Topological symmetry restoration

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We examine spontaneously broken N-component $\lambda\phi^4$ theory at finite temperature on a static manifold whose equal-time hypersurfaces are homogeneous but may be topologically nontrivial. The alterations in the infrared structure of the field theory caused by the nontrivial topology can induce a transition from the ordered to the disordered phase, even at zero temperature. Results derived include a general expression for the zero-temperature one-loop effective potential on a topologically trivial homogeneous curved manifold and a calculation of the free energy of the self-interacting scalar field at finite temperature on a static universe whose spatial section is S^3/Γ, where Γ is a discrete group.

I. INTRODUCTION

The past few years have witnessed steady progress in quantum field theory in curved spacetime (for a review see the forthcoming book by Birrell and Davies[1]). A particularly important development, recently achieved by two groups,[2-6] has been the successful transcription from Minkowski space to a general curved spacetime, of the usual perturbative techniques for simple interacting field theories, and the proof that (in the perturbative sense) such theories remain renormalizable in their curved-space setting (see Ref. 7 for a concise review).

Discussions of the cosmological development of grand unified theories and their associated hierarchy of broken symmetries,[8] however, base their predictions and numerical estimates on models of the field-theoretic phase transition which, though well explored,[9,10] are only valid in Minkowski space. With a sufficiently sophisticated curved-space interacting-field-theory formalism at hand, now appears to be the time to employ it in removing this inconsistency and simultaneously upgrading the confidence to be placed in such discussions. Developments in this direction in any case are imperative if one is to seriously face up to the back-reaction problem[11,12] for grand unified theories in the early universe.

A first tentative step required of such a program is an analysis of the criteria which govern the onset of a phase transition in a field theory with a spontaneously broken symmetry propagating in a curved spacetime. Since phase transitions are associated with large correlation lengths and alterations in the infrared structure of the field theory, and hence will be presumably rather insensitive to changes in local objects such as the Riemann tensor, one would not expect such an investigation to be particularly illuminating. If, however, the manifold possesses some global features such as a compact section, as in the $k=1$ Robertson-Walker (RW) models, or a nontrivial topology, the consequent alteration in the infrared structure of the field theory may influence the possible occurrence of a phase transition. We are thereby lured to an investigation of topological effects on symmetry restoration in its own right.

With some foresight, therefore, we might wish to pay particular attention to a rather more exotic class of cosmologies than the standard RW models. In fact, as Ellis[13] has pointed out, the usual arguments[14] which suggest the observed universe is RW type, or nearly so, are purely local, and allow one the freedom to consider a wide class of $t=$ constant hypersurfaces which are only locally isometric to $R^3(k=0)$, $S^3(k=1)$, or $H^3(k=-1)$. Such hypersurfaces are the Clifford-Klein (CK) space forms,[15] which we denote by $M_3=\tilde{M}_3/\Gamma$, where $\tilde{M}_3=R^3$, S^3, or H^3 is the covering manifold and Γ is a discrete group of isometries of $\tilde{M}_3$ which acts freely and without fixed points. The corresponding spacetimes we label CKRW, $\Gamma=1$, the identity, specifying the usual RW models. In considering these cosmologies, which are of some interest within classical general relativity,[16] we take the attitude that since such exotic topologies can influence the occurrence of a phase transition, it may well be possible to discount some on grounds of incompatibility with present views on the hierarchical cosmological evolution of broken symmetries in grand unified theories.[8]

Quite apart from the cosmological setting has been the considerable recent interest in field theory on topologically nontrivial spacetimes.[17-25] References 20–25 are of particular relevance to the present paper since these are concerned with interacting fields. In fact Ford and Yoshimura[21] found that nontrivial topological features can influence the mass renormalization in simple massless $\lambda\phi^4$ theory, and the topological effects on symmetry restoration considered here are manifested by the same mechanism. The simplest of the cases treated here, that of zero-temperature field theory on $S^1\times R^3$, has already been investigated by Denardo and Spallucci[24] who established the existence of a critical length of the spatial periodicity below which a spontaneously broken symmetry becomes restored.

In the following section we set up some basic formalism and calculate the zero-temperature effective potential to one-loop order for a spontaneously broken N-component scalar field in a topologically trivial homogeneous curved manifold. Section III considers the effect of introducing a finite temperature and nontrivial topological features in the special cases of the static homogeneous CKRW universes with $k=0$, ±1, and ascertains whether or not such global features can induce a phase transition. Discussion and extensions are to be found in Sec. IV.

II. SPONTANEOUSLY BROKEN $\lambda\phi^4$ THEORY IN CURVED SPACETIME

The model under investigation, that of an $O(N)$-invariant scalar field, $\tilde{\phi}^i(x)$, with a quartic self-interaction, is described by the Lagrangian density[2-7]

$$\mathfrak{L} = -\tfrac{1}{2}\tilde{\phi}^i(\Box+\xi_B R+m_B{}^2)\tilde{\phi}^i - \frac{\lambda_B}{4!}\tilde{\phi}^4,$$

$$\tilde{\phi}^4\equiv(\tilde{\phi}^2)^2\equiv(\tilde{\phi}^i\tilde{\phi}_i)^2. \tag{2.1}$$

