu\nu}][dB_\mu] \times\exp\left\{-\int\left[\frac{g^2}{4}(C_{\mu\nu}+eJ_{\mu\nu})^2+\frac{g^2}{2m_V{}^2}(\partial_\mu C_{\mu\nu})^2+\frac{\beta_1 g^2}{4}(2\pi a_N C^*_{\mu\nu}+\partial_\mu B_\nu-\partial_\nu B_\mu+\sigma^*_{\mu\nu})^2+\frac{\beta_2 g^2 m_s{}^2}{2}B_\mu{}^2\right]\right\}, \tag{3.17}$$

where the combination of integer fields $m_{\mu\nu}$ and m_μ has been converted into $\sigma^*_{\mu\nu}$. Here,

$$\sigma_{\mu\nu}=\frac{2\pi}{N}\sum_q m_q\int d\sigma\,d\tau\,\frac{\partial(z^q_\mu,z^q_\nu)}{\partial(\sigma,\tau)}\delta^{(4)}(x-z^q) \tag{3.18a}$$

and

$$\partial_\mu\sigma_{\mu\nu}=2\pi\sum_p\hat{m}_p\int d\sigma\,\dot{z}^p_\nu\delta^{(4)}(x-z^p)\,. \tag{3.18b}$$

They are interpreted as world sheets and lines swept by electric strings and charges, respectively. The notation $\int[d(\text{string})]$ stands for an integration over all possible configurations of these strings and charges. Parameters β_k are given by $\beta_1=1/4\alpha_1 b^2 m_V{}^2$ and $\beta_2=1/2\alpha_2 b^3 m_s{}^3$ in terms of lattice spacing b.

We note that electric string singularities (3.18) and magnetic string singularities (2.11) have exactly the same expression. Namely, (3.18) represents a system of electric monopoles with m units and electric strings with m/N units, m being an integer. They give rise to topological excitations in the strong-coupling regime, which are electric monopoles with m units and electric vortex loops with m/N units. Recall that the Abelian charge of the SU(N) Higgs field in the adjoint representation has been defined to be one unit. Now we show that such a monopole excitation can annihilate an integer part of the external charge. We may interpret this phenomenon as a Debye screening. Let us denote the integer nearest to e by $[e]$. Then, we make a change of variable as

$$C_{\mu\nu}\to C_{\mu\nu}-[e]J_{\mu\nu}$$

in (3.17). We obtain

$$Z=\int[d(\text{string})][dC_{\mu\nu}][dB_\mu]\exp\left\{-\int\left[\frac{g^2}{4}(C_{\mu\nu}+\epsilon J_{\mu\nu})^2+\frac{g^2}{2m_V{}^2}(\partial_\mu C_{\mu\nu}+[e]j_\nu)^2 +\frac{\beta_1 g^2}{4}(2\pi a_N C^*_{\mu\nu}+\partial_\mu B_\nu-\partial_\nu B_\mu+\hat{\sigma}^*_{\mu\nu})^2+\frac{\beta_2 g^2 m_s{}^2}{2}B_\mu{}^2\right]\right\}, \tag{3.19}$$

where

$$\epsilon = e - [e], \tag{3.20}$$

and we have set

$$\tilde{\sigma}^*_{\mu\nu} = \sigma^*_{\mu\nu} - 2\pi a_N [e] J^*_{\mu\nu}. \tag{3.21}$$

It is obvious that we can shift the variable of integration from $\sigma_{\mu\nu}$ to $\tilde{\sigma}_{\mu\nu}$ and remove the term $2\pi a_N[e]J^*_{\mu\nu}$ from (3.19). This change of variable is possible because both $\sigma^*_{\mu\nu}$ and $J^*_{\mu\nu}$ represent world sheets swept by electric Dirac strings. Indeed, $\sigma_{\mu\nu}$ is defined by (3.18), while $J_{\mu\nu}$ is defined by (3.3), (3.4), and (3.5).

