Effective-action approach to mean-field non-Abelian statics, and a model for bag formation

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I propose a simple set of equations for mean-field non-Abelian statics with c-number sources, at general inverse temperature β, working from the Euclidean path-integral representation of the Hamiltonian partition function. The problem of finding the background-field configuration, and the mean-field potential, for point sources can be reduced to a classical differential equation problem involving a suitably defined thermal effective action functional. As an application I study the interaction of a pair of static classical sources coupled to a quantized SU(2) gauge field, using the simplified model defined by keeping only the leading-logarithm renormalization-group improvement to the local Euclidean action functional. I prove that the mean-field potential in this model grows at least linearly with the source separation, giving a simple model for bag formation. The use of these methods to construct a leading approximation to the $q\bar{q}$ binding problem in SU(3) quantum chromodynamics is discussed in two appendices. Appendix A describes the use of color-charge-algebra methods to generate an equivalent classical source problem, while Appendix B develops the properties of the transformation to a running coupling constant for which the one-loop renormalization group is exact. As a consistency check, in Appendix C I calculate the total mean-field ground-state energy, with source kinetic terms included, and show that it has the expected form.

I. EFFECTIVE-ACTION FORMALISM FOR NON-ABELIAN STATICS

I analyze in this paper the question of calculating the mean-field potential of classical point sources coupled to a quantized SU(2) gauge field, at zero and at finite temperature. This problem is of interest both in itself as a mathematical model, and because arguments based on the use of color-charge algebras suggest[1] that c-number source models should give a leading approximation to the problem of calculating the heavy quark-antiquark static potential in quantum chromodynamics.

My analysis proceeds from a field-theoretic generalization of the Euclidean (imaginary-time) version of Feynman's sum over histories. In potential scattering in one dimension, with Minkowski Lagrangian

$$L_M=\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\,, \tag{1}$$

the Euclidean sum over histories reads

$$\langle x_f | e^{-\beta H} | x_i\rangle = N\int_{x_i}^{x_f}[dx]e^{-S}\,. \tag{2}$$

On the left-hand side of Eq. (2) $|x_i\rangle$ and $|x_f\rangle$ are position eigenstates and H is the Hamiltonian, while on the right-hand side N is a normalization constant and S is the Euclidean action

$$S=\int_0^\beta dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right], \tag{3}$$

and $\int[dx]$ denotes a functional integration over all paths $x(t)$ obeying the boundary conditions $x(0)=x_i$, $x(\beta)=x_f$. Setting $x_f=x_i$ and integrating over initial states gives a formula for the partition function,

$$\begin{aligned}\mathrm{Tr}(e^{-\beta H}) &= \int dx_i\langle x_i | e^{-\beta H} | x_i\rangle \\ &= N\int dx_i \int_{x_i}^{x_i}[dx]e^{-S}\,,\end{aligned} \tag{4}$$

where the paths in Eq. (4) now run from $x(0)=x_i$ back to $x(\beta)=x_i$. The generalization of Eq. (4) to a boson field theory containing spin-0 scalar fields and spin-1 gauge fields, denoted collectively by ϕ, can be written as

$$Z=\mathrm{Tr}(e^{-\beta H}) = N\int d\phi_i \int_{\phi_i}^{\phi_i}[d\phi]e^{-S_E}\,. \tag{5}$$

On the left-hand side of Eq. (5), H is the Hamiltonian operator defined from the stress-energy tensor

$$H=\int d^3x\, T^{00}\,, \tag{6}$$

while on the right, S_E is the Euclidean action

$$S_E=\int_0^\beta dt\int d^3x\,\mathfrak{L}_E\,, \tag{7}$$

obtained by continuing g_{00} from -1 to 1 in the generally covariant form of the Minkowski Lagrangian density[2]

$$\mathfrak{L}_E = -\mathfrak{L}_M^{\text{gen cov}}\big|_{-1=g_{00}\to 1}\,. \tag{8}$$

The trace on the left is understood to be evaluated in any canonical gauge, where the Hilbert space contains only physical states, while the path integral on the right again extends over periodic paths, with $\phi(0)=\phi(\beta)=\phi_i$. The following observation makes the form of Eq. (5) intuitively plausible: For a field theory of scalars and spin-1 gauge fields, the generally covariant Lagrangian density is linear in g_{00},

$$\mathcal{L}_M^{\text{gen cov}} = \mathcal{L}_{(0)} + \mathcal{L}_{(1)} g_{00}\,, \tag{9}$$

with $\mathcal{L}_{(0,1)}$ independent of g_{00}, whence from Eq. (8) we have

$$\begin{aligned}\mathcal{L}_M &= \mathcal{L}_{(0)} - \mathcal{L}_{(1)}\,,\\ \mathcal{L}_E &= -(\mathcal{L}_{(0)} + \mathcal{L}_{(1)})\,.\end{aligned} \tag{10}$$

But forming the Minkowski energy density T^{00},

$$\begin{aligned}T^{00} &= g^{00}\mathcal{L}_M - 2\,\frac{\delta \mathcal{L}_M}{\delta g_{00}}\\ &= (-1)(\mathcal{L}_{(0)} - \mathcal{L}_{(1)}) - 2\mathcal{L}_{(1)} = \mathcal{L}_E\,,\end{aligned} \tag{11}$$

we see that it is identical to the Euclidean Lagrangian density $\mathcal{L}_E$. Hence, the Euclidean action of Eq. (7) is a functional representation of the operator βH, just as in the potential theory case. A detailed justification of Eq. (5) can be obtained by a transformation from the conventional canonical formalism given by Bernard.[3]

I now apply Eq. (5) to an SU(2) gauge theory (with gauge potential $\vec{b}_\mu$ and electric and magnetic fields $\vec{E}^j$ and $\vec{B}^j$) coupled to a system of massive sources, and replace the source current density by its expectation, represented by a time-independent c-number external source $\vec{j}_\mu$. The equilibrium gauge field can be studied by keeping only the terms in H and S_E which explicitly depend on the gauge field variables,[4] while omitting the source dynamics (hence, H in the following formulas is a truncated Hamiltonian, and not the Hamiltonian for a closed system). With this simplification, we have

$$Z[\vec{j}_\mu] = \mathrm{Tr}(e^{-\beta H}) = N\int d\vec{b}_{\mu i}\int_{\vec{b}_{\mu i}}^{\vec{b}_{\mu i}} d[\vec{b}_\mu]\,e^{-S_E}\,, \tag{12}$$

where on the left

$$H = \int d^3x\left(\frac{1}{g^2}\frac{1}{2}(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j) - \vec{b}_\mu\cdot\vec{j}_\mu\right) \tag{13}$$

is an operator, while on the right

$$S_E = \int_0^\beta dt\int d^3x\left(\frac{1}{g^2}\frac{1}{2}(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j) - \vec{b}_\mu\cdot\vec{j}_\mu\right) \tag{14}$$

is a functional. The mean-field potential[5,6] associated with the static external source distribution $\vec{j}_\mu = (\vec{j}_0 \neq 0, \vec{j}_i = 0)$, including self-energies, is defined as

$$\begin{aligned}\delta V_{\text{mean field}} &\equiv \left\langle \int d^3x\,\vec{b}_0\cdot\delta\vec{j}_0\right\rangle\\ &= \mathrm{Tr}\left(e^{-\beta H}\int d^3x\,\vec{b}_0\cdot\delta\vec{j}_0\right)\Big/\mathrm{Tr}(e^{-\beta H})\,.\end{aligned} \tag{15}$$