The equations

$$Z[J]=\exp(iW[J]) = \left\langle\beta\left|T\exp\left(i\int d^{2\omega}x\tilde{\phi}^i(x)J_i(x)\right)\right|\beta\right\rangle, \tag{2.2a}$$

$$\Gamma[\bar{\phi}]=W[J]-\int d^{2\omega}x\bar{\phi}^i(x)J_i(x) \equiv\int d^{2\omega}x\sqrt{-g}\,\mathfrak{L}_{\rm eff}(x), \tag{2.2b}$$

$$\bar{\phi}_i(x)=\frac{\delta W[J]}{\delta J^i(x)}, \tag{2.2c}$$

$$J_i(x)=-\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}^i(x)} \tag{2.2d}$$

specify the generating functional for connected single-particle irreducible finite-temperature Green's functions $W[J]$ and the finite-temperature effective action $\Gamma[\bar{\phi}]$ (Refs. 26 and 27), where we have adopted the notation

$$\langle\beta|O|\beta\rangle\equiv{\rm Tr}[e^{-\beta H}O],\quad \beta\equiv(kT)^{-1}, \tag{2.3}$$

for any operator O, Tr being a Fock-space trace, and have presumed that the field propagates at finite temperature T on a manifold M, of dimension 2ω, which may be topologically nontrivial. Then

$$\hat{\phi}_i(x)\equiv\bar{\phi}_i(x)|_{J=0}=\frac{\langle\beta|\tilde{\phi}_i(x)|\beta\rangle}{\langle\beta|\beta\rangle}\equiv\langle\tilde{\phi}_i(x)\rangle^\beta, \tag{2.4}$$

$$\tilde{G}_{ij}(x,y)=\left.\frac{\delta^2 W[J]}{\delta J^i(x)\delta J^j(y)}\right|_{J=0}=i\langle T\{\phi_i(x)\phi_j(y)\}\rangle^\beta \tag{2.5}$$

define the thermal expectation value of $\tilde{\phi}_i$ and the complete two-point Green's function, where $\phi_i(x)\equiv\tilde{\phi}_i(x)-\hat{\phi}_i(x)$ denotes the quantum fluctuations. $\hat{\phi}_i$ plays the role of the order parameter: when $\hat{\phi}_i(x)\neq0$ the system exists in an ordered phase, the simple $O(N)$ invariance of $\mathfrak{L}$ is not an invariance of the equilibrium state of the system, and the symmetry is said to be spontaneously broken.

We assume that the ordered phase exists at zero temperature on a topologically trivial manifold $\bar{M}$, locally isometric to M; in DeWitt's condensed notation,[28]

$$\bar{\Gamma}_{,i}=0 \text{ for } \hat{\phi}_i\neq0. \tag{2.6a}$$

(We shall throughout indicate by a bar the zero temperature and trivial topology associated with $\bar{M}$.) Our main concern in the present paper is to determine whether global features such as a finite temperature or a nontrivial topology can influence the variational Eq. (2.2d) such that the only solution is the disordered one,

$$\Gamma_{,i}=0 \text{ for } \hat{\phi}_i=0. \tag{2.6b}$$

Expressed otherwise, we wish to investigate whether alteration of the infrared structure of the field theory can induce an ordered to disordered phase transition.

To proceed we need a handle on Γ. This object generates the inverse propagator and proper vertices of the renormalized theory in a manner analogous to the generation of the bare quantities by the classical action S, e.g.,

$$S_{,ij}G^{jk}=-\delta_i^k, \tag{2.7a}$$

$$\Gamma_{,ij}\tilde{G}^{jk}=-\delta_i^k \tag{2.7b}$$

define the bare and complete two-point Green's functions.[28] In terms of a loop expansion in the bare propagator and vertices,[28, 29]

$$\Gamma[\hat{\phi}]=S[\hat{\phi}]-\frac{i}{2}\ln{\rm Det}G+\Gamma', \tag{2.8}$$

where the functional determinant summarizes the one-loop processes and Γ' represents higher loops.

Substituting (2.1) in (2.7a) leads to

$$\left\{\delta_{ij}\left[\Box+\xi_B R+m_B{}^2+\frac{\lambda_B}{6}\hat{\phi}^2(x)\right]+\frac{\lambda_B}{3}\hat{\phi}_i(x)\hat{\phi}_j(x)\right\}\times G^{jk}(x,x')=\delta_i^k\,\delta(x,x')\ , \quad (2.9)$$

where $\delta(x,x')$ is the covariant δ function.[30] In general this equation has no simple solution. If, however, we restrict M to be homogeneous, in which case $\hat{\phi}_i$ is independent of x, a simple matrix inversion gives[29]

$$G^{jk}(x,x')=\frac{\hat{\phi}^j\hat{\phi}^k}{\hat{\phi}^2}G(x,x';\xi_B R+M_1{}^2)+\left[\delta^{jk}-\frac{\hat{\phi}^j\hat{\phi}^k}{\hat{\phi}^2}\right]G(x,x';\xi_B R+M_2{}^2)\ , \quad (2.10)$$

where

$$M_1{}^2=m_B{}^2+\frac{\lambda_B}{2}\hat{\phi}^2\ , \quad (2.11a)$$

$$M_2{}^2=m_B{}^2+\frac{\lambda_B}{6}\hat{\phi}^2\ , \quad (2.11b)$$

and $G(x,x';\xi_B R+M^2)$ satisfies

$$(\Box+\xi_B R+M^2)G(x,x';\xi_B R+M^2)=\delta(x,x')\ . \quad (2.12)$$

In this case the effective potential $V(\hat{\phi})\equiv-\mathcal{L}_{\text{eff}}(\hat{\phi})$ is a more convenient function, in terms of which (2.8) now becomes

$$V(\hat{\phi})=V_0(\hat{\phi})+V_1(\hat{\phi})+V'(\hat{\phi})\ , \quad (2.13)$$

where V_0 is the classical potential and

$$V_1(\hat{\phi})=\frac{i}{2}[\langle x|\ln G(M_1{}^2)|x\rangle+(N-1)\langle x|\ln G(M_2{}^2)|x\rangle]=\frac{i}{2}\int_0^\infty\frac{d\tau}{\tau}[e^{-iM_1{}^2\tau}+(N-1)e^{-iM_2{}^2\tau}]K(x,x;\tau) \quad (2.14)$$

accounts for the one-loop effects. Here we have adopted a spacetime matrix notation, in which $\langle x|G(M^2)|x'\rangle\equiv G(x,x';\xi_B R+M^2)$, and have utilized the standard proper-time parameter integral representation based on the Schrödinger equation[26]

$$\left(i\frac{\partial}{\partial\tau}-\Box-\xi_B R\right)K(x,x';\tau)=i\delta(\tau)\delta(x,x') \quad (2.15)$$

for the quantum-mechanical propagator $K(x,x';\tau)$. The dependence of $V(\hat{\phi})$ on global features such as a finite temperature or nontrivial topology is related, through (2.14), to the boundary conditions we impose on (2.15).[27]