We have derived (3.19) from (3.7) by summing up all topological excitations. Physical substances of formulas (3.7) and (3.19) are as follows: In the weak-coupling regime the system is in the superconducting phase, where electric charges ($e=1$) are condensed and topological excitations are magnetic Z_N vortices. These vortices are known to be relevant to the N-ality confinement of magnetic monopoles.[5] On the other hand, in the strong-coupling regime the system is in the normal phase, where these electric charges are liberated as topological excitations into an incoherent plasma state. As a result, there is a Debye screening for the integer part of external charges. The system also contains electric Z_N vortices, which will be shown later to be relevant to the N-ality confinement of electric charges.

Formula (3.19) is equivalent to (3.7), and it is valid for all values of g^2. It contains topological excitations described by $\tilde{\sigma}_{\mu\nu}$ in the strong-coupling regime. Obviously, the chemical potential of such an excitation is proportional to $(mg)^2$, where mg is a charge of the excitation. We shall derive the effective Lagrangian governing the vacuum thereof, by setting $\tilde{\sigma}^*_{\mu\nu}=0$. In this case, we may transform (3.19) into the following equivalent form:

$$Z = \int [dF_{\mu\nu}][dA_\mu][dB_\mu]\delta\left(F_{\mu\nu} - \partial_\mu A_\nu + \partial_\nu A_\mu + i\frac{\gamma g^2}{2\pi a_N}\epsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta\right)$$
$$\times \exp\left\{-\int\left[\frac{1}{4(1+\gamma)g^2}F_{\mu\nu}{}^2 + \frac{m_V{}^2}{2g^2}A_\mu{}^2 + \frac{\gamma g^2}{16\pi^2 a_N{}^2}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta)^2 + \frac{\beta_2 g^2 m_s{}^2}{2}B_\mu{}^2 \right.\right.$$
$$\left.\left. - \frac{i}{2}\frac{\epsilon}{1+\gamma}F_{\mu\nu}J_{\mu\nu} - i[e]A_\mu j_\mu + \frac{\gamma\epsilon^2 g^2}{4(1+\gamma)}J_{\mu\nu}{}^2\right]\right\}, \tag{3.22}$$

with $\gamma = 4\pi^2 a_N{}^2\beta_1$, where we have used an equation similar to (3.10). In this way, we have derived the effective Lagrangian

$$L^{\text{eff}} = -\frac{1}{4(1+\gamma)g^2}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta)^2 + \frac{m_A{}^2}{2(1+\gamma)g^2}A_\mu{}^2 + \left(e - \frac{\gamma\epsilon}{1+\gamma}\right)A_\mu j_\mu$$
$$- \frac{\gamma g^2}{4(1+\gamma)}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta)^2 + \frac{\gamma m_B{}^2 g^2}{2(1+\gamma)}B_\mu{}^2 + \frac{\gamma\epsilon g^2}{1+\gamma}\partial_\mu B_\nu J^*_{\mu\nu}, \tag{3.23}$$

where $m_A = m_V(1+\gamma)^{1/2}$ and $m_B = 2\pi m_s a_N[\beta_2(1+\gamma)/\gamma]^{1/2}$, and we have rescaled B_μ by factor $1/2\pi a_N$. The electromagnetic field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \gamma g^2 \epsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta \tag{3.24}$$

in terms of two potentials A_μ and B_μ.

Note that the appearance of the electromagnetic field tensor in the form of (3.24) is not surprising. It has been argued[16] that such a general form is necessary in order to describe electric and magnetic field quantities simultaneously. Namely, when magnetic vortices are isolated topological excitations as in the weak-coupling regime, the standard form of $F_{\mu\nu}$ is just enough. However, this is no longer true when the vacuum is a condensed state of magnetic vortices as in the strong-coupling regime. We emphasize that we have started our theory with the standard definition of the electromagnetic field tensor and obtained the general form[16] as an effective representation in the strong-coupling regime.

Formula (3.23) is the main result of this paper. Here, we recall that typical values of external charge e are as follows: (i) $e=1$ if it is introduced in the adjoint representation, and (ii) $e=1/N$ if it is introduced in the fundamental representation. It is remarkable in (3.23) that, when $\epsilon=0$, field B_μ decouples entirely from the system. In this case the net effects of magnetic condensation are just renormalizations of coupling constant g and mass m_V, and external charge e is screened completely. We now show that, when $\epsilon\neq 0$, there are topological excitations in B_μ which are electric vortices and confine the external charges.