Since

$$\int d^3x\,\vec{b}_0\cdot\delta\vec{j}_0 = -\int d^3x\,\delta\vec{j}_0\cdot\frac{\delta}{\delta\vec{j}_0}H\,, \tag{16}$$

we can reexpress the mean-field potential directly in terms of the partition function[6]

$$\begin{aligned}\delta V_{\text{mean field}} &= \frac{1}{\beta}\int d^3x\,\delta\vec{j}_0\cdot\frac{\delta}{\delta\vec{j}_0}\ln Z[\vec{j}_\mu]\\ &= \delta\,\frac{1}{\beta}\ln Z[\vec{j}_\mu]\,,\end{aligned} \tag{17a}$$

$$\Rightarrow V_{\text{mean field}} = \frac{1}{\beta}\,\{\ln Z[\vec{j}_\mu] - \ln Z[\vec{0}]\}\,, \tag{17b}$$

where I have fixed the constant of integration so that $V_{\text{mean field}}$ vanishes for vanishing source density. The problem of calculating $Z[\vec{j}_\mu]$ can be further reexpressed in terms of a classical differential equation problem involving a classical background field $\vec{c}_\mu$ and a vacuum effective action functional $\Gamma[\vec{c}_\mu]$. To do this, we write

$$Z[\vec{j}_\mu] = e^{-\beta W[\vec{j}_\mu]}\,, \tag{18}$$

and we introduce the time-independent classical background field $\vec{c}_\mu(x)$ induced by the time-independent external source distribution $\vec{j}_\mu(x)$,

$$\begin{aligned}\vec{c}_\mu(x) &\equiv -\frac{\delta W[\vec{j}_\mu]}{\delta\vec{j}_\mu(x)}\\ &= Z^{-1}N\int d\vec{b}_{\mu i}\int_{\vec{b}_{\mu i}}^{\vec{b}_{\mu i}} d[\vec{b}_\mu]\left(\frac{1}{\beta}\int_0^\beta dt\,\vec{b}_\mu(x)\right)e^{-S_E}\,.\end{aligned} \tag{19}$$

In this notation, the mean-field potential is given by

$$V_{\text{mean field}} = -W[\vec{j}_\mu] + W[\vec{0}]\,. \tag{20}$$

Defining the Legendre-transformed functional $\Gamma[\vec{c}_\mu]$ by

$$W[\vec{j}_\mu] = \Gamma[\vec{c}_\mu] - \int d^3x\,\vec{c}_\mu(x)\cdot\vec{j}_\mu(x)\,, \tag{21}$$

a standard calculation[7] shows that

$$\frac{\delta\Gamma[\vec{c}_\mu]}{\delta\vec{c}_\mu(x)} = \vec{j}_\mu(x)\,. \tag{22}$$

Equations (18)–(22) are the principal result of this section. They show that the mean-field potential, for any inverse temperature β, can be calculated by solving the classical differential equation problem of minimizing the functional $\Gamma - \int d^3x\,\vec{c}_\mu\cdot\vec{j}_\mu$, with Γ the thermal effective action functional.[8,9] In the limit $\beta\to\infty$, where Γ reduces to the Euclidean vacuum effective action functional, this minimum problem reproduces the variational principle of the "Euclidean statics" method which I have advocated elsewhere,[1] but with some significant differences in physical interpretation.[10]

According to Eqs. (28)–(22), the problem of studying the mechanism for confinement in the model discussed here can be rephrased in terms of the following two related questions.

(1) Is there a physically reasonable class of vacuum action functionals for which Eqs. (18)–(22) give a confining potential for static point sources? This question is answered in the affirmative in the following section.

(2) Does the exact vacuum action functional calculated from the functional integral of Eq. (20) belong to the confining class?

The methods appropriate to studying these questions are quite different. For a given functional or class of functionals Γ, the first question is one of classical analytic or numerical methods for investigating partial differential equations. In the following section, Eq. (22) is investigated analytically for the leading-logarithm approximation to the renormalization-group improved local effective action functional, for which Γ takes the simple form $\Gamma[\vec{c}_\mu]=\Gamma(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)$; numerical methods of solution applicable to this class of functionals are currently being developed.[1,11] The second question is probably best studied by numerical Monte Carlo methods for doing the functional integral. Since confinement is an infrared effect, it should suffice to establish the properties of Γ for slowly varying source currents $\vec{j}_\mu$ and background fields $\vec{c}_\mu$. In this case, appropriate lattice transcriptions of the functional integral of Eq. (19) may give quantitatively accurate estimates of the behavior of the continuum effective action.

II. A SIMPLE MODEL FOR BAG FORMATION

As an illustration of the formalism developed in Sec. I, I analyze the following simple model, obtained by keeping only the leading-logarithm renormalization-group improvement[9] to the local Euclidean action functional

$$\Gamma[\vec{c}_\mu]=\int d^3x(\mathcal{L}_{\text{eff}}-\mathcal{L}_{\text{eff}}^{\min}), \tag{23}$$

with

$$\begin{aligned}
\mathcal{L}_{\text{eff}}&=\mathcal{L}_{\text{eff}}(F^2)\\
&=\frac{1}{2}\,\frac{(\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j)}{g^2}\left[1+\tfrac{1}{4}b_0g^2\ln\!\left(\frac{\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j}{\mu^4}\right)\right]\\
&=\tfrac{1}{8}b_0F^2\ln[F^2/(e\kappa^2)]\,,\\
\mathcal{L}_{\text{eff}}^{\min}&=\mathcal{L}_{\text{eff}}(\kappa^2)=-\tfrac{1}{8}b_0\kappa^2\,,\\
\kappa^2&=\frac{\mu^4}{e}e^{-4/(b_0g^2)}\,,\quad b_0=\frac{1}{8\pi^2}\,\frac{11}{3}\,C_2[\mathrm{SU}(2)]=\frac{1}{4\pi^2}\,\frac{11}{3}\,,\\
\vec{E}^j&=-\frac{\partial}{\partial x^j}\vec{c}^{\,0}-\vec{c}^{\,j}\times\vec{c}^{\,0}\equiv-\mathfrak{D}_j\vec{c}^{\,0}\,,\\
\vec{B}^j&=\epsilon^{jkl}\left(\frac{\partial}{\partial x^k}\vec{c}^{\,l}+\frac{1}{2}\vec{c}^{\,k}\times\vec{c}^{\,l}\right),\quad F^2\equiv\vec{E}^j\cdot\vec{E}^j+\vec{B}^j\cdot\vec{B}^j\,.
\end{aligned}\tag{24}$$

As has been extensively discussed in the literature,[12] the minimum of $\mathcal{L}_{\text{eff}}$ occurs at the nonzero field strength $F=\kappa$. The source density $\vec{j}_0$ is taken to be a pair of classical sources of equal magnitude,

$$\begin{aligned}
\vec{j}_0&=\vec{Q}_1\delta^3(x-x_1)+\vec{Q}_2\delta^3(x-x_2)\,,\\
|x_1-x_2|&=R\,,\quad |\vec{Q}_1|=|\vec{Q}_2|=Q\,.
\end{aligned}\tag{25}$$

In analyzing the model defined by Eqs. (18)–(25), I make the physically plausible technical assumption that it suffices to minimize over potentials $\vec{c}_\mu$ for which $\vec{E}^j\cdot\vec{E}^j$ is axially symmetric around the line joining the sources.

The variational equations following from Eqs. (22)–(24) are

$$\mathfrak{D}_j(\epsilon\vec{E}^j)=\vec{j}_0\,, \tag{26a}$$

$$\epsilon^{kjm}\mathfrak{D}_j(\epsilon\vec{B}^m)=\vec{c}^{\,0}\times(\epsilon\vec{E}^k)\,, \tag{26b}$$

$$\epsilon=\epsilon(F^2)=\frac{\partial\mathcal{L}_{\text{eff}}}{\partial(\frac{1}{2}F^2)}=\tfrac{1}{4}b_0\ln(F^2/\kappa^2)\,. \tag{26c}$$

Acting with $\mathfrak{D}_k$ on Eq. (26b) and using Eq. (26a) gives the constraint

$$0=\vec{c}^{\,0}\times\vec{j}_0\,; \tag{27}$$

hence, if we write

$$\vec{c}^{\,0}(x)=\hat{c}(x)c(x)\,,\quad \hat{c}\cdot\hat{c}=1\,, \tag{28}$$

we have

$$\hat{c}(x_1)\times\vec{Q}_1=\hat{c}(x_2)\times\vec{Q}_2=0\,, \tag{29}$$

which implies that

$$\vec{c}_0(x)\cdot\vec{j}_0(x)=c(x)j(x)\,, \tag{30}$$

with either

$$j(x)=\pm Q[\delta^3(x-x_1)-\delta^3(x-x_2)] \tag{31a}$$

or

$$j(x)=\pm Q[\delta^3(x-x_1)+\delta^3(x-x_2)]\,. \tag{31b}$$

From here on it will be convenient to work in the gauge with $\hat{c}(x)=\hat{z}$, in which we have

$$F^2=\left(\frac{\partial c}{\partial x^j}\right)^2+c^2(\vec{c}^j\times\hat{z})^2+\vec{B}^j\cdot\vec{B}^j\,. \tag{32}$$

