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Interior C-metric

Hamid Farhoosh* and Robert L. Zimmerman
Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403
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An interior solution for a uniformly accelerating particle is constructed and joined continuously with the exterior vacuum C-metric. At the boundary of the interior solution there exists a discontinuity in the pressure which is responsible for the object's uniform acceleration.

I. INTRODUCTION

The static vacuum C-metric represents the geometric properties of space-time about a uniformly accelerating particle. The C-metric solution is the generalization of the Schwarzschild solution which includes uniform linear acceleration. Even though this solution describes an accelerating object which is emitting gravitational radiation, it can be expressed in a static coordinate system which is rigidly attached to the uniformly accelerating particle. It is the purpose of this paper to construct an interior solution for this vacuum C-metric.

The vacuum C-metric was first constructed in 1918 by Levi-Civita.[1] It was rediscovered in 1961 by Newman and Tamburino[2] and its properties and physical interpretations have been explored by several authors.[3-16]

The question of what provides the acceleration of the particle is an interesting one. Kinnersley and Walker[5] have shown that in the exact solution the two-surface surrounding the particle possesses a conical singularity at the north or south pole. In an asymptotic flat coordinate system this conical singularity manifests itself as a line singularity.[13] Kinnersley and Walker suggested that the nodal singularity is a manifestation of the neglect of the force necessary to accelerate a massive particle. Ernst[7] showed that for the charged C-metric solution the nodal singularity could be removed for small acceleration by the addition of an external electric field.

In this paper, we suggest an alternate view of what might produce the acceleration of the particle. We conjecture that the nodal singularity is a manifestation of the uniform acceleration and is not a direct consequence of neglecting the external force that is necessary to accelerate the object. We suggest that the acceleration of the object is a consequence of the reaction of the emission of gravitational radiation that the particle is anisotropically emitting. We will support this claim by constructing an interior solution and joining it continuously onto the exterior vacuum C-metric. In so doing, we will form a discontinuity in the pressure at the surface of the object. It is this discontinuity in the pressure which gives rise to the force which is necessary for the acceleration of the object.

II. INTERIOR SOLUTION

The vacuum C-metric assumes its simplest mathematical form in the (t,q,p,ω) coordinate system[14]

$$ds^2=\frac{1}{A^2(p+q)^2}\left[A^2F(q)dt^2-\frac{dq^2}{F(q)}-\frac{dp^2}{G(p)}-G(p)d\omega^2\right], \tag{1a}$$

where

$$G(p)=1-p^2-2Amp^3\,, \tag{1b}$$

$$F(q)=-1+q^2-2Amq^3\,. \tag{1c}$$

m and A are related, respectively, to the mass and acceleration of the particle. The (t,q,p,ω) are related to the more familiar spherical coordinates (t,r,θ,ϕ) by the transformations

$$r=\frac{1}{A(p+q)}\,, \tag{2a}$$

$$G(p)=\sin^2\theta\,, \tag{2b}$$

$$\omega=\phi\,. \tag{2c}$$

In terms of the (t,r,θ,ϕ) coordinates, the line element in Eq. (1) becomes

$$ds^2=H\,dt^2-\frac{dr^2}{H}+\frac{2\sin\theta\cos\theta}{Hp(1+3Amp)}Ar^2dr\,d\theta -\frac{r^2\cos^2\theta}{p^2(1+3Amp)^2}\left(1+\frac{A^2r^2\sin^2\theta}{H}\right)d\theta^2-r^2\sin^2\theta\,d\phi^2\,, \tag{3a}$$

where

$$H=1-2Arp-A^2r^2(1-p^2)-\frac{2m}{r}(1-Arp)^3 =A^2r^2F \tag{3b}$$

and p is related to θ by Eq. (2b).

If we set $A=0$, the line element in Eq. (3) reduces to the Schwarzschild solution in the standard spherical coordinates. If, instead, we let m go to zero in Eq. (3) we get a flat-space line element as expressed in a spherical coordinate system whose origin is undergoing uniform acceleration.