For this purpose we rewrite (3.23) as

$$L^{\text{eff}} = L_A^{\text{eff}} + L_B^{\text{eff}}, \tag{3.25}$$

$$(1+\gamma)g^2L_A^{\rm eff}=-\tfrac{1}{4}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta)^2+\tfrac{1}{2}m_A{}^2A_\mu{}^2+(e+[e]\gamma)g^2A_\mu j_\mu\,, \tag{3.26}$$

$$(1+\gamma)\gamma^{-1}g^{-2}L_B^{\rm eff}=-\tfrac{1}{4}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha B_\beta-\epsilon J_{\mu\nu})^2+\tfrac{1}{2}m_B{}^2B_\mu{}^2\,. \tag{3.27}$$

With respect to Lagrangian (3.26), there is a standard solution of screening type with penetration depth $\sim m_A{}^{-1}$. On the other hand, we note that Lagrangian (3.27) is precisely the one which generates open vortices with penetration depth $\sim m_B{}^{-1}$, as has been discussed in Ref. 13. Here, vortices are parametrized by electric Dirac strings $J_{\mu\nu}$ given by (3.5). We now argue that these vortices lead to the N-ality confinement of electric charges.

Let us suppose that there exist N external charges each of which carries $1/N$ unit. Note that these charges are accompanied with electric Dirac strings. We choose these strings to meet at one point. Then, according to (3.27), we obtain N physical electric vortices with finite length. At the joint there arises a charge with one unit, but such a charge is screened. Thus, the SU(N) Higgs model presents an ideal scheme for the N-ality confinement of electric charges in the strong-coupling regime.

Finally, we wish to remark that effective Lagrangian (3.23) includes *electric* gauge symmetry

$$A_\mu\to A_\mu+\partial_\mu f\,, \tag{3.28}$$

which has been broken by the Higgs mechanism, and *magnetic* gauge symmetry

$$B_\mu\to B_\mu+\partial_\mu f\,, \tag{3.29}$$

which has been broken dynamically. We could associate the origin of the magnetic gauge symmetry with the vortex condensation, while its spontaneous breakdown with the monopole condensation. We now explain this statement.

We notice that field B_μ has been introduced as the Lagrange multiplier to assure magnetic flux conservation (2.11b). Let us argue that, except for Z_N monopoles, the magnetic gauge symmetry would be exact, or that B_μ would remain massless. If there are no magnetic monopole excitations, topological excitations are only composed of closed magnetic vortex loops. Here, we introduce magnetic monopoles carrying one Dirac unit of charges into the system as test particles, in place of external electric charges in (3.1). Since the Meissner effect is operating in the weak-coupling regime, a pair of monopoles is confined by a magnetic vortex. As vortex-loop excitations dominate the vacuum fluctuation, this confining magnetic vortex is under the influence of the vacuum fluctuation and may take an arbitrarily long shape. This implies that there is a long-range Coulomb interaction between magnetic monopoles, which is quite easily shown to be mediated by field B_μ. However, the SU(N) Higgs model actually contains magnetic Z_N monopoles, which are topological excitations in the weak-coupling regime but are condensed in the vacuum in the strong-coupling regime. These condensed magnetic charges may screen external magnetic charges. Thus, field B_μ must be massive and the magnetic gauge symmetry is broken spontaneously.

IV. CHARGE CONFINEMENT IN 2+1 DIMENSIONS

The Lagrangian density of the present model is defined formally by the same formula as in the (3 + 1)-dimensional model. Moreover, we may analyze the model essentially in the same way as in the (3 + 1)-dimensional case. Here, we only present those equations which differentiate these two models.