I now will show that the minimization of W with respect to the vector potential $\vec{c}^j$ can be carried out explicitly, with the result

$$\begin{aligned}&\min_{\vec{c}^j} W\equiv\tilde{W}[c]=\int d^3x\left\{\tilde{\mathcal{L}}_{\rm eff}\left[\left(\frac{\partial c}{\partial x^j}\right)^2\right]-c(x)j(x)\right\},\\&\tilde{\mathcal{L}}_{\rm eff}\left[\left(\frac{\partial c}{\partial x^j}\right)^2\right]=0 \text{ for } \left(\frac{\partial c}{\partial x^j}\right)^2\leq\kappa^2,\\&\tilde{\mathcal{L}}_{\rm eff}\left[\left(\frac{\partial c}{\partial x^j}\right)^2\right]=\mathcal{L}_{\rm eff}\left[\left(\frac{\partial c}{\partial x^j}\right)^2\right]\\&\qquad\qquad-\mathcal{L}_{\rm eff}^{\rm min} \text{ for } \left(\frac{\partial c}{\partial x^j}\right)^2\geq\kappa^2.\end{aligned} \tag{33}$$

To prove Eq. (33), we note that $\mathcal{L}_{\rm eff}(F^2)-\mathcal{L}_{\rm eff}(\kappa^2)$ is a monotonic decreasing function of its argument for $0\leq F^2\leq\kappa^2$, and is a monotonic increasing function of its argument for $\kappa^2\leq F^2$. Hence, W is minimized by the following choice of vector potential,

$$\begin{aligned}&\vec{c}^j=\hat{z}\hat{z}^j a(\rho,z)\,,\ \ \rho=(x^2+y^2)^{1/2}\,,\\&a(\rho,z)=\int_0^\rho d\rho'\,A(\rho',z)\,,\\&A(\rho',z)=0\,,\ \text{ where } \left(\frac{\partial c}{\partial x^j}\right)^2\geq\kappa^2\,,\\&A(\rho',z)=\left[\kappa^2-\left(\frac{\partial c}{\partial x^j}\right)^2\right]^{1/2},\ \text{ where } \left(\frac{\partial c}{\partial x^j}\right)^2\leq\kappa^2\,,\end{aligned} \tag{34}$$

which gives (with $\hat{\phi}^j$ the azimuthal unit vector)

$$\begin{aligned}&\vec{c}^j\times\hat{z}=0\,,\\&\vec{B}^j=-\hat{z}\hat{\phi}^j A(\rho,z)\Rightarrow\\&F^2=\max\left[\left(\frac{\partial c}{\partial x^j}\right)^2,\kappa^2\right],\end{aligned} \tag{35}$$

and from which Eq. (33) immediately follows. (Note that it is at this point in the argument where the axial symmetry assumption has been used.) What is happening is that wherever the color-electric field is less than κ in magnitude, a color-magnetic field fills in to bring the total squared field strength up to the value κ^2 at which $\mathcal{L}_{\rm eff}$ is minimized.

We are now left with the purely Abelian problem of minimizing $\tilde{W}[c]$, to which we apply simple flux conservation estimates introduced by 't Hooft.[13] Varying $\tilde{W}$, we get the flux conservation equation

$$\begin{aligned}&\frac{\partial}{\partial x^j}D^j=j(x)\,,\\&D^j\equiv\bar{\epsilon}E^j\,,\ \ E^j=-\frac{\partial}{\partial x^j}c\,,\end{aligned} \tag{36}$$

with

$$\bar{\epsilon}=\begin{cases}\epsilon(E^jE^j) & \text{where } E^jE^j\geq\kappa^2\,,\\ 0 & \text{where } E^jE^j\leq\kappa^2\,.\end{cases} \tag{37}$$

Evidently, wherever the E field strength is less than κ, the D field vanishes. This fact can be exploited to get a lower bound on $V_{\rm mean\ field}$ and an upper bound on $\tilde{W}$, which by Eqs. (33), (36), and (37) can be rewritten as

$$\tilde{W}_{\rm at\ equilibrium}=\int_{D>0}d^3x[\tilde{\mathcal{L}}_{\rm eff}(E^jE^j)-E^jD^j]\,, \tag{38}$$

with the integral extending only over the region where D^j is nonvanishing. Dividing the integrand of Eq. (38) by $D=(D^jD^j)^{1/2}$, we get

$$D^{-1}[E^jD^j-\tilde{\mathcal{L}}_{\rm eff}]=\kappa[\tfrac{1}{2}u+\tfrac{1}{2}f(u)] \tag{39}$$

with[14] (in the domain where $D>0$)

$$\begin{aligned}&u=(E^jE^j)^{1/2}/\kappa\geq1\,,\\&f(u)=\frac{u^2-1}{u\ln u^2}\geq1\,.\end{aligned} \tag{40}$$

To turn Eq. (38) into a meaningful inequality, it is necessary to exclude the divergent self-energies of the charges by defining $\tilde{W}^r$ to be the contribution to Eq. (38) coming from the exterior of small spheres of radius r centered on the charges. We then get

$$\begin{aligned}&\tilde{W}^r\leq-\kappa I^r\,,\\&V^r_{\rm mean\ field}\geq\kappa I^r\,,\\&I^r=\int_\Lambda d^3x\,D\,,\quad \Lambda=\text{domain}\begin{cases}D>0\\|x-x_1|\geq r\\|x-x_2|\geq r\,.\end{cases}\end{aligned} \tag{41}$$

The final step of the argument is to write $d^3x=dl\,dA$ with l the length along the flux lines of D^j and dA an element of area perpendicular to the flux lines. Denoting the flux by Φ, we have

$$\begin{aligned}&dA\,D=d|\Phi|\geq d\Phi\,,\\&\int_\Lambda d^3x\,D\geq\int_\Lambda d\Phi\,l(\Phi)\geq\Phi_{\rm tot}\,l_{\rm min}\end{aligned} \tag{42}$$

with $l_{\rm min}$ the length of the shortest flux line. For the charge orientations of Eq. (31b), where the flux lines terminate at infinity, we have

$$\Phi_{\rm tot}=2Q\,,\quad l_{\rm min}=\infty\,, \tag{43}$$

and I^r is infinite.[15] For the charge orientations of Eq. (31a), the flux lines run from the positive to the negative charge, giving

$$\begin{aligned}\Phi_{\rm tot}&=Q\,,\\ l_{\rm min}&=R-2r\,,\\ \tilde{W}^r&\leq -\kappa Q(R-2r)\,,\\ V^r_{\rm mean\ field}&\geq \kappa Q(R-2r)\,,\end{aligned} \tag{44}$$

which proves that the mean-field potential increases at least linearly for large R. In the limit of small R, a simple calculation shows that

$$V_{\rm mean\ field}-\text{self-energies}\approx -\frac{Q^2}{4\pi R}\,\frac{g^2}{1+b_0g^2\ln(1/R\mu)}\,, \tag{45}$$

as expected from a leading-logarithm renormalization-group improved formalism. Hence, the simple model of Eqs. (18)–(25) interpolates smoothly between asymptotically free behavior at small source separations, and "baglike,"[16] confining behavior at large separations. The above analysis readily generalizes to the full renormalization-group improved local effective action functional,[9] provided that the effective action minimum remains at nonzero Euclidean field strength κ.[17] More generally, the results obtained above support the conjecture that a bag will form for large source separations, irrespective of the functional form of $\Gamma[\vec{c}_\mu]$, whenever the minimum of Γ occurs at potentials $\vec{c}_\mu$ with nonvanishing mean-square field strength.