The line elements in Eqs. (1) and (3) represent an accelerating Schwarzschild-type object. It is the purpose of this paper to produce an interior solution which can be joined continuously across some boundary with this exterior vacuum solution. Because of the acceleration this interior solution will not only contain information about the structure of matter but it will also include the properties of the source which is producing the acceleration.

We require that the stress-energy tensor that describes the matter portion of the interior C-metric solution be analogous to the Schwarzschild interior solution of constant density. As we let the acceleration vanish in the C-metric interior solution the line element will reduce to the interior Schwarzschild solution

$$ds^2 = \left(1 - \frac{2mr^2}{r_0^3}\right)\left[\frac{3}{2}\left(\frac{1-2m/r_0}{1-2mr^2/r_0^3}\right)^{1/2} - \frac{1}{2}\right]^2 dt^2 - \frac{dr^2}{1-2mr^2/r_0^3} - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 . \quad (4)$$

Requiring the matter to have constant density will uniquely determine the properties of the source which is responsible for the acceleration.

Equation (4) represents the line element for a spherically symmetric object with constant density $\rho = 3m/4\pi r_0^3$. The metric tensor and its first partial derivatives as defined by Eq. (4) join continuously across the boundary at $r = r_0$ with the vacuum Schwarzschild solution. These conditions imply the Lichnerowicz junction conditions, which state that for a perfect fluid the pressure must vanish at the boundary and that, if n_μ is the unit normal to the boundary at $r = r_0$, the fluid four-velocity u^μ must satisfy $u^\mu n_\mu = 0$ on the boundary.[17] These Lichnerowicz junction conditions are too restrictive for accelerating objects because we must have a discontinuity in the pressure at the boundary in order to produce a force which causes the particle to accelerate.

In constructing the interior C-metric solution it is more convenient to work with the (t, q, p, ω) coordinates where the line element assumes the form

$$ds^2 = \frac{1}{A^2(p+q)^2}(B\,dt^2 + C\,dq^2 + D\,dp^2 + E\,d\omega^2) . \quad (5)$$

In this coordinate system the surface of infinite red-shift in the exterior C-metric is located at q = constant and the metric terms for the angular coordinates (p, ω) assume a simple form. As was the case for the interior Schwarzschild solution we assume the shape of the object to be analogous to the shape of the surface of infinite red-shift. Therefore we assume the boundary of the object to be described by $q = \text{constant} \equiv q_0$. Also, requiring the axial symmetry of the interior solution to be the same as the vacuum line element requires

$$E = D^{-1} = G(p) . \quad (6)$$

In addition, we assume that the metric coefficients B and C are only a function of q. Therefore, the interior line element is assumed to be of the form

$$ds^2 = \frac{1}{A^2(p+q)^2} \times \left(A^2\overline{F}(q)Q(q)dt^2 - \frac{dq^2}{\overline{F}(q)} - \frac{dp^2}{G(p)} - G(p)d\omega^2\right) , \quad (7)$$

where $\overline{F}(q)$ and $Q(q)$ are to be determined by the field equations.

The stress-energy tensor is assumed to be described in terms of an energy density and pressure. Part of the contribution to the stress-energy tensor will be due to the structure of matter and the other part will be from the source which is responsible for the acceleration. This separation of the matter and source terms is not covariant, however, it is useful from a mathematical point of view in constructing a solution which is analogous to a constant density of matter. The components of the stress-energy tensor $T_{\mu\nu}$ are assumed to be of the form

$$T_{tt} = \rho g_{tt} = (\rho_m + \rho_s) g_{tt} , \quad (8a)$$

$$T_{qq} = P_1 g_{qq} , \quad (8b)$$

$$T_{pp} = P_2 g_{pp} , \quad (8c)$$

$$T_{\omega\omega} = P_3 g_{\omega\omega} . \quad (8d)$$

ρ_m is the energy density of matter, ρ_s is the energy density of the source which produces the acceleration, and P_i are the anisotropic components of the pressure due to both the matter and source terms.