The present model also contains magnetic Z_N vortices and Z_N monopoles in three-dimensional Euclidean space. These topological excitations are characterized by the boundary condition given in terms of Dirac strings as in (2.10):

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+\rho_{\mu\nu}\,. \tag{4.1}$$

where

$$\rho_\mu^*(x)=2\pi a_N\sum_q n_q\int d\sigma\,\dot y_\mu^q\,\delta^{(3)}(x-y^q) \tag{4.2a}$$

parametrizes vortex centers, and

$$\partial_\mu\rho_\mu^*(x)=2\pi N a_N\sum_p \hat n_p\delta^{(3)}(x-y^p) \tag{4.2b}$$

parametrizes monopole centers. The chemical potentials of these excitations are

$$\mathfrak{M}_1(x)=\alpha_1\frac{m_V{}^2}{g^2}\sum_q n_q{}^2\int d\sigma\,(\dot y_\mu^q\dot y_\mu^q)^{1/2}\,\delta^{(3)}(x-y^q) \tag{4.3a}$$

and

$$\mathfrak{M}_2(x)=\alpha_2\frac{m_s}{g^2}N^2\sum_p \hat n_p{}^2\delta^{(3)}(x-y^p)\,. \tag{4.3b}$$

Formula (4.2b), representing magnetic flux conservation, introduces a scalar field B just as in (3.14). Then, the integration over all possible configurations of excitations is performed similarly as in (3.15), and we obtain

$$\int [d(\text{string})]\exp\left[-\int (\mathfrak{M}_1+\mathfrak{M}_2+\tfrac{1}{2}iC_{\mu\nu}\rho_{\mu\nu})\right]$$
$$=\sum_m\sum_{m_\mu}\prod_x\int[dB]\exp\left[\frac{-bg^2}{4\alpha_1 m_V{}^2}\left(2\pi a_N C_\mu^* + \partial_\mu B+\frac{2\pi}{b}m_\mu\right)^2-\frac{g^2}{4\alpha_2 N^2 m_s}(NB-2\pi m)^2\right]. \quad (4.4)$$

After making a change of variable

$$B\to B-\frac{2\pi}{N}m,$$

we take the continuum limit of (4.4) and obtain

$$Z=\int[d(\text{string})][dC_{\mu\nu}][dB]\exp\left\{-\int\left[\frac{g^2}{4}(C_{\mu\nu}+eJ_{\mu\nu})^2+\frac{g^2}{2m_V{}^2}(\partial_\mu C_{\mu\nu})^2 +\frac{\beta_1 g^2}{4}(2\pi a_N C_\mu^*+\partial_\mu B+\sigma_\mu^*)^2+\frac{\beta_2 g^2 m_s{}^2}{4}B^2\right]\right\}, \quad (4.5)$$

which corresponds to (3.17), where $\beta_1=1/\alpha_1 m_V{}^2 b^2$, $\beta_2=1/\alpha_2 m_s{}^3 b^3$, and

$$\sigma_{\mu\nu}=\frac{2\pi}{N}\sum_q m_q\int d\sigma\, d\tau\,\frac{\partial(z_\mu^q,z_\nu^q)}{\partial(\sigma,\tau)}\,\delta^{(3)}(x-z^q), \quad (4.6\text{a})$$

with

$$\partial_\mu\sigma_{\mu\nu}=2\pi\sum_p \hat{m}_p\int d\sigma\,\dot{z}_\nu^p\,\delta^{(3)}(x-z^p). \quad (4.6\text{b})$$

These singularities give rise to excitations of electric Z_N vortices and monopoles. Typical examples are a closed vortex with $1/N$ unit, and an open vortex with one unit. However, in the $(2+1)$-dimensional case, there is an interesting phenomenon associated with a closed vortex with $1/N$ unit. We could interpret that such a closed loop creates a domain wall[1] to separate two different vacuums in Minkowski space-time.

In order to explain this observation, it is more convenient to perform a summation over m in (4.4). Since we are interested in the strong-coupling limit $g^2\to\infty$, we may use a Villain approximation:

$$\sum_m\exp\left[-\frac{ag^2}{2}(NB+2\pi m)^2\right]\approx\exp(ag^2\cos NB). \quad (4.7)$$