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APPENDIX A: CONNECTION WITH SU(3) QUANTUM CHROMODYNAMICS

I briefly describe in this appendix how the methods of the text, together with the color-charge-algebra analysis of Ref. 1, can be applied to give a leading approximation to the $q\bar{q}$ binding problem in SU_3 quantum chromodynamics (QCD). In the static quark limit,[1] the gluon source current for the $q\bar{q}$ binding problem in QCD is

$$\begin{aligned}j^{Aj}&=0\,,\\ j^{A0}&=Q_q^A\delta^3(x-x_1)+Q_{\bar{q}}^A\delta^3(x-x_2)\end{aligned} \tag{A1}$$

with Q_q^A and $Q_{\bar{q}}^A$ the quark and antiquark color-charge matrices. As discussed in Ref. 1, the gluon source current is now a 9×9 matrix operator acting on the nine-dimensional Hilbert space spanned by the $q,\bar{q}$ color states. The analysis of Sec. I can be extended to this case by including a factor $\frac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}$ in all formulas and symmetrizing all inner products, so that Eq. (12) becomes

$$\begin{aligned}Z[j_\mu^A]&=\tfrac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}\,{\rm Tr}_{\rm gluon}(e^{-\beta H})\\ &=N\tfrac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}\left(\int db^A_{\mu i}\int_{b^A_{\mu i}}^{b^A_{\mu i}} d[b^A_\mu]e^{-S_B}\right),\end{aligned} \tag{A2}$$

where on the left

$$H=\int d^3x\left[\frac{1}{g^2}\,\frac{1}{2}(E^{Aj}E^{Aj}+B^{Aj}B^{Aj})-\frac{1}{2}(b^A_\mu j^A_\mu+j^A_\mu b^A_\mu)\right] \tag{A3}$$

is an operator in the product of the $q,\bar{q}$ and gluon Hilbert spaces, while on the right

$$S_B=\int_0^\beta dt\int d^3x\left[\frac{1}{g^2}\,\frac{1}{2}(E^{Aj}E^{Aj}+B^{Aj}B^{Aj})-\frac{1}{2}(b^A_\mu j^A_\mu+j^A_\mu b^A_\mu)\right] \tag{A4}$$

is a functional in its dependence on the gluon variables, but is still a matrix operator in the finite dimensional $q,\bar{q}$ color Hilbert space.[18] Using cyclic invariance of the trace, the steps leading to Eqs. (18)–(22) go through just as before, giving

$$Z[j^A_\mu]=e^{-\beta W[j^A_\mu]}\,, \tag{A5}$$

$$V_{\rm mean\ field}=-W[j^A_\mu]+W[0^A]\,, \tag{A6}$$

$$\delta W[j^A_\mu]\equiv -\tfrac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}\left(\int d^3x\,c^A_\mu(x)\delta j^A_\mu(x)\right), \tag{A7}$$

$$W[j^A_\mu]=\Gamma[c^A_\mu]-\tfrac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}\left(\int d^3x\,c^A_\mu(x)j^A_\mu(x)\right), \tag{A8}$$

$$\delta\Gamma[c^A_\mu]=\tfrac{1}{9}\,{\rm tr}_q\,{\rm tr}_{\bar{q}}\left(\int d^3x\,\delta c^A_\mu(x)j^A_\mu(x)\right). \tag{A9}$$

Note that Eq. (A7) defines c^A_μ to be a potential which, like the source current j^A_μ, is matrix valued in the nine-dimensional $q\bar{q}$ Hilbert space. To construct a QCD analog of the analysis of Sec. II, we must calculate a leading approximation to the effective action. In the classical limit, the effective action density is given by[19]

$$\mathcal{L}_{\rm cl} = \frac{1}{2g^2} F^2 ,$$
$$F^2 = \tfrac{1}{9} \operatorname{tr}_q \operatorname{tr}_{\bar q} (E^{Aj}E^{Aj} + B^{Aj}B^{Aj}) ,$$
$$E^{Aj} = -\frac{\partial}{\partial x^j} c^{A0} + iP_f^A(c^j, c^0) , \qquad \text{(A10)}$$
$$B^{Aj} = \epsilon^{jkl}\left(\frac{\partial}{\partial x^k} c^{Al} - \frac{i}{2} P_f^A(c^k, c^l)\right) ,$$
$$P_f^A(u, v) \equiv \frac{i}{2} f^{ABC}(u^B v^C + v^C u^B) .$$

The renormalization-group improvement of Eq. (A10) is obtained by taking g^2 to be a running-coupling function of the argument $g^2\mathcal{L}_{\rm cl}$ giving, in leading-logarithm approximation (cf. remarks in Ref. 28),

$$\Gamma[c_\mu^A] = \int d^3x [\mathcal{L}_{\rm eff}(F^2) - \mathcal{L}_{\rm eff}(\kappa^2)] , \qquad \text{(A11)}$$

$$\mathcal{L}_{\rm eff}(F^2) = \tfrac{1}{8} b_0 F^2 \ln(F^2/e\kappa^2) , \quad \kappa^2 = \frac{\mu^4}{e} e^{-4/(b_0 g^2)} ,$$
$$b_0 = \frac{1}{8\pi^2}\frac{11}{3} C_2[\mathrm{SU}(3)] = \frac{11}{8\pi^2} . \qquad \text{(A12)}$$

To carry out the remainder of the analysis of Sec. II, we must reexpress Eqs. (A8)–(A12) in terms of number valued, as opposed to matrix valued, source density and gluon variables. To do this, let us recall[1] that the $q\bar q$ color-charge algebra is spanned by a basis $w_1^A, \ldots, w_4^A$, which satisfies the $\mathrm{SU}(2)\times\mathrm{U}(1)$ outer product algebra

$$P_f(w_r, w_s) = i\,\frac{1}{2}\epsilon_{rst} w_t , \quad r, s, t = 1, 2, 3$$
$$P_f(w_r, w_4) = 0 , \qquad \text{(A13)}$$

is orthonormal in the color-trace inner product,

$$\tfrac{1}{9} \operatorname{tr}_q \operatorname{tr}_{\bar q}(w_r^A w_s^A) = \tfrac{8}{27}\delta_{rs} , \qquad \text{(A14)}$$

and over which the quark and antiquark color charges have the expansions

$$Q_q^A = \tfrac{3}{2} w_1^A + w_2^A + \frac{\sqrt{5}}{2} w_4^A ,$$
$$Q_{\bar q}^A = \tfrac{3}{2} w_1^A - w_2^A - \frac{\sqrt{5}}{2} w_4^A . \qquad \text{(A15)}$$

[As a check, we note that $\frac{1}{9}\operatorname{tr}_q \operatorname{tr}_{\bar q}(Q_q^A Q_q^A) = (8/27)\times(18/4) = 4/3$.]

Expanding $Q_q^A, Q_{\bar q}^A, j_\mu^A, c_\mu^A, E^{Aj}$, and B^{Aj} over the basis w_r^A, with c-number coefficients, reduces the variational problem

$$\delta\Gamma[c_\mu^A] - \tfrac{1}{9}\operatorname{tr}_q \operatorname{tr}_{\bar q}\left(\int d^3x\, \delta c_\mu^A(x) j_\mu^A(x)\right) = 0 \qquad \text{(A16)}$$

to a classical $\mathrm{SU}(2)\times\mathrm{U}(1)$ problem, analogous to that discussed in Sec. II. According to Eq. (A15), the U(1) effective q and $\bar q$ charges are opposite in sign, while the SU(2) effective charges have equal magnitudes. Hence, the quark and antiquark effective charges can be made antiparallel by an SU(2) gauge transformation,[20] leading to a solution with the same form as that obtained from Eq. (31a) in Sec. II, apart from the substitutions[21,22]

$$Q \to (\tfrac{4}{3})^{1/2} ,$$
$$b_0 \to \frac{11}{8\pi^2} . \qquad \text{(A17)}$$

APPENDIX B: TRANSFORMATION OF THE RUNNING COUPLING CONSTANT TO EXACT LEADING-LOGARITHM FORM

As already noted, the argument given in the text for a confining mean-field potential generalizes to the full renormalization-group improved local-effective-action functional, provided that the effective-action minimum remains at nonzero Euclidean field strength. When expressed in terms of the β function, this condition translates[9] into the requirement that the integral

$$\int^\infty \frac{dg'}{\beta(g')} \qquad \text{(B1)}$$

should be convergent at its upper limit. Assuming convergence of the integral in Eq. (B1), I show in this appendix that one can make a nonanalytic transformation to a new running coupling constant g_R for which the one-loop renormalization-group structure is exact. The transformation is simply (with $\alpha_R = g_R^2,\ \alpha = g^2$)

$$\frac{1}{\alpha_R} = -\tfrac{1}{2} b_0 \int_\alpha^\infty \frac{d\alpha'}{\bar\beta(\alpha')} , \qquad \text{(B2)}$$

where $\bar\beta = g\beta$ has the power-series expansion [with the coefficients given for SU(3) QCD with N_f light quark flavors[23]]

$$\bar\beta(\alpha) = -[\tfrac{1}{2} b_0\alpha^2 + b_1\alpha^3 + O(\alpha^4)] ,$$
$$b_0 = \frac{1}{8\pi^2}(11 - \tfrac{2}{3}N_f) , \quad b_1 = \frac{1}{2}\frac{1}{(8\pi^2)^2}(51 - \tfrac{19}{3}N_f) . \qquad \text{(B3)}$$