Using the line element in Eq. (7) and the stress-energy tensor in Eq. (8), the field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \quad (9)$$

become

$$T_{tt} = (\rho_m + \rho_s) g_{tt} = -\frac{A^2}{16\pi}\left[(6\overline{F} + 6 - 2q\overline{F}_{,q} - 2q^2) + 2p(2q - \overline{F}_{,q} - 6Amq^2)\right] g_{tt} , \quad (10a)$$

$$T_{qq}=P_1 g_{qq}=\frac{A^2}{16\pi}\left[\left(6\overline{F}+6-2q\overline{F}_{,q}-2q^2-2q\overline{F}\frac{Q_{,q}}{Q}\right)+2p\left(2q-\overline{F}_{,q}-6Amq^2-\overline{F}\frac{Q_{,q}}{Q}\right)\right]g_{qq}, \tag{10b}$$

$$T_{pp}=P_2 g_{pp}=\frac{g_{pp}}{g_{\omega\omega}}T_{\omega\omega}=\frac{g_{pp}}{g_{\omega\omega}}(P_3 g_{\omega\omega})$$

$$=\frac{A^2}{16\pi}\left\{\left[6\overline{F}+6-4q\overline{F}_{,q}-2q\overline{F}\frac{Q_{,q}}{Q}+q^2\left(\overline{F}_{,qq}+\tfrac{3}{2}\overline{F}_{,q}\frac{Q_{,q}}{Q}+\overline{F}\frac{Q_{,qq}}{Q}-\tfrac{1}{2}\overline{F}\frac{Q_{,q}^2}{Q^2}\right)\right]\right.$$

$$\left.+2p\left[2q-2\overline{F}_{,q}-\overline{F}\frac{Q_{,q}}{Q}-p+6Ampq+(\tfrac{1}{2}p+q)\left(\overline{F}_{,qq}+\tfrac{3}{2}\overline{F}_{,q}\frac{Q_{,q}}{Q}+\overline{F}\frac{Q_{,qq}}{Q}-\tfrac{1}{2}\overline{F}\frac{Q_{,q}^2}{Q^2}\right)\right]\right\}g_{pp}, \tag{10c}$$

where a comma represents an ordinary derivative.

We notice that the field equations in (10) contain angular terms as represented by the p variable. This result is to be expected because of the anisotropy produced by the acceleration. We can now conveniently separate the energy density into two parts. On the right-hand side of Eq. (10a) we identify the p-independent term with the energy density of matter, and the p-dependent term we define to be the energy density of the source ρ_s, i.e.,

$$\rho_m=-\frac{A^2}{16\pi}(6\overline{F}+6-2q\overline{F}_{,q}-2q^2) \tag{11}$$

and

$$\rho_s=-\frac{A^2}{8\pi}p(2q-\overline{F}_{,q}-6Amq^2). \tag{12}$$

To construct a solution which is analogous to the interior Schwarzschild solution of constant density we require ρ_m to be a constant. From Eq. (10a) we get the following differential equation for $\overline{F}(q)$:

$$6\overline{F}-2q\overline{F}_{,q}-2q^2+6=-\frac{16\pi}{A^2}\rho_m. \tag{13}$$

The solution of this equation is

$$F(g)=-\left(1+\frac{8\pi}{3A^2}\rho_m\right)+q^2+Kq^3. \tag{14}$$

K is a constant of integration and is set equal to zero since it is analogous to the singular term in the interior Schwarzschild solution which is dropped in order that the interior solution be nonsingular.