In terms of $V(\hat{\phi})$, which must be a function solely of $\hat{\phi}^2$, Eqs. (2.6) are rewritten

$$\frac{\partial\bar{V}(\hat{\phi})}{\partial\hat{\phi}^2}=0 \text{ for } \hat{\phi}^2\neq 0\ , \quad (2.16a)$$

$$\frac{\partial V(\hat{\phi})}{\partial\hat{\phi}^2}\neq 0 \text{ for } \hat{\phi}^2\neq 0\ . \quad (2.16b)$$

To ensure that field oscillations be bounded, $V(\hat{\phi})$ must become large and positive for large $\hat{\phi}^2$, therefore, if M is homogeneous,

$$\left.\frac{\partial V(\hat{\phi})}{\partial\hat{\phi}^2}\right|_{\hat{\phi}=0}\geq 0 \quad (2.17)$$

indicates our criterion for symmetry restoration, equality specifying the critical values of the global parameters at which the phase transition occurs.[9] Since in practice $V(\hat{\phi})$ can only be calculated approximately one must check that the approximation is consistent with

$$\text{Im}V(0)=0\ , \quad (2.18)$$

which is required for stability of the disordered phase. Only then can the approximate $V(\hat{\phi})$ be used in (2.17) to determine the critical global parameters.

Before embarking on detailed calculations of $V(\hat{\phi})$ for particular cases we note that our alteration of the global field theory structure has left the local structure, and hence the ultraviolet divergences, unchanged. The renormalization program for removal of these divergences at finite temperature on M is therefore identical to that applied at zero temperature on $\bar{M}$. In the remainder of this section we illustrate this program by calculating the renormalized effective potential to one-loop order.

Following Birrell[5] and Bunch *et al.*[2-4] we choose dimensional regularization to handle the infinities in (2.14) and adopt the 't Hooft mass-independent renormalization scheme[31] whereby the bare and renormalized parameters are related through

$$m_B{}^2=m_R{}^2\left[1+\sum_{\nu=1}^\infty\sum_{j=\nu}^\infty\frac{b_{\nu j}\lambda_R{}^j}{(2\omega-4)^\nu}\right], \quad (2.19a)$$

$$\xi_B=\xi_R+\sum_{\nu=1}^\infty\sum_{j=\nu}^\infty\frac{d_{\nu j}\lambda_R{}^j}{(2\omega-4)^\nu}\ , \quad (2.19b)$$

$$\lambda_B=\mu^{4-2\omega}\left[\lambda_R+\sum_{\nu=1}^\infty\sum_{j=\nu}^\infty\frac{a_{\nu j}\lambda_R{}^j}{(2\omega-4)^\nu}\right]\ , \quad (2.19c)$$

$$Z(\lambda_R;2\omega)=1+\sum_{\nu=1}^\infty\sum_{j=\nu}^\infty\frac{c_{\nu j}\lambda_R{}^j}{(2\omega-4)^\nu}\ . \quad (2.19d)$$

Z is the field renormalization constant and μ is the unit of mass[31] required to keep the dimensionality of λ_R fixed in 2ω dimensions. Given the well-known asymptotic expansion[28, 32] for the solution to (2.15) on $\bar{M}$,

$$\bar{K}(x,x;\tau) = (-4\pi i\tau)^{-1/2}(4\pi i\tau)^{1/2-\omega}\sum_{l=0}^{\infty} a_l\,(i\tau)^l \quad (\tau\to 0_+)\,, \tag{2.20}$$

where the a_l are the Schwinger–DeWitt coefficients, (2.13), (2.14), and (2.19) quickly result in the expression, to one-loop order,

$$\bar{V}(\hat\phi)=\frac{1}{2}(m_B{}^2+\xi_B R)\hat\phi^2+\frac{\lambda_B}{4!}\hat\phi^4-\frac{1}{2(4\pi)^\omega}\times\sum_{l=0}^{\infty}a_l\,\Gamma(l-\omega)[M_{1R}{}^{2(\omega-l)}+(N-1)M_{2R}{}^{2(\omega-l)}]+\text{h.l.p.} \tag{2.21}$$

for the effective potential, where h.l.p. abbreviates higher-loop processes, and

$$M_{1R}{}^2=m_R{}^2+\frac{\mu^{4-2\omega}}{2}\lambda_R\hat\phi^2\,, \tag{2.22a}$$

$$M_{2R}{}^2=m_R{}^2+\frac{\mu^{4-2\omega}}{6}\lambda_R\hat\phi^2\,. \tag{2.22b}$$

To renormalize (2.21) by the general prescription of the Utiyama and DeWitt,[33] a bare gravitational Lagrangian density

$$\begin{aligned}\mathcal{L}_B^G&=(16\pi G_B)^{-1}R-\Lambda_B+\alpha_{1B}R^2+\alpha_{2B}R_{\mu\nu}R^{\mu\nu}+\alpha_{3B}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}\\&\equiv\mu^{2\omega-4}[(16\pi\tilde G_B)^{-1}R-\tilde\Lambda_B+\tilde\alpha_{1B}R^2+\tilde\alpha_{2B}R_{\mu\nu}R^{\mu\nu}+\tilde\alpha_{3B}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}]\end{aligned} \tag{2.23}$$

is now introduced such that the total

$$\begin{aligned}\mathcal{L}_{\text{tot}}&=\mathcal{L}_B^G-\bar V(\hat\phi)\\&\equiv\mathcal{L}_R^G-\bar V_R(\hat\phi)\end{aligned} \tag{2.24}$$

is finite as $\omega\to 2$. (Note the appearance of μ to keep the dimensions of coupling constants fixed.) The renormalized constants in $\mathcal{L}_R^G$ obtained in this way must be real[34] (in the limit $\omega\to 2$), and, of course, $\hat\phi$ independent. Hence

$$\bar V_R(\hat\phi)\equiv\bar V(\hat\phi)-\text{Re}\bar V_{\text{DIV}}(0)\,, \tag{2.25}$$

where $\bar V_{\text{DIV}}(0)$ denotes the $\hat\phi$-independent divergent piece to be removed from $\bar V(\hat\phi)$.