Then, (4.5) is replaced by

$$Z=\int[d(\text{string})][dC_{\mu\nu}][dB]\exp\left\{-\int\left[\frac{g^2}{4}(C_{\mu\nu}+eJ_{\mu\nu})^2+\frac{g^2}{2m_V{}^2}(\partial_\mu C_{\mu\nu})^2 +\frac{\beta_1 g^2}{4}(2\pi a_N C_\mu^*+\partial_\mu B+\tilde{\sigma}_\mu^*)^2-\frac{\beta_2 g^2 m_s{}^2}{2N^2}\cos NB\right]\right\}, \quad (4.8)$$

where

$$\tilde{\sigma}_{\mu\nu}=2\pi\sum_q m_q\int d\sigma\, d\tau\,\frac{\partial(z_\mu^q,z_\nu^q)}{\partial(\sigma,\tau)}\,\delta^{(3)}(x-z^q), \quad (4.9)$$

with the same formula as (4.6b) for $\partial_\mu\tilde{\sigma}_{\mu\nu}$. Now, these singularities describe electric vortices and monopoles carrying only integer units of charges. Domain-wall excitations are topological excitations of field variable B, which we shall discuss by making use of the effective Lagrangian.

We may derive the effective Lagrangian in the strong-coupling regime, just as we did for the $(3+1)$-dimensional model. We obtain

$$L^{\text{eff}}=L_A^{\text{eff}}+L_B^{\text{eff}}, \quad (4.10)$$

$$(1+\gamma)g^2L_A^{\text{eff}}=-\tfrac{1}{4}(\epsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta)^2+\tfrac{1}{2}m_A{}^2A_\mu{}^2 +(e+\gamma[e])g^2A_\mu j_\mu, \quad (4.11)$$

$$(1+\gamma)\gamma^{-1}g^{-2}L_B^{\text{eff}}=\tfrac{1}{2}(\partial_\mu B-\epsilon J_\mu^*)^2 -\eta(1-\cos 2\pi N a_N B), \quad (4.12)$$

as in (3.25), (3.26), and (3.27), where $\gamma=2\pi^2 a_N{}^2\beta_1$, $m_A=m_V(1+\gamma)^{1/2}$, $\eta=m_s{}^2(1+\gamma)\beta_2 g^4/2\gamma N^2$. The electromagnetic field tensor is given by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+\gamma g^2\epsilon_{\mu\nu\alpha}\partial_\alpha B. \quad (4.13)$$

With respect to Lagrangian (4.11), there is a standard solution of screening type. On the other hand, we note that (4.12) is exactly the Lagrangian which generates domain walls,[1] as has been discussed in Ref. 9. For instance, a domain wall separates two vacuums characterized by $B = 0$ and $B = g^2/Na_N$. Such a domain wall carries an electric flux corresponding to $\Delta B = 1/Na_N$, and this flux is precisely the one which the external source with $e = 1/N$ carries. Namely, external charges with $e = 1/N$ are confined with domain walls, while external charges with $e = 1$ are simply screened. Thus, the SU(N) Higgs model presents an ideal scheme for the N-ality confinement of electric charges in the strong-coupling regime.

Finally, we wish to remark that effective Lagrangian (4.10) includes *electric* gauge symmetry

$$A_\mu \rightarrow A_\mu + \partial_\mu f \,, \tag{4.14}$$

which has been broken by the Higgs mechanism, and *magnetic* gauge symmetry

$$B \rightarrow B + \text{constant}, \tag{4.15}$$

which has been broken dynamically.

V. DISCUSSIONS

In this paper, using a method proposed in Refs. 9–11, we have derived effective Lagrangian (3.23) of the frozen SU(N) Higgs model in the strong-coupling regime, where the vacuum is a condensed phase of magnetic vortices together with associated magnetic monopoles. The remarkable feature is that vortex condensation generates magnetic gauge symmetry (3.29) while monopole condensation leads to a spontaneous breakdown of this symmetry. Here, the magnetic gauge symmetry is accompanied with a new field denoted by B_μ which couples with the external charge introduced in the fundamental representation.

We may interpret this phenomenon as follows. In the weak-coupling regime the system is in the superconducing phase, where electric charges ($e = 1$) are condensed. As the coupling constant gets larger, the vacuum fluctuation is gradually dominated by excitations of Z_N vortices. Eventually, in the strong-coupling regime the system is in the normal phase, where those electric charges are liberated into an incoherent plasma state. As a result, there is a Debye screening for integer charges in the strong-coupling regime as well. Thus, field A_μ remains massive. However, such a plasma of electric charges cannot screen fractional charges ($e = 1/N$). Hence, there are long-range forces between fractional charges, which must be created either by electric vortices or by Coulomb interactions. The criterion is whether field B_μ is massive or massless.