Substituting the expansion of Eq. (B3) into Eq. (B2), we learn that for small running coupling constant,

$$\alpha_R = \alpha - a\alpha^2(\ln\alpha + \text{const}) + \cdots , \quad a = \frac{2b_1}{b_0} \qquad \text{(B4)}$$

and so $\alpha_R \to 0$ when $\alpha \to 0$. On the other hand, the convergence of Eq. (B1) implies that

$$\lim_{\alpha\to\infty} \int_\alpha^\infty \frac{d\alpha'}{\bar\beta(\alpha')} = 0 , \qquad \text{(B5)}$$

and so $\alpha_R \to \infty$ as $\alpha \to \infty$. Hence the transformation of Eq. (B2) gives a nonsingular mapping from the half

(B2) gives a nonsingular mapping from the half line $0 \leq \alpha < \infty$ to the half line $0 \leq \alpha_R < \infty$. The renormalization-group structure in the new variable α_R is determined by $\bar{\beta}_R(\alpha_R)$, given by

$$\bar{\beta}_R(\alpha_R) = \bar{\beta}(\alpha)\,\frac{\partial \alpha_R}{\partial \alpha} = \bar{\beta}(\alpha)(-\alpha_R{}^2)\,\frac{\partial(\alpha_R{}^{-1})}{\partial \alpha} = -\tfrac{1}{2}\,b_0\alpha_R{}^2\,, \tag{B6}$$

and so has exactly one-loop form.

A particularly interesting case of Eq. (B2) is obtained when $\bar{\beta}(\alpha)$ terminates at two-loop order,

$$\bar{\beta}(\alpha) = -(\tfrac{1}{2}\,b_0\alpha^2 + b_1\alpha^3)\,, \tag{B7}$$

a situation which can always[24] be achieved [provided Eq. (B1) converges[25]] by an analytic transformation of the running coupling constant [i.e., by a rearrangement of the perturbation series which does not introduce coupling-constant logarithms]. In this case, Eq. (B2) can be explicitly integrated to give the transformation

$$\frac{1}{\alpha_R} = \frac{1}{\alpha} - a\left[\ln\left(\frac{1}{a\alpha}\right) + \ln(1 + a\alpha)\right], \tag{B8}$$

which for small $a\alpha$ can be developed into a series expansion

$$\frac{1}{\alpha_R} = \frac{1}{\alpha} - a\ln\left(\frac{1}{a\alpha}\right) + a\sum_{n=1}^{\infty}\frac{(-a\alpha)^n}{n}\,. \tag{B9}$$

The series of Eq. (B9) can be inverted by substituting

$$\alpha = \alpha_R(1 + \alpha_R f)\,, \tag{B10}$$

which after some algebra gives

$$0 = \sum_{n=1}^{\infty}(-f)^n\alpha_R{}^{n-1} + a\ln(a\alpha_R) - a\sum_{n=1}^{\infty}\frac{(-\alpha_R f)^n}{n} + a\sum_{n=1}^{\infty}\frac{(-a\alpha_R)^n(1+\alpha_R f)^n}{n}\,. \tag{B11}$$

Substituting

$$f = \sum_{k=0}^{\infty}\alpha_R^k f_k \tag{B12}$$

into Eq. (B10), and equating the coefficients of like powers of α_R, gives explicit expressions for the coefficients f_k as polynomials in $\ln(a\alpha_R)$,

$$f_0 = a\ln(a\alpha_R)\,,$$
$$f_1 = f_0{}^2 + af_0 - a^2,\ \ldots\ . \tag{B13}$$

Hence, starting from any convenient calculational scheme (for example, minimal subtraction in dimensional regularization), the QCD perturbation series can be reexpressed in terms of α_R by a two-step transformation: First, one transforms to a running coupling constant for which $\beta(\alpha)$ is given by Eq. (B7), and then one substitutes the inverse transformation given by Eqs. (B10)–(B13), yielding a series expansion in powers of α_R and $\ln(a\alpha_R)$.[26] In this series, the terms of order α_R^n contain only powers $0, 1, \ldots, n-1$ of $\ln(a\alpha_R)$.

An important property of the one-loop running coupling α_R is that it simultaneously maximizes the domains of analyticity of the renormalization-group improved local effective action $\mathcal{L}_{\text{eff}}(F^2)$ and of the $\bar{\beta}$ function $\bar{\beta}(\alpha)$. For a general running coupling $\alpha(t)$, the renormalization-group improved local effective action density is given by[9]

$$\mathcal{L}_{\text{eff}}(F^2) = \frac{1}{2}\,\frac{F^2}{\alpha(t)}\,, \qquad t = \tfrac{1}{4}\ln(F^2/e\kappa^2)\,. \tag{B14}$$

Substituting the one-loop running coupling α_R,

$$\frac{1}{\alpha_R(t)} = b_0 t \tag{B15}$$

gives

$$\mathcal{L}_{\text{eff}\,R}(F^2) = \tfrac{1}{8}\,b_0 F^2\ln(F^2/e\kappa^2)\,, \tag{B16}$$

as used in Eq. (23) of the text. As a function of complex F^2, Eq. (B16) is analytic apart from a cut in the F^2 plane running along the negative real axis from $F^2 = 0$ to $F^2 = -\infty$. Such a timelike cut is expected from unitarity, and so $\mathcal{L}_{\text{eff}\,R}$ has the maximum allowed analyticity domain in F^2. To study the analyticity properties of the general $\mathcal{L}_{\text{eff}}(F^2)$, let us calculate the derivative

$$\frac{d}{d(F^2)}\,\mathcal{L}_{\text{eff}}(F^2) = \frac{1}{2}\,\frac{1}{\alpha(t)} + \frac{1}{8}\,\frac{d}{dt}\left(\frac{1}{\alpha(t)}\right). \tag{B17}$$

From Eqs. (B2) and (B15), we get

$$\frac{d}{dt}\left(\frac{1}{\alpha(t)}\right) = \frac{\bar{\beta}(\alpha)}{-\frac{1}{2}\alpha^2}\,, \tag{B18}$$

and so

$$\frac{d}{d(F^2)}\,\mathcal{L}_{\text{eff}}(F^2) = \frac{1}{2}\,\frac{1}{\alpha(t)} - \frac{1}{4}\,\frac{\bar{\beta}(\alpha(t))}{\alpha(t)^2}\,. \tag{B19}$$

We have seen above that at the spacelike F^2 where t vanishes, both α_R and α become infinite. Hence, $d\mathcal{L}_{\text{eff}}(F^2)/d(F^2)$ is singular at spacelike F^2 unless $\bar{\beta}(\alpha)/\alpha^2$ is bounded as α becomes infinite. This is possible with $\bar{\beta}(\alpha)$ an entire function [which corresponds to the maximum allowed analyticity domain for the function $\bar{\beta}(\alpha)$] only if $\bar{\beta}(\alpha)/\alpha^2$ is a constant. Hence, the one-loop running coupling gives the maximal analytic extension of the renormalization-group substructure of QCD.[27] This result suggests that the one-loop model of Sec. II may give a universal, leading, semiclassical approximation to the confinement problem.[28]

APPENDIX C: TOTAL GROUND-STATE ENERGY

As a consistency check on the formalism of Sec. I, I show here that when source kinetic terms are included, the ground-state expectation of the *total* Hamiltonian for a system of two well-localized sources, in mean-field approximation, is

$$\langle 0|H_T|0\rangle = V_{\text{mean field}}(x_1, x_2) + \text{recoil terms} + \text{constant}. \tag{C1}$$