For free fields, where $\hat\phi=0$, Bunch[35] has examined (2.24) to one loop within the framework of dimensional regularization. His work suggests that we remove all terms involving four or less derivatives of the metric, i.e.,

$$\bar V_{\text{DIV}}(0)=-\frac{N}{2(4\pi)^\omega}\sum_{l=0}^{2}a_l\,\Gamma(l-\omega)m_R{}^{2(\omega-l)}+\text{h.l.p.} \tag{2.26}$$

From Eqs. (2.21), (2.25), and (2.26) a straightforward calculation then shows

$$\begin{aligned}\bar V_R(\hat\phi)=&\tfrac12(m_R{}^2+\xi_R R)\hat\phi^2+\frac{\lambda_R}{4!}\hat\phi^4\\&-\frac{1}{32\pi^2}\Bigg\{\sum_{l=0}^{2}\frac{(-1)^{3-l}}{(2-l)!}\Bigg[a_l\left(M_{1R}{}^{2(2-l)}\ln\frac{M_{1R}{}^2}{4\pi\mu^2}+(N-1)M_{2R}{}^{2(2-l)}\ln\frac{M_{2R}{}^2}{4\pi\mu^2}-Nm_R{}^{2(2-l)}\ln\frac{|m_R{}^2|}{4\pi\mu^2}\right)\\&\qquad+[a_l'-\psi(3-l)a_l][M_{1R}{}^{2(2-l)}+(N-1)M_{2R}{}^{2(2-l)}-Nm_R{}^{2(2-l)}]\Bigg]\\&\qquad+\sum_{l=3}^{\infty}a_l\,\Gamma(l-2)[M_{1R}{}^{2(2-l)}+(N-1)M_{2R}{}^{2(2-l)}]\Bigg\}+\text{h.l.p.},\end{aligned} \tag{2.27}$$

where the prime denotes $d/d\omega$ (we allow for the possibility that a_l carries an ω dependence through ξ) and the $\omega\to 2$ limit has been taken. In arriving at (2.27) the one-loop counterterms in (2.19) have been identified as

$$b_{11}=-\frac{1}{16\pi^2}\left[1+\frac{(N-1)}{3}\right], \tag{2.28a}$$

$$d_{11}=\frac{1}{16\pi^2}\left(\frac16-\xi_R\right)\left[1+\frac{(N-1)}{3}\right], \tag{2.28b}$$

$$a_{12}=-\frac{3}{16\pi^2}\left[1+\frac{(N-1)}{9}\right]\,, \tag{2.28c}$$

which agree with Birrell[5] and Bunch *et al.*[2-4] in the $N=1$ case.

No particular physical significance is attached to the R-subscripted quantities in (2.19). The 't Hooft renormalization prescription is designed to handle the infinities which arise in perturbation theory without worrying about finite pieces. This property is an added advantage in an arbitrarily curved manifold since such finite pieces will not in general be constants[7] and so cannot be absorbed through additional finite renormalizations. Of course, in the restrictive class of homogeneous manifolds considered here the finite pieces are guaranteed to be constant, but in the more general case they imply that particle masses, for example, will be complicated functions of the spacetime geometry.

It is possible to reexpress (2.27) in terms of the physical mass and coupling constants defined as[34,29,9]

$$m^2+\xi R=2\,\mathrm{Re}\,\frac{\partial \overline{V}_R}{\partial \hat{\phi}^2}(\hat{\phi})\bigg|_{\hat{\phi}=0}, \tag{2.29a}$$

$$\lambda=\mathrm{Re}\,\frac{\partial^4 \overline{V}_R}{\partial \hat{\phi}^4}(\hat{\phi})\bigg|_{\hat{\phi}=0}, \tag{2.29b}$$

there being no field renormalization to one loop.[34,36] Evaluation of (2.29) leads to

$$m_R^{\,2}=m^2-\frac{\lambda m^2}{32\pi^2}\left[1+\frac{(N-1)}{3}\right]\left[\gamma-1+\ln\frac{|m^2|}{4\pi\mu^2}+\sum_{l=2}^{\infty}a_l\,\Gamma(l-1)m^{-2l}\right]+O(\lambda^2)\,, \tag{2.30a}$$

$$\xi_R=\xi+\frac{\lambda}{32\pi^2}\left[1+\frac{(N-1)}{3}\right]\left[\left(\frac{1}{6}-\xi\right)\left(\gamma+\ln\frac{|m^2|}{4\pi\mu^2}\right)-\xi'\right]+O(\lambda^2)\,, \tag{2.30b}$$

$$\lambda_R=\lambda-\frac{3\lambda^2}{32\pi^2}\left[1+\frac{(N-1)}{9}\right]\left[\gamma+\ln\frac{|m^2|}{4\pi\mu^2}-\sum_{l=1}^{\infty}a_l\,\Gamma(l)m^{-2l}\right]+O(\lambda^3)\,, \tag{2.30c}$$