Requiring that the g_{qq} component of the metric join continuously across the $q=q_0$ boundary gives us a relation between the mass and density. Comparing Eqs. (1) and Eq. (7), where $\overline{F}$ is given by Eq. (14), we have

$$\rho_m=\frac{3m}{4\pi}A^3 q_0^{\,3}. \tag{15}$$

We still have a great deal of freedom in the description of the stress-energy tensor through the choice of $Q(q)$. We choose the simplest function of q which will allow the interior metric to join continuously to the exterior metric at $q=q_0$ and will reduce to the Schwarzschild interior solution as the acceleration vanishes. The simplest choice is

$$Q(q)=\left[\frac{3}{2}\left(\frac{1-2Amq_0}{1-2Amq_0^{\,3}/q^2}\right)^{1/2}-\frac{1}{2}\right]^2. \tag{16}$$

We have now determined all the metric components for the interior C-metric solution. The interior C-metric solution is given by the Eq. (7), where $G(p)$ and $Q(q)$ are defined in Eqs. (1b) and (16), respectively, and $\overline{F}(q)$ is given by

$$\overline{F}(q)=-1+q^2-2Amq_0^{\,3}, \tag{17}$$

where q_0 is a constant defining the boundary of the accelerating object. The matter energy density is given by (15) and from (12), (10b), (10c), and (17) the density of the source causing the acceleration and the interior anisotropic pressure are given by

$$\rho_s=\frac{3m}{4\pi}A^3pq^2, \tag{18a}$$

$$T_{qq}=P_1 g_{qq}=\frac{3m}{4\pi}\left[A^3(q_0^{\,3}+pq^2)+\tfrac{2}{3}A^3\frac{q_0^{\,3}}{q^3}(p+q)(1+\tfrac{1}{2}Q^{-1/2})\frac{1-q^2(1-2Amq_0^{\,3}/q^2)}{1-2Amq_0^{\,3}/q^2}\right]g_{qq}, \tag{18b}$$

$$T_{pp}=P_2 g_{pp}=\frac{g_{pp}}{g_{\omega\omega}}T_{\omega\omega}=\frac{3m}{4\pi}\left\{A^3(q_0^{\,3}+pq^2)+\tfrac{2}{3}A^3\frac{q_0^{\,3}}{q^3}(p+q)(1+\tfrac{1}{2}Q^{-1/2})\left[\frac{1-q^2(1-2Amq_0^{\,3}/q^2)}{1-2Amq_0^{\,3}/q^2}\right]\right.$$

$$\left.+A^3(p+q)\left[\frac{q_0^{\,3}}{q^4}\frac{(p+q)(1+\tfrac{1}{2}Q^{-1/2})}{1-2Amq_0^{\,3}/q^2}-pq\right]\right\}g_{pp}. \tag{18c}$$

At the surface of the object there is a discontinuity in the pressure. Evaluating Eqs. (18b) and (18c) at $q=q_0$, the discontinuity in the pressure is

$$(P_1)_0=\frac{3m}{4\pi}\frac{A^3(q_0+p)}{1-2Amq_0}, \tag{19a}$$

$$(P_2)_0=\frac{3m}{4\pi}\frac{A^3(q_0+p)}{1-2Amq_0}\left[1+\tfrac{3}{2}\left(1+\frac{p}{q_0}\right)-pq_0(1-2Amq_0)\right]=(P_3)_0, \tag{19b}$$

where the index 0 indicates the fact that the quantities are being evaluated on the surface.

P_1 represents the pressure normal to the surface $q=q_0$, and Eq. (19a) represents the discontinuity in this pressure at the surface. This discontinuity represents a force normal to the surface. Integrating this force over the surface of the object will give the net force acting on the surface of the object. In the forward hemisphere $P<0$ and in the backward hemisphere $P>0$. Therefore, upon integrating (19a) over the surface of the object there will be a net force directed along the symmetry axis.

It is probably more familiar if we transform Eqs. (7), (15), and (18) to spherical coordinates (t,r,θ,ϕ). Using transformation equations (2) we have

$$ds^2=A^2r^2\overline{F}Q\,dt^2-\frac{1}{A^2r^2\overline{F}}dr^2+\frac{2\sin\theta\cos\theta}{A\overline{F}p(1+3Amp)}dr\,d\theta-\frac{r^2\cos^2\theta}{p^2(1+3Amp)^2}\left(1+\frac{\sin^2\theta}{\overline{F}}\right)d\theta^2-r^2\sin^2\theta\,d\phi^2, \tag{20a}$$