To most simply parallel the discussion of the text, I consider only the case of massive distinguishable fermion sources, with classical[29] SU_2 charges, for which H_T has the form

$$\begin{aligned} H_T &= \int d^3x\, T^{00} = \mathcal{H} + \mathcal{H}_{\text{kin}}, \\ \mathcal{H} &= \int d^3x \left(\frac{1}{g^2}\frac{1}{2}(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j) - \vec{b}_0\cdot\vec{\mathcal{J}}_0\right), \\ \mathcal{J}_0 &= \psi_1^\dagger\vec{Q}_1\psi_1 + \psi_2^\dagger\vec{Q}_2\psi_2, \\ \mathcal{H}_{\text{kin}} &= \int d^3x(\psi_1^\dagger i\partial_0\psi_1 + \psi_2^\dagger i\partial_0\psi_2). \end{aligned} \tag{C2}$$

Taking the ground-state expectation of Eq. (C2), we have

$$\langle 0|H_T|0\rangle = \langle 0|\mathcal{H}|0\rangle + \langle 0|\mathcal{H}_{\text{kin}}|0\rangle. \tag{C3}$$

To apply mean-field theory, one assumes a Hartree factorization of the ground state (g = gluon, s = source)

$$|0\rangle = |0\rangle_g|0\rangle_s \tag{C4}$$

with

$$\begin{aligned} {}_g\langle 0|0\rangle_g &= {}_s\langle 0|0\rangle_s = 1, \\ {}_s\langle 0|\mathcal{J}_0|0\rangle_s &= \vec{Q}_1\delta^3(x-x_1) + \vec{Q}_2\delta^3(x-x_2) = \vec{j}_0. \end{aligned} \tag{C5}$$

Hence, for $\langle 0|\mathcal{H}|0\rangle$ we get

$$\begin{aligned} \langle 0|\mathcal{H}|0\rangle &= {}_g\langle 0|H|0\rangle_g, \\ H &= \int d^3x\left(\frac{1}{g^2}\frac{1}{2}(\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j) - \vec{b}_0\cdot\vec{j}_0\right), \end{aligned} \tag{C6}$$

which involves the truncated Hamiltonian introduced in Sec. I. Since in the limit $\beta\to\infty$ only the ground state contributes to the partition function, from Eqs. (12), (18), and (20) of the text and Eq. (C6) we learn that

$$\langle 0|\mathcal{H}|0\rangle = W[\vec{j}_0] = -V_{\text{mean field}}(x_1,x_2) + \text{constant}. \tag{C7}$$

To evaluate the second term in Eq. (C3), we use the source field equations of motion

$$\begin{aligned} i\partial_0\psi_1 &= \vec{Q}_1\psi_1\cdot\vec{b}_0(x) + \text{recoil terms}, \\ i\partial_0\psi_2 &= \vec{Q}_2\psi_2\cdot\vec{b}_0(x) + \text{recoil terms}, \end{aligned} \tag{C8}$$

which, together with Eqs. (C4) and (C5), give

$$\langle 0|\mathcal{H}_{\text{kin}}|0\rangle = {}_g\langle 0|\int d^3x\,\vec{b}_0(x)\cdot\vec{j}_0(x)|0\rangle_g. \tag{C9}$$

The right-hand side of Eq. (C9) can be reexpressed in terms of the $\beta\to\infty$ limit of the partition function and then further rewritten using Eq. (19) of the text, giving

$${}_g\langle 0|\int d^3x\,\vec{b}_0\cdot\vec{j}_0(x)|0\rangle_g = \lim_{\beta\to\infty}\frac{1}{\beta}\left(\frac{\partial}{\partial\lambda}\ln Z[\lambda\vec{j}_0]\right)\bigg|_{\lambda=1} \tag{C10a}$$

$$= \int d^3x\,\vec{c}_0(x)\cdot\vec{j}_0(x). \tag{C10b}$$

Hence, we have

$$\langle 0|\mathcal{H}_{\text{kin}}|0\rangle = \vec{c}_0(x_1)\cdot\vec{Q}_1 + \vec{c}_0(x_2)\cdot\vec{Q}_2, \tag{C11}$$

and we can complete the proof of Eq. (C1) by showing that

$$\vec{c}_0(x_1)\cdot\vec{Q}_1 = V_{\text{mean field}}(x_1,x_2) + \text{constant}, \tag{C12a}$$

$$\vec{c}_0(x_2)\cdot\vec{Q}_2 = V_{\text{mean field}}(x_1,x_2) + \text{constant}. \tag{C12b}$$

To prove Eq. (C12a), let us write $\vec{c}_0(x)$ in the form

$$\vec{c}_0(x) = \vec{c}_0^{(A)}(x,x_1,x_2) + \vec{c}_0^{(B)}(x,x_1) \tag{C13}$$

with $\vec{c}_0^{(B)}$ chosen so that $\vec{c}_0^{(A)}$ is regular near $x=x_1$ and satisfies

$$[\delta_{x_1}\vec{c}_0^{(A)}(x,x_1,x_2)]|_{x=x_1} = 0. \tag{C14}$$

Using Eq. (15) of the text, in the $\beta\to\infty$ limit, we get

$$\begin{aligned} \delta_{x_1}V_{\text{mean field}}(x_1,x_2) &= \int d^3x\,\vec{c}_0(x)\cdot\vec{Q}_1\delta_{x_1}\delta^3(x-x_1) \\ &= \int d^3x\,\vec{c}_0^{(A)}(x,x_1,x_2)\cdot\vec{Q}_1\delta_{x_1}\delta^3(x-x_1) + \int d^3x\,\vec{c}_0^{(B)}(x,x_1)\cdot\vec{Q}_1\delta_{x_1}\delta^3(x-x_1). \end{aligned} \tag{C15}$$

The first term on the right of Eq. (C15) can be rewritten, by use of Eq. (C14), as

$$\delta_{x_1}\int d^3x\,\vec{c}_0^{(A)}(x,x_1,x_2)\cdot\vec{Q}_1\delta^3(x-x_1) - \int d^3x[\delta_{x_1}\vec{c}_0^{(A)}(x,x_1,x_2)]\cdot\vec{Q}_1\delta^3(x-x_1) = \delta_{x_1}[\vec{c}_0^{(A)}(x_1,x_1,x_2)\cdot\vec{Q}_1], \tag{C16}$$

while the second term on the right of Eq. (C15) is independent of x_2. Hence, Eq. (C15) implies

$$V_{\text{mean field}}(x_1, x_2) - \vec{Q}_1 \cdot \vec{c}_0(x_1) = V_1(x_1) + V_2(x_2)\,, \qquad \text{(C17)}$$

with V_1 independent of x_2 and V_2 independent of x_1. But since translational, rotational, and local SU_2 gauge invariance imply that both terms on the left-hand side of Eq. (C17) depend only on the relative distance $x_1 - x_2$, the terms V_1 and V_2 on the right must be constants, proving Eq. (C12a). A similar proof, using δ_{x_2}, gives Eq. (C12b).

According to Eqs. (C11) and (C12), a consistent mean-field approximation to the source wave equations is given by

$$\begin{aligned} i\partial_0\psi_1 &= \vec{c}_0(x_1)\cdot\vec{Q}_1\psi_1 + \text{recoil terms} \\ &= [V_{\text{mean field}}(x_1, x_2) + \text{constant}]\psi_1 + \text{recoil terms}\,, \\ i\partial_0\psi_2 &= \vec{c}_0(x_2)\cdot\vec{Q}_2\psi_2 + \text{recoil terms} \\ &= [V_{\text{mean field}}(x_1, x_2) + \text{constant}]\psi_2 + \text{recoil terms}\,. \end{aligned} \qquad \text{(C18)}$$

These are just the usual one-body wave equations obtained from the potential theory of two sources, interacting through a potential $V_{\text{mean field}}(x_1, x_2)$, in the limit that the sources are well localized. Hence, the formalism of Sec. I reproduces all of the expected potential theory results.[30]

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[1]R. Giles and L. McLerran, Phys. Lett. 79B, 447 (1978); S. L. Adler, Phys. Rev. D 17, 3212 (1978); S. L. Adler, *ibid.* 20, 3273 (1979). For a review, see S. L. Adler, in *The High Energy Limit*, proceedings of the of the 18th International School of Subnuclear Physics, "Ettore Majorana", edited by A. Zichichi (Plenum, New York, to be published). Further references are given here. [In QCD, the quantized gauge field is the underlying SU(3) gauge field, while the effective c-number sources lie in an unquantized, overlying SU(2) × U(1) gauge field. See Appendix A for a detailed discussion.]