and hence (2.27) is rewritten

$$\begin{aligned}\overline{V}_R(\hat{\phi})={}&\tfrac{1}{2}(m^2+\xi R)\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4\\&-\frac{1}{32\pi^2}\Bigg[\sum_{l=0}^{2}\frac{(-1)^{3-l}}{(2-l)!}a_l\left[M_{1R}^{\,2(2-l)}\ln\frac{M_{1R}^{\,2}}{|m^2|}+(N-1)M_{2R}^{\,2(2-l)}\ln\frac{M_{2R}^{\,2}}{|m^2|}\right]\\&+\frac{3}{4}\left[(M_{1R}^{\,2}-\tfrac{2}{3}m^2)^2+(N-1)(M_{2R}^{\,2}-\tfrac{2}{3}m^2)^2-\frac{Nm^4}{9}\right]-\frac{a_1}{2m^2}\left[M_{1R}^{\,4}+(N-1)M_{2R}^{\,4}-Nm^4\right]\\&+\sum_{l=2}^{\infty}\Gamma(l-1)\Bigg(a_l\left\{-\frac{(l+1)}{2}Nm^{2(2-l)}+l\,m^{2(1-l)}(M_{1R}^{\,2}+(N-1)M_{2R}^{\,2})+\tfrac{1}{2}(1-l)m^{-2l}[M_{1R}^{\,4}+(N-1)M_{2R}^{\,4}]\right\}\\&+a_{l+1}[M_{1R}^{\,2(1-l)}+(N-1)M_{2R}^{\,2(1-l)}]\Bigg)\Bigg]+\text{h.l.p.}\end{aligned} \tag{2.31}$$

This equation generalizes the results of Refs. 9, 34, and 36 to the case of a topologically trivial homogeneous curved manifold.

The appearance of $|m^2|$ rather than m^2 is, of course, associated with Re in (2.25). If $m^2<0$, which in Minkowski space indicates symmetry breaking in the tree approximation, we immediately see from either (2.31) or (2.27) that $\mathrm{Im}\overline{V}_R(0)\neq 0$ to one-loop order and hence the configuration with $\hat{\phi}=0$ is unstable against decay. In this respect our renormalization prescription differs slightly from that in Refs. 9 and 34 which instead involve $\ln m^2$ and therefore necessarily imply both that $\mathrm{Im}\overline{V}_R(0)=0$ and that Λ_B is complex if $m^2<0$.

Finally we note, in accordance with our previous remarks, that the renormalization program (2.23)–(2.25) is equally valid in the case of finite-temperature field theory on M, leading to the renormalized effective potential

$$V_R(\hat{\phi})\equiv V(\hat{\phi})-\mathrm{Re}\overline{V}_{\mathrm{DIV}}(0)\,. \tag{2.32}$$

III. SYMMETRY RESTORATION IN STATIC HOMOGENOUS CLIFFORD-KLEIN ROBERTSON-WALKER UNIVERSES

Having detailed how the renormalization prescription operates at the one-loop level for field theory at zero temperature on $\bar{M}$, we can now investigate what interesting physics arises when global features, such as a finite temperature and a nontrivial topology are introduced. The particular models we choose to study are the static homogeneous CKRW universes, described in the introduction, which may locally be specified by the line element[37]

$$ds^2 = dt^2 - \rho(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \qquad (3.1)$$

where $\rho(r) = (1 - kr^2a^{-2})^{-1}$ for $k = 0, \pm 1$. The t = constant hypersurfaces $M_3 = \tilde{M}_3/\Gamma$ are locally isometric to $\tilde{M}_3$ where $\tilde{M}_3$ labels the covering manifold $R^3(k=0)$, $S^3(k=1)$, or $H^3(k=-1)$ and Γ is a discrete group of isometries of $\tilde{M}_3$, points on $\tilde{M}_3$ equivalent under the action of Γ being identified to give the multiply connected hypersurface M_3.[15]

The restriction that M be static is necessary in order that we have a globally valid Fock-space construction and hence a concept of thermal equilibrium.[27] The effective potential is then just the free energy density,[27] $f(\hat{\phi}) = V_R(\hat{\phi})$, whose minimization is embodied in the variational equation $dV_R(\hat{\phi})/d\hat{\phi} = 0$. The solution to (2.15) appropriate at finite temperature can be written as an image sum of zero-temperature solutions,[38, 27] K^∞,

$$\begin{aligned} K(t,\vec{x};t,\vec{x}';\tau) &= \sum_{m=-\infty}^{\infty} K^\infty(t,\vec{x};t-im\beta,\vec{x}';\tau) \\ &= \sum_{m=-\infty}^{\infty} (-4\pi i\tau)^{-1/2}\exp\left(\frac{im^2\beta^2}{4\tau}\right)K_3(\vec{x},\vec{x}';\tau)\,, \end{aligned} \qquad (3.2)$$

where

$$\left(i\frac{\partial}{\partial\tau} + \Delta_2 - \xi R\right)K_3(\vec{x},\vec{x}';\tau) = i\delta(\tau)\delta(\vec{x},\vec{x}')\,, \qquad (3.3)$$

supplemented by the boundary conditions appropriate to the spatial section M_3, determines the spatial quantum-mechanical propagator $K_3(\vec{x},\vec{x}';\tau)$. Δ_2 here is the Laplace-Beltrami operator on the spatial section.

K_3 may also be expressed as an image sum,[39]

$$K_3(\vec{x},\vec{x}';\tau) = \sum_{\gamma\in\Gamma} a(\gamma)\tilde{K}_3(\vec{x},\vec{x}'\gamma;\tau)\,, \qquad (3.4)$$

of covering manifold propagators $\tilde{K}_3$, where the group elements γ are chosen to act on the right and $a(\gamma)$ is a unitary one-dimensional representation of Γ. In the cases $k = 0, -1$ the covering manifold is noncompact, and $R \times \tilde{M}_3$ serves as our topologically trivial manifold $\bar{M}$ locally isometric to M. In the case $k=1$ the covering manifold is compact, a feature which $\tilde{K}_3$ exhibits via a sum over contributions from indirect geodesics which lap S^3 an arbitrary number of times [see (3.34)]. Retaining only the direct geodesic contribution gives the quantum-mechanical propagator on a manifold which is locally isometric to S^3 but devoid of global features: this we identify with the $\bar{M}$ spatial section.