where

$$\overline{F}=\frac{1}{A^2r^2}\left[(1-Arp)^2-\frac{2mr^2}{R_0^3}-A^2r^2\right], \tag{20b}$$

$$Q=\left[\frac{3}{2}\left(\frac{1-2m/R_0}{1-2mr^2/R_0^3(1-Arp)^2}\right)^{1/2}-\frac{1}{2}\right]^2, \tag{20c}$$

and

$$T_{tt}=\frac{3m}{4\pi R_0^3}\left[1+\frac{ApR_0^3}{r^2}(1-Arp)^2\right]g_{tt}, \tag{21a}$$

$$T_{rr}=\frac{g_{rr}}{g_{r\theta}}T_{r\theta}=\frac{3m}{4\pi R_0^3}\left\{1+\frac{ApR_0^3}{r^2}(1-Arp)^2-\frac{2}{3}\frac{1+\frac{1}{2}Q^{-1/2}}{(1-Arp)^3}\frac{(1-Arp)^2[1-2mr^2/R_0^3(1-Arp)^2]-A^2r^2}{1-2mr^2/R_0^3(1-Arp)^2}\right\}g_{rr}, \tag{21b}$$

$$T_{\theta\theta}=\frac{3m}{4\pi R_0^3}\left\{1+\frac{ApR_0^3}{r^2}(1-Arp)^2-\frac{2}{3}\frac{1+\frac{1}{2}Q^{-1/2}}{(1-Arp)^3}\frac{(1-Arp)^2[1-2mr^2/R_0^3(1-Arp)^2]-A^2r^2}{1-2mr^2/R_0^3(1-Arp)^2}\right.$$
$$\left.+\frac{\overline{F}}{\overline{F}+\sin^2\theta}\left[\frac{A^2r^2}{(1-Arp)^4}\frac{1+\frac{1}{2}Q^{-1/2}}{1-2mr^2/R_0^3(1-Arp)^2}-\frac{Ap(1-Arp)R_0^3}{r^2}\right]\right\}g_{\theta\theta}, \tag{21c}$$

$$T_{\phi\phi}=\frac{3m}{4\pi R_0^3}\left\{1+\frac{ApR_0^3}{r^2}(1-Arp)^2-\frac{2}{3}\frac{1+\frac{1}{2}Q^{-1/2}}{(1-Arp)^3}\frac{(1-Arp)^2[1-2mr^2/R_0^2(1-Arp)^2]-A^2r^2}{1-2mr^2/R_0^3(1-Arp)^2}\right.$$
$$\left.+\frac{A^2r^2}{(1-Arp)^4}\frac{1+\frac{1}{2}Q^{-1/2}}{1-2mr^2/R_0^3(1-Arp)^2}-\frac{Ap(1-Arp)R_0^3}{r^2}\right\}g_{\phi\phi}. \tag{21d}$$

In these equations R_0 is the value of the radial coordinate r on the surface on the equatorial plane (where $p=0$) and is related to q_0 by

$$q_0=\frac{1}{AR_0}. \tag{22}$$

As the acceleration goes to zero the interior metric and the stress-energy tensor given by Eqs. (20) and (21) reduce to those of the interior Schwarzschild solution in the standard spherical coordinates.

III. CONCLUSIONS

The interior C-metric solution given by Eqs. (7), (1b), (16), and (17) [or equivalently by Eq. (20)] is an interesting solution because it joins an interior solution of an accelerating object with a vacuum solution that contains gravitational radiation.[13] To the authors' knowledge this is the only example of its kind. There are two rather unusual properties illustrated by this solution: (1) The first derivative of the g_{tt} and g_{qq} components of the metric are discontinuous across the boundary; (2) the stress-energy tensor becomes singular at the origin.