[2]I define $\mathcal{L}_M^{\text{gen cov}}$ to be a scalar, so that

$$S_M = \int d^4x\,\sqrt{-g}\,\mathcal{L}_M^{\text{gen cov}}.$$

[3]C. W. Bernard, Phys. Rev. D 9, 3312 (1974). I thank L. Dolan for bringing this reference to my attention. For a discussion of the generalization of Eq. (5) to the case when fermions are present, see D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981). Equations (12)–(14) remain valid for massive fermion sources at rest.

[4]The functional measure in Eq. (12) is understood to include the exponential of the gauge-fixing term, and the compensating Faddeev-Popov determinant (which can be represented as an additional functional integral over ghost fields). When the kinetic terms and functional integrals for the source fields producing $\vec{j}_\mu$ are included, S_E is properly gauge invariant, justifying use of the Faddeev-Popov functional measure. In the situation studied in this paper, where the only sources present are infinitely massive sources at rest, the source current can always be made time independent by an appropriate time-dependent gauge transformation. In such static-source gauges, the source functional integral can be omitted, leaving the expression for the partition function given in Eqs. (12)–(14), with $\vec{j}_\mu = (\vec{j}_0 \neq 0,\ \vec{j}_i = 0)$ and with $\vec{j}_0$ time independent. The static-source formalism is no longer invariant under all gauge transformations, but remains invariant under the subclass of time-independent gauge transformations. Since in a stationary state we have $d\langle\vec{b}_j\rangle/dt = d\vec{c}_j/dt = 0$, we are assured that the mean-field potential can be calculated from the expectation of the scalar potential $\langle\vec{b}_0\rangle = \vec{c}_0$.

[5]In a linear system the incremental potential $\delta V_{\text{mean field}}$ can be defined as either $\langle\int d^3x\,\vec{b}_0\cdot\delta\vec{j}_0\rangle$, which gives the mean energy change when an increment in source density $\delta\vec{j}_0$ is brought in from infinity, or as $\langle\delta\int d^3x\,\tfrac{1}{2}\vec{E}^j\cdot\vec{E}^j/g^2\rangle$, but in general these expressions are not equivalent: only the former can be used for nonlinear systems and is renormalization group invariant. When we study the nonrelativistic motion of the sources, the leading coupling of the sources to the gluon field involves only the values of $\vec{b}_0$ at the source positions. Hence, an average potential calculated from $\langle\int d^3x\,\vec{b}_0\cdot\delta\vec{j}_0\rangle$ is the correct starting point for a mean field, potential theory analysis of the source motion. See Appendix C for further details.

[6]From Eq. (17) we can see that the zero temperature ($\beta\to\infty$) mean-field potential is *not* the same as the static potential calculated from the Wilson loop, which in the notation used here is

$$V_{\text{static}} = \lim_{\beta\to\infty}\{W[i\vec{j}_\mu] - W[\vec{0}]\}.$$

The physical interpretation of V_{static} is that it is the ground-state eigenenergy of a static $q\bar{q}$ system. [See, for example, the derivation of the Wilson-loop formula given by L. S. Brown and W. I. Weisberger, Phys. Rev. D 20, 3239 (1979).] Since eigenenergies are defined only by continuation back to Minkowski spacetime, it is not surprising that an imaginary source occurs when we formally represent V_{static} by a Euclidean path integral. The motivation for introducing $V_{\text{mean field}}$ is that it can be calculated strictly within the Euclidean formalism. In a perturbation expansion in the external source strength $\vec{j}_\mu$, the mean-field and Wilson-loop potentials agree in order $(\vec{j}_\mu)^2$, but differ beyond this order. In the Abelian case, there are no terms of higher order than $(\vec{j}_\mu)^2$, and so the two formalisms give the same static potential. In the non-Abelian case, the formalisms are inequivalent, and give different formulations of the confinement problem. It appears that the simple effective action approach to confinement developed in this paper can be obtained only by using a mean value formalism. I wish to thank R. F. Dashen for several discussions of these points.

(See also Appendix C and Ref. 30 below.)

[7]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[8]Since Γ is not gauge invariant, the gauge-fixing conditions used in solving for $\bar{c}_\mu$ must be chosen to be compatible with the gauge noninvariance of Γ.

[9]The use of an effective action in this context was first suggested by H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978).

[10]In particular, $\bar{c}_\mu$ is not to be used as Minkowski space Cauchy data and time evolved, as was implied in Ref. 1. In canonical gauges, the physical interpretation of $\bar{c}_\mu$ is that it is the expectation of $\bar{b}_\mu$, and is Minkowski time independent. Also, in Ref. 1 I used the incorrect, renormalization-group noninvariant formula for the potential (see Ref. 5 above).

[11]S. L. Adler and T. Piran, in *High Energy Physics—1980*, proceedings of the XXth International Conference, Madison, Wisconsin, edited by L. Durand and L. G. Pondrom (AIP, New York, 1981), p. 958.

[12]I. A. Batalin, S. G. Matinyan, and G. K. Savvidi, Yad. Fiz. 26, 407 (1977) [Sov. J. Nucl. Phys. 26, 214 (1977)]; G. K. Savvidy, Phys. Lett. 71B, 133 (1977); H. Pagels and E. Tomboulis, Ref. 9. See J. Ambjörn and P. Olesen, Nucl. Phys. B170, 60 (1980) for a discussion of corrections to the local effective action approximation, and extensive references. Many of these references consider only constant color fields, which has tended to obscure the fact that the gauge theory vacuum leading to the effective action of Eq. (23) is Lorentz invariant, with $\langle 0|\vec{b}^\mu|0\rangle = \langle 0|\vec{E}^j|0\rangle = \langle 0|\vec{B}^j|0\rangle = 0$ in the absence of sources. The vanishing of these expectations is reflected in the fact that the minimization of $\Gamma[\bar{c}_\mu]$ of Eq. (23) leads to a partially indeterminate variational problem, solved by any random color-electric and magnetic fields $\vec{E}^j$ and $\vec{B}^j$ satisfying $\vec{E}^j\cdot\vec{E}^j + \vec{B}^j\cdot\vec{B}^j = \kappa^2$. When sources are added, the variational problem of Eq. (22) is fully determinate only in the interior of the "bag", where $D>0$, but remains partially indeterminate (in the sense described above) in the exterior region where $D=0$. As a result, one cannot argue that there are nonvanishing gluon gauge potential or gauge field vacuum expectations by considering the limit of the exterior solution as a weak source is turned off; this line of reasoning applies only when the variational problem in the presence of sources is fully determinate in all of space.

[13]G. 't Hooft, in *Recent Progress in Lagrangian Field Theory and Applications*, proceedings of the Marseilles Colloquium, 1974, edited by C. P. Korthes-Altes *et al.* (Centre de Physique Theorique, Marseilles, 1975).

[14]To prove $f(u)\geq 1$, let $\psi = f-1$, $\phi = 2/(u\ln u^2)$. A simple calculation shows that ψ satisfies the differential equation

$$\frac{d\psi}{du} + \phi\psi = \frac{(u-1)^2}{u^2\ln u^2} > 0 \text{ for } u>1.$$

Integrating up from $u=1$ (where $\psi=0$), this implies that ψ is positive for $u>1$.

[15]In this case, one cannot neglect the surface term (as was done in the text) in the integration by parts leading from Eq. (33) to Eq. (38) but rather, one must work directly from Eq. (33). For the charge orientations of Eq. (31b), simple estimates (see H. Pagels and E. Tomboulis, Ref. 9) show that $\tilde{W}$ has a positive infinite infrared divergence at equilibrium, corresponding to a vanishing partition function Z. Hence, the configuration with nonvanishing color flux at infinity is automatically excluded from the physical spectrum. Note that when the correspondence with QCD is made as in Appendix A, charge-conjugation symmetry or permutation symmetry will select either the effective charge orientations of Eq. (31a) or those of Eq. (31b), but not both. In the $q\bar{q}$ problem, the averaged potentials c_μ^A are charge-conjugation odd, selecting Eq. (31a). For the qq sector, the averaged potentials c_μ^A are symmetric under permutation of the sources, selecting Eq. (31b), and giving a vanishing partition function contribution. This is the expected result for a system which cannot be in a color singlet state.