In all cases we can therefore write

$$K(t,\vec{x};t,\vec{x};\tau) = \sum_{m=-\infty}^{\infty}\sum_{\gamma\in\Gamma}\sum_{\gamma\in\tilde{\Gamma}} \bar{K}(t,\vec{x};t-im\beta,\vec{x}\gamma\tilde{\gamma};\tau)\,, \qquad (3.5)$$

where $\tilde{\Gamma}$ includes elements besides the identity only in the case of $k=1$. The trivial representation, $a(\gamma)=1$, has been chosen here since only this is consistent with our assumption $\hat{\phi}^i$ = constant. Incidentally, from theorem 2.7.5 of Wolf,[15] when M_3 is homogenous every element of Γ is a Clifford translation of $\tilde{M}_3$ and hence (3.5) is $\vec{x}$ independent.

A. $k=0$ in the one-loop approximation

From theorem 2.7.1 of Wolf[15] the only homogeneous M_3 hypersurfaces are $M_3 = R^{4-p} \times T^{p-1}$, where T^{p-1} is the $(p-1)$-torus $S^1 \times S^1 \times \cdots \times S^1$ (p-1 factors) and p takes integer values from 1 to 4 inclusive. The corresponding finite-temperature quantum-mechanical propagator on M readily follows from (3.2) and (3.4) with $\Gamma = Z_\infty \times Z_\infty \times \cdots \times Z_\infty$ ($p-1$ factors) by inserting image coordinates[17, 19] in the standard Minkowski-space propagator[28]

$$\bar{K}(x,x';\tau) = (-4\pi i\tau)^{-1/2}(4\pi i\tau)^{-3/2} \times \exp\left\{-\frac{i}{4\tau}\left[(t-t')^2 - (\vec{x}-\vec{x}')^2\right]\right\}. \qquad (3.6)$$

In terms of the p-dimensional θ function defined in the Appendix, it takes the concise form

$$K(x,x;\tau) = (-4\pi i\tau)^{-1/2}(4\pi i\tau)^{-3/2}\theta_{(p)}\left|{}^0_0\right|\left((4\pi\tau)^{-1}, A\right)\,, \qquad (3.7)$$

where $A = \mathrm{diag}(4\pi^2a_1^2, 4\pi^2a_2^2, \ldots, 4\pi^2a_{p-1}^2, \beta^2)$ is a diagonal $p \times p$ matrix and the a_i are the tori radii. Clearly $A = \infty$ returns us to zero-temperature field theory in Minkowski space ($\bar{M}$).

The one-loop effective potential follows from (2.14) and (3.7). Continuing to 2ω dimensions and setting $N=1$ momentarily for convenience we find

$$V_1(\hat{\phi}; 2\omega; N=1) = \frac{i}{2}\int_0^\infty \frac{d\tau}{\tau} e^{-iM_1{}^2\tau}(-4\pi i\tau)^{-1/2}(4\pi i\tau)^{1/2-\omega}\theta_{(p)}\left|{0 \atop 0}\right|((4\pi\tau)^{-1}, A)$$

$$= \frac{i}{2(\det A)^{1/2}}\int_0^\infty \frac{d\tau}{\tau} e^{-iM_1{}^2\tau}(-4\pi i\tau)^{-1/2}(4\pi i\tau)^{p/2+1/2-\omega}\theta_{(p)}\left|{0 \atop 0}\right|(-4\pi\tau, A^{-1})$$

$$= -\frac{(4\pi)^{p/2-\omega}}{2(\det A)^{1/2}}\Big[M_1{}^{2(\omega-p/2)}\Gamma(p/2-\omega) + \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(4\pi^2)^{\omega-p/2-r} M_1{}^{2r}\Gamma(p/2+r-\omega)Z_p\left|{0 \atop 0}\right|(p+2r-2\omega, A^{-1})\Big], \quad (3.8)$$

where (A2) has been used in passing to (3.8), an expansion valid when $M_1{}^2$ is small compared to a typical element of A^{-1}. Adding $N-1$ identical expressions with $M_2{}^2$ replacing $M_1{}^2$, making use of the properties of the Epstein ζ function $Z_p\left|{0 \atop 0}\right|$ described in the Appendix, and renormalizing the theory according to (2.31) and (2.32) there results in the limit $\omega\to 2$, p *even*:

$$V_R(\hat{\phi}) = \tfrac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{1}{64\pi^2}\Big\{[M_{1R}{}^4 + (N-1)M_{2R}{}^4]\Big[2\psi(1) - \ln\frac{|m^2|}{4} - 2Z_p'\left|{0 \atop 0}\right|(0, A)\Big] + 2m^2[M_{1R}{}^2 + (N-1)M_{2R}{}^2] - \frac{N}{2}m^4\Big\} - \frac{1}{2(\det A)^{1/2}}$$
$$\times\Bigg(\frac{(-1)^{3-p/2}}{(2-p/2)!}(4\pi)^{p/2-2}\Big\{[M_{1R}{}^{2(2-p/2)} + (N-1)M_{2R}{}^{2(2-p/2)}][\psi(1) - \psi(3-p/2)] + M_{1R}{}^{2(2-p/2)}\ln\frac{M_{1R}{}^2}{4\pi^2} + (N-1)M_{2R}{}^{2(2-p/2)}\ln\frac{M_{2R}{}^2}{4\pi^2}\Big\}$$
$$+ \sum_{r=0}^{2-p/2}(-1)^{p/2}\frac{2^{1-2r}\pi^{2-2r-p/2}}{r!\,(2-p/2-r)!}[M_{1R}{}^{2r} + (N-1)M_{2R}{}^{2r}]Z_p'\left|{0 \atop 0}\right|(p+2r-4, A^{-1})$$
$$+ \Big(\sum_{r=3-p/2}^{1} + \sum_{r=3}^{\infty}\Big)\frac{(-1)^r}{r!}2^{-2r}\pi^{2-2r-p/2}[M_{1R}{}^{2r} + (N-1)M_{2R}{}^{2r}]\Gamma(p/2+r-2)Z_p\left|{0 \atop 0}\right|(p+2r-4, A^{-1})\Bigg) + \text{h.l.p.} \quad (3.9a)$$