To proceed with our interpretations, we now introduce magnetic monopoles into the system as test particles. These monopoles are supposed to carry one Dirac unit of magnetic charges. As is well known, the Meissner effect is operating in the superconducting phase where a pair of these magnetic monopoles is confined by a vortex. However, as vortex-loop excitations get abundant in the vacuum fluctuation, the above confining vortex is also subject to the fluctuation and may take an arbitrarily long shape. Therefore, but for Z_N monopoles, there would be a long-range Coulomb interaction between these magnetic monopoles, which is easily shown to be mediated by field B_μ. However, the SU(N) Higgs model actually contains magnetic Z_N monopoles, which are topological excitations in the weak-coupling regime but are condensed in the vacuum in the strong-coupling regime. These condensed magnetic charges may screen external magnetic charges. Thus, field B_μ must be massive, or the magnetic gauge symmetry must be broken spontaneously. We emphasize that field B_μ solely couples either with magnetic monopoles or with fractional electric charges.

We have obtained the following conclusions: (i) if external charges are introduced in the fundamental representation, they are confined by way of electric vortex strings, and (ii) if external charges are introduced in the adjoint representation, they are screened completely. Moreover, we have shown that electric vortices are nothing but domain walls[1] separating two different vacuums in 2 + 1 dimensions. Therefore, the present model gives an ideal example of N-ality confinement of magnetic monopoles in the weak-coupling regime, and N-ality confinement of electric charges in the strong-coupling regime.

In this paper, we have analyzed a special class of SU(N) Higgs models, where we have frozen all the gauge degrees of freedom except for the component that is necessary to define Z_N vortices in continuous space-time. It is important for our program to recover the complete gauge degrees of freedom. Then, we might conjecture that vortex condensation would generate magnetic SU(N) gauge symmetry[17] in such a complete SU(N) Higgs model. We are currently investigating this problem.

[1]G. 't Hooft, Nucl. Phys. B138, 1 (1978); B153, 141 (1979).
[2]C. P. Korthals Altes, Nucl. Phys. B142, 315 (1978); T. Yoneya, *ibid.* B144, 195 (1978); A. Casher, *ibid.* B151, 353 (1979).
[3]G. Mack and V. B. Petkova, Ann. Phys. (N. Y.) 123, 447 (1979); 125, 117 (1980).
[4]S. Mandelstam, Phys. Lett. 53B, 476 (1975).
[5]Z. F. Ezawa and H. C. Tze, Nucl. Phys. B100, 1 (1975); Phys. Rev. D 14, 1006 (1976).
[6]Z. F. Ezawa and H. C. Tze, J. Math. Phys. 17, 2228 (1976); Phys. Rev. D 15, 1647 (1977).
[7]M. Stone and P. R. Thomas, Phys. Rev. Lett. 41, 351 (1978); D. Foerster, Phys. Lett. 77B, 211 (1978); 78B, 473 (1978).
[8]J. M. Cornwall, Nucl. Phys. B157, 392 (1979).
[9]Z. F. Ezawa, Phys. Lett. 85B, 87 (1979).
[10]Z. F. Ezawa, Phys. Lett. 86B, 313 (1979).
[11]Z. F. Ezawa, Nucl. Phys. B157, 444 (1979); K. Bardacki and E. Rabinovici, Phys. Rev. D 20, 1360 (1979).
[12]R. Brout, F. Englert, and W. Fischler, Phys. Rev. Lett. 36, 649 (1976).
[13]Y. Nambu, Phys. Rev. D 10, 4262 (1974).
[14]A. M. Polyakov, Nucl. Phys. B120, 429 (1977).
[15]A. M. Polyakov, Phys. Lett. 59B, 82 (1975); T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977).
[16]N. Cabibbo and E. Ferrari, Nuovo Cimento 23, 1147 (1962); R. Tevikyan, Zh. Eksp. Teor. Fiz. 50, 911 (1966) [Sov. Phys.-JETP 23, 606 (1966)]; D. K. Ross, Phys. Rev. 181, 2055 (1969).
[17]S. Mandelstam, Phys. Rev. D 19, 2391 (1979).