[16]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[17]If $\mathcal{L}_{\rm eff}$ attains its minimum at $F=0$, and vanishes there as F^α, a simple estimate shows that the asymptotic behavior of the potential is $V_{\rm mean\ field} \sim R^{(\alpha-3)/(\alpha-1)}$. This confines for $\alpha>3$, but gives a linear potential only in the limit $\alpha\to\infty$. (See H. Pagels and E. Tomboulis, Ref. 9.)

[18]I am assuming a standard canonical quantization, in which only the constrained components of b_μ^A are matrix valued. Hence, the differentials $db_{\mu i}^A$ and $d[b_\mu^A]$ in Eq. (A2) are ordinary numbers. The assumption of canonical quantization is consistent with the conclusion reached at the end of the analysis, that the mean matrix-valued potential c_μ^A is Abelian apart from a time-independent gauge transformation. This means that c_0^A is nonzero, while c_j^A contains only a single spatial degree of freedom. The two spatial degrees of freedom in b_j^A which are orthogonal to c_j^A can then be canonically quantized by the standard Dirac bracket procedure. For example, taking the q and $\bar{q}$ to lie on the z axis, the gauge transformation rotating the $\bar{q}$ effective charge to be antiparallel to the q effective charge can be chosen to depend on z only, giving c_0^A, $c_z^A \neq 0$, but $c_{x,y}^A = 0$. This matrix-valued structure in the potentials is compatible with axial-gauge quantization.

[19]The classical limit of the effective action can be read off from Eq. (A2) by approximating e^{-S_E} by

$$e^{-S_E} \approx 1 - S_E,$$

and so is given by the field-strength terms in Eq. (A4), acted on by the quark color trace $\frac{1}{9}\,\mathrm{tr}_q\,\mathrm{tr}_{\bar{q}}$.

[20]Color-charge-algebra solutions of this form have been discussed by I. Bender, D. Gromes, and H. J. Rothe, Z. Phys. 5C, 151 (1980).

[21]In the formulation of Ref. 1, there arose the issue of how to fix the integration constants $K_{(I)}$ in the Lagrangian for the overlying algebra. The present analysis corresponds to taking the K_I's all equal, which differs from the rule which I had originally postulated.

[22]The effective Lagrangian analysis of the $q\bar{q}$ binding problem has the following Feynman diagram interpretation: Working in Coulomb gauge, the effective Lagrangian for the Coulomb gluons arises from Feynman diagrams which may be characterized as a central "blob," containing one or more closed gluon loops, from which $n\geq 2$ Coulomb gluons emerge. The effective Lagrangian contribution to $q\bar{q}$ binding is obtained

by stringing such "blobs" between q and $\bar{q}$ lines, attaching each emerging Coulomb gluon to either the q or the $\bar{q}$ line. This procedure yields the usual renormalization-group improved one-Coulomb-gluon exchange graph, and its nonlinear generalizations, which are responsible for the weak field-strength modification in the effective action which leads to confinement.

[23]Previously in this paper, I have taken N_f to be 0.

[24]G. 't Hooft, in *The Whys of Subnuclear Physics*, proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979), pp. 943–971. 't Hooft restricts his discussion to the case of analytic running coupling constant transformations. Nonanalytic transformations similar to those of Eq. (B8) have been recently investigated by Y. Frishman, R. Horsely, and U. Wolff, Phys. Lett. (to be published) and Weizmann Institute report (unpublished).

[25]N. N. Khuri and O. A. McBryan, Phys. Rev. D 20, 881 (1979).

[26]Similar coupling-constant logarithms have been found in three-dimensional QCD (which is related to the behavior of the four-dimensional theory at high-temperature phase transitions) by R. Jackiw and S. Templeton, Phys. Rev. D 23, 2291 (1981), and in chiral perturbation theory by H. Pagels, Phys. Rep. 16C, 219 (1975). In using the modified expansion to evaluate Euclidean Green's functions, it may be important to keep the $-i\epsilon$ in the Feynman denominators even after continuation to the Euclidean section. This gives a definite prescription for circling the spacelike pole in α_R and chooses a definite branch of the spacelike cut in $\ln\alpha_R$. The rearranged power series will in general contain imaginary contributions to the Euclidean Green's functions in each order, but (under the conventional assumption that the Euclidean Green's functions in QCD are real) these will cancel when the entire series is summed. Hopefully, the rearranged series will give real contributions to the Euclidean Green's functions which converge fast enough to give useful estimates (as, for example, is the case in the Wilson-Fisher expansion in critical phenomena when applied in 3 or 2 dimensions). Good convergence of the rearranged series would be an indication that the infrared behavior of QCD is effectively controlled by a weak coupling regime.

[27]There appears to be a close analogy between transformations of the radial coordinate in the theory of Schwarzschild black holes in general relativity, and transformations of the running coupling constant in QCD, with the concept of maximal analytic extension playing a key role in both cases. In both theories the natural coordinate (or coupling) in which one does calculations does not give the maximal analytic extension. Moreover, the transformations which yield the maximal extension have very similar functional form: Eq. (B8) relating $\alpha_R{}^{-1}$ to α^{-1} closely resembles the transformation $r^* = r + 2M\ln|r/2M - 1|$ which is used to remove the coordinate singularity at the horizon in black hole physics.

[28]A second interesting analogy is the fact that the leading logarithm effective action

$$\Gamma \propto \int d^4x F^2(x) \ln F^2(x)$$

has the same structure as the quantum-mechanical entropy

$$S = -k_B \operatorname{Tr} \rho \ln \rho ,$$

which has many special and useful formal properties [see A. Wehrl, Rev. Mod. Phys. 50, 221 (1978)]. Perhaps this analogy can be exploited to understand the thermodynamic aspects of hadronic behavior. As a simple application of the entropy analogy, suppose that in the discussion of Appendix A we had applied the renormalization group improvement argument locally in the $q, \bar{q}$ color space, thus obtaining

$$l_{\text{eff}}(f^2) = \tfrac{1}{8} b_0 \operatorname{tr}_q \operatorname{tr}_{\bar{q}} [f^2 \ln(f^2/e\kappa^2)] ,$$

$$f^2 = \tfrac{1}{9}(E^{Aj}E^{Aj} + B^{Aj}B^{Aj}) ,$$

instead of Eqs. (A10) and (A11), which in terms of f^2 read

$$\mathcal{L}_{\text{eff}}(F^2) = \tfrac{1}{8} b_0 (\operatorname{tr}_q \operatorname{tr}_{\bar{q}} f^2) \ln (\operatorname{tr}_q \operatorname{tr}_{\bar{q}} f^2/e\kappa^2) .$$

Since l_{eff} yields the same stress-energy tensor trace anomaly as does $\mathcal{L}_{\text{eff}}$, it is also an acceptable form for the effective action density. By some simple algebra, we find

$$l_{\text{eff}}(f^2) - \mathcal{L}_{\text{eff}}(F^2) = \tfrac{1}{8} b_0 (\operatorname{tr}_q \operatorname{tr}_{\bar{q}} f^2) \operatorname{tr}_q \operatorname{tr}_{\bar{q}} (\rho \ln \rho) ,$$

$$\rho = f^2/(\operatorname{tr}_q \operatorname{tr}_{\bar{q}} f^2) , \quad \operatorname{tr}_q \operatorname{tr}_{\bar{q}} \rho = 1 .$$

Since ρ is a color density matrix, we can use the positivity of the entropy to conclude that

$$l_{\text{eff}}(f^2) \le \mathcal{L}_{\text{eff}}(F^2) ,$$

and so the use of l_{eff} would give at least as strong a linear potential as is obtained with $\mathcal{L}_{\text{eff}}$.

[29]The discussion of Appendix C is readily generalized to the QCD case by taking ${}_s\langle 0|\cdots|0\rangle_s$ to be an expectation with respect to the source spatial (but not color) wave functions, and including the source color wave functions in $|0\rangle_g$. Following Appendix A, the only changes are then the replacement of arrows by octet color indices, and the inclusion of a factor $\frac{1}{9}\operatorname{tr}_q \operatorname{tr}_{\bar{q}}$ in the inner products involving c_0 appearing in Eqs. (C10)–(C18).

[30]In contrast to the mean field approach, the Wilson-loop formula evaluates $\langle 0|H_T|0\rangle$ directly, without approximation, in terms of a Euclidean functional integral with imaginary sources. (See also the remarks in Ref. 6 above.)