p *odd*:

$$V_R(\hat{\phi}) = \tfrac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{1}{64\pi^2}\Big\{[M_{1R}{}^4 + (N-1)M_{2R}{}^4]\Big[2\psi(1) - \ln\frac{|m^2|}{4} - 2Z_p'\left|{0 \atop 0}\right|(0, A)\Big] + 2m^2[M_{1R}{}^2 + (N-1)M_{2R}{}^2] - \frac{N}{2}m^4\Big\} - \frac{1}{2(\det A)^{1/2}}$$
$$\times\Big\{(4\pi)^{p/2-2}\Gamma(p/2-2)[M_{1R}{}^{2(2-p/2)} + (N-1)M_{2R}{}^{2(2-p/2)}] + \Big(\sum_{r=0}^{1} + \sum_{r=3}^{\infty}\Big)\frac{(-1)^r}{r!}2^{-2r}\pi^{2-2r-p/2}[M_{1R}{}^{2r} + (N-1)M_{2R}{}^{2r}]$$
$$\times\Gamma(p/2+r-2)Z_p\left|{0 \atop 0}\right|(p+2r-4, A^{-1})\Big\} + \text{h.l.p.} \quad (3.9b)$$

In particular for $p=1$, which describes finite-temperature field theory in Minkowski space, (3.9b) yields the high-temperature expansion

$$V_R(\hat\phi)=\tfrac{1}{2}m^2\hat\phi^2+\frac{\lambda}{4!}\hat\phi^4-\frac{N\pi^2}{90\beta^4}+\frac{1}{24\beta^2}[M_{1R}^{\ 2}+(N-1)M_{2R}^{\ 2}]-\frac{1}{12\pi\beta}[M_{1R}^{\ 3}+(N-1)M_{2R}^{\ 3}]$$
$$+\frac{1}{64\pi^2}\left\{[M_{1R}^{\ 4}+(N-1)M_{2R}^{\ 4}]\left[2\psi(1)-\ln\frac{|m^2|\beta^2}{16\pi^2}\right]+2m^2[M_{1R}^{\ 2}+(N-1)M_{2R}^{\ 2}]-\frac{N}{2}m^4\right\}$$
$$+\sum_{r=3}^{\infty}\frac{(-1)^{r+1}}{r!}2^{-2r}\pi^{3/2-2r}\Gamma(r-3/2)\zeta(2r-3)[M_{1R}^{\ 2r}+(N-1)M_{2R}^{\ 2r}]\beta^{2r-4}+\text{h.l.p.}\,, \qquad (3.10)$$

the first few terms of which Dolan and Jackiw[9] obtained by a rather more arduous route.

Our intention is now to use (3.9) in (2.17) to determine whether a phase transition occurs on alteration of the parameters β, a_i, and, if so, the values of the critical parameters. Here a failing of the one-loop approximation becomes manifest; if $m^2<0$, $\text{Im}V_R(0)\neq 0$ to one-loop order, whatever the elements of A. This is apparent from (3.9): for p even, the $\ln M_i^{\ 2}$ terms are responsible, while for p odd $M_i^{\ 2(2-p/2)}$ is the culprit.

Dolan and Jackiw[9] also noticed this nonzero imaginary part in the case of $p=1$ and circumvented the difficulty by making the further approximation that β be small, allowing the neglect of terms in (3.10) of $O(\beta^{-1})$ and higher. In this regime the equality (2.17) yields the usual relation[9,10]

$$\beta_c^{\ 2}=-\frac{\lambda}{24m^2}\left[1+\frac{(N-1)}{3}\right] \qquad (3.11)$$

for the critical temperature.

The symmetry of $Z_p\left|{}^0_0\right|(s,A)$ under permutations of the A elements allows us to view the $p=1$ version of (3.9b) as describing zero-temperature field theory in Minkowski space periodically identified in one direction ($M_3=R^2\times S^1$) with period $2\pi a$. With (2.17) we then find that a symmetry initially broken in the Minkowski space limit ($a=\infty$) will, as a is reduced, become restored at a critical value

$$a_c^{\ 2}=-\frac{\lambda}{96\pi^2m^2}\left[1+\frac{(N-1)}{3}\right], \qquad (3.12)$$

where once again we have assumed a is small to avoid troublesome imaginary parts. This result, which demonstrates the influence of topology on the occurence of a phase transition, has also recently been obtained by Denardo and Spallucci.[24]

Unfortunately, for $p\neq 1$ one cannot use a similar small-A-type argument to discard the unwanted imaginary part. The failing of the one-loop approximation in the vicinity of the phase transition is there to stay and another approximation scheme for calculating $V_R(\hat\phi)$ is required.

B. The large-N approximation

The large-N approximation,[40-42] familiar from the realm of statistical mechanics, enables one to sum the loop expansion for $V(\hat\phi)$ to all orders, retaining only the bubble graphs dominant for large N in each order. A detailed account is given in the elegant paper by Schnitzer[42] and is easily generalized to the present context of a homogenous curved manifold. Paralleling the steps which lead to Eq. (3.15) of Schnitzer one finds the unrenormalized equation in 2ω dimensions

$$\frac{\partial V}{\partial\hat\phi_i}=\frac{\partial V_0}{\partial\hat\phi_i}-\frac{iN\lambda_B}{6}\hat\phi_i\tilde B_{2\omega}\left(m_B^{\ 2}+\xi_BR+\frac{\lambda_B}{6}\hat\phi^2\right), \qquad (3.13)$$

where

$$\tilde B_{2\omega}(m_B^{\ 2}+\xi_BR)\equiv\tilde G(x,x;m_B^{\ 2}+\xi_BR) \qquad (3.14)$$

and $\tilde G(x,x';m_B^{\ 2}+\xi_BR)$ satisfies, in 2ω dimensions,

$$\left[\Box+\xi_BR+m_B^{\ 2}-\frac{iN\lambda_B}{6}\tilde B_{2\omega}(m_B^{\ 2}+\xi_BR)\right]\tilde G(x,x';m_B^{\ 2}+\xi_BR)=\delta(x,x'), \qquad (3.15)$$

together with the boundary conditions appropriate to the finite-temperature[27] or nontrivial topology.[17-19] Defining

$$B_{2\omega}(m_B^{\ 2}+\xi_BR)\equiv G(x,x;m_B^{\ 2}+\xi_BR), \qquad (3.16)$$

(2.12) and (3.15) yield

$$\tilde B_{2\omega}(m_B^{\ 2}+\xi_BR)=B_{2\omega}\left[m_B^{\ 2}+\xi_BR-\frac{iN\lambda_B}{6}\tilde B_{2\omega}(m_B^{\ 2}+\xi_BR)\right]. \qquad (3.17)$$