Although the components of the metric tensor are continuous across the boundary $q=q_0$, the first derivatives of g_{tt} and g_{qq} have discontinuity across the boundary which are

$$\Delta(g_{tt,q})_0 \equiv [(g_{tt,q})_{\text{inside}}]_0 - [(g_{tt,q})_{\text{outside}}]_0$$

$$= \frac{6Am}{(p+q_0)^2(1-2Amq_0)} ,$$

$$\Delta(g_{qq,q})_0 \equiv [(g_{qq,q})_{\text{inside}}]_0 - [(g_{qq,q})_{\text{outside}}]_0$$

$$= \frac{6Amq_0^2}{A^2(p+q_0)^2(-1+q_0^2-2Amq_0^3)^2} .$$

These discontinuities manifest themselves in the stress-energy tensor as a discontinuity in the pressure across the surface of the object, as given by Eqs. (19). As the acceleration goes to zero these discontinuities vanish, as they should.

That the first derivatives of the metric components and the pressure are discontinuous across the boundary are not believed to be peculiar to this solution but must be true in many cases of accelerating solutions which are joined to the vacuum. The discontinuity in the first derivatives of the metric are necessary in order to produce the discontinuity in the pressure at the surface of the object. A discontinuity in the pressure must exist in order to create the force necessary for the acceleration. Furthermore, it is conjectured that there must exist discontinuous first derivatives at the boundary for solutions which join onto exterior vacuum solutions which contain gravitational radiation in order to create the force necessary to accelerate the object which is required for the emission of radiation.

It has been shown by Lichnerowicz[17] that, given an interior solution of the field equations for which there exists a hypersurface S on which the pressure is zero and $u^\mu n_\mu = 0$, then there exists a physically unique vacuum metric on the other side of S for which $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$ are continuous across S. It would be interesting if a similar statement could be made about accelerating solutions. That is, for accelerating solutions where the first derivatives of the metric tensor are not continuous across S does there exist a unique vacuum on the other side of S for which the metric is continuous across S?

The second unusual property of this solution is that the stress-energy tensor becomes singular at the center of the object. Furthermore, near the center ($r=0$) and at those directions where p is negative the energy density becomes negative. These properties are immediately evident in Eqs. (21). It is felt that this behavior is peculiar only to solution for uniform acceleration. Uniform acceleration is physically very artificial and must have an infinite energy source in order to preserve the *indefinite* accelerating motion. In a physically realistic accelerating system one does not have continuous acceleration for an indefinite period of time.

It is well known that the vacuum C-metric represents an object undergoing uniform acceleration. We have joined this solution to an interior solution which has a force acting on the surface of the object. We do not know the meaning of this force nor its origin. Since this force is the only force in the solution one would certainly expect it to be responsible for the acceleration of the object. It is conceivable that part of this force may also act to produce a surface tension on the object.

We still have not answered the interesting question about what causes the particle to accelerate; that is: What is the origin for the discontinuity in the pressure? Does the discontinuity in the pressure represent a nongravitational force which causes the particle to accelerate which in turn will cause the emission of radiation, or does the emission of the gravitational radiation produce a recoil which causes the particle to accelerate which in turn will cause a discontinuity in the pressure? This latter explanation is rather bizarre but similar to the runaway solutions that exist in electromagnetic theories.

In order to study the effects of the radiation reaction one must go to an asymptotically flat coordinate system. In such a coordinate system the radiation and its effects can be identified. This was done in Ref. 13 where the radiation, mass loss, and momentum for the vacuum C-metric were investigated. The disadvantage of this approach is that in going to an asymptotically flat coordinate system the answers cannot be expressed in a closed form but must be represented by an expansion. This expansion causes the conical singularity to manifest itself as a line singularity. The information about the effects of the radiation reaction is contained in the line singularity. In addition, the line singularity leads to divergent results in the expressions for the momentum and mass loss. It is also not clear just how these vacuum results couple back to the properties of the interior solution to produce a radiation reaction.

If this is indeed an example of a runaway solution it is a highly idealized case where the gravitational radiation just self-sustains the uniform acceleration. In general, this idealized example will not be plausible; however, it does give rise to speculation of runaway solutions that exist for particles in electromagnetic theory. If spacetime does allow for such runaway solutions or singular structure, such circumstances would lead to unusual conclusions about the final states of matter.

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*Present address: Department of Physics, University of British Columbia, Vancouver, British Columbia, Canada.

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