Renormalization of (3.13) is carried out most easily when $\text{Im}\overline{V}(0)=0$, i.e., the disordered phase exists at zero temperature on $\overline{M}$. We then introduce renormalized mass and coupling constants through[42]

$$(m^2+\xi R)\delta_{ij}=\frac{\partial^2\overline{V}(\hat\phi)}{\partial\hat\phi_i\partial\hat\phi_j}\bigg|_{\hat\phi=0}, \tag{3.18a}$$

$$\frac{\lambda}{3}(\delta_{ij}\delta_{kl}+\delta_{jk}\delta_{il}+\delta_{ik}\delta_{jl})=\frac{\partial^4\overline{V}(\hat\phi)}{\partial\hat\phi_i\partial\hat\phi_j\partial\hat\phi_k\partial\hat\phi_l}\bigg|_{\hat\phi=0}. \tag{3.18b}$$

Combined with (3.13) these equations yield

$$m^2+\xi R=m_B{}^2+\xi_B R-\frac{iN\lambda_B}{6}\overline{B}_{2\omega}(m^2+\xi R), \tag{3.19a}$$

$$\lambda=\lambda_B\left[1-\frac{iN\lambda_B}{6}\frac{d}{dz}\overline{\overline{B}}_{2\omega}(m_B{}^2+\xi_B R+z)\bigg|_{z=0}\right], \tag{3.19b}$$

where, as usual, a bar indicates boundary conditions appropriate to zero temperature and trivial topology.

Equation (3.13) is now written in terms of these renormalized parameters as

$$\frac{\partial V}{\partial\hat\phi_i}=\hat\phi_i\left[m^2+\xi R+\frac{\lambda}{6}\hat\phi^2-\frac{iN\lambda}{6}B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2)\right], \tag{3.20}$$

where

$$\lambda B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2)\equiv\lambda_B\left[\overline{B}_{2\omega}(m_B{}^2+\xi_B R+\tfrac{1}{6}\lambda_B\hat\phi^2)-\overline{B}_{2\omega}(m_B{}^2+\xi_B R)-\frac{\lambda_B}{6}\hat\phi^2\frac{d}{dz}\overline{B}_{2\omega}(m_B{}^2+\xi_B R+z)\bigg|_{z=0}\right]. \tag{3.21}$$

Using (3.17) and (3.19), (3.21) may also be re-expressed entirely in terms of renormalized parameters as

$$\left[1-\frac{iN\lambda}{6}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}\right]B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2).$$

$$=B_{2\omega}(\mu^2+\xi R)-\overline{B}_{2\omega}(m^2+\xi R)-\frac{\lambda\hat\phi^2}{6}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}, \tag{3.22}$$

where

$$\mu^2\equiv m^2+\frac{\lambda}{6}\hat\phi^2-\frac{iN\lambda}{6}B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2). \tag{3.23}$$

To generalize to the case of $m^2<0$ we continue (3.22) in m^2, being careful to take only the imaginary parts of the barred quantities since we know from the unrenormalized equation (3.13) that $B^R_{2\omega}(\mu^2+\xi R)$ must be pure imaginary for $\mu^2\geqslant 0$. The following equation for $B^R_{2\omega}$ results (compare Ref. 42)

$$\left\{1-\mathrm{Re}\left[\frac{iN\lambda}{6}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}\right]\right\}B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2)$$

$$=B_{2\omega}(\mu^2+\xi R)-i\,\mathrm{Im}\overline{B}_{2\omega}(m^2+\xi R)-\frac{i\lambda\hat\phi^2}{6}\mathrm{Im}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}. \tag{3.24}$$

We must now ascertain that (3.24) does indeed have a finite solution $B^R_{2\omega_0}(m^2+\xi R;\frac{1}{6}\lambda\hat\phi^2)$ in the limit $\omega\to\omega_0$, where for generality we allow the spacetime dimensionality $2\omega_0$ to be an arbitrary integer. For odd dimensions no infinities arise, so we need only study the case of integer ω_0.

We first separate out the local from the global contributions in[17-19, 27]

$$B_{2\omega}(\mu^2+\xi R)=\overline{B}_{2\omega}(\mu^2+\xi R)+B'_{2\omega}(\mu^2+\xi R). \tag{3.25}$$

The ultraviolet divergences which appear as $\omega\to\omega_0$ then reside solely in the first term on the right-hand-side: $B'_{2\omega}(\mu^2+\xi R)$ will generally have a finite limit as $\omega\to\omega_0$, although it may exhibit infrared divergences for particular choices of μ^2. (3.24) is then rewritten

$$B^R_{2\omega}(m^2+\xi R;\tfrac{1}{6}\lambda\hat\phi^2)=B'_{2\omega}(\mu^2+\xi R)+C_{2\omega}(\mu^2;m^2), \tag{3.26}$$

where

$$C_{2\omega}(\mu^2;m^2) \equiv \overline{B}_{2\omega}(\mu^2+\xi R) + \mathrm{Re}\left[\frac{iN\lambda}{6}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}\right] B^R_{2\omega}(m^2+\xi R;\tfrac{1}{8}\lambda\hat{\phi}^2)$$
$$-i\,\mathrm{Im}\,\overline{B}_{2\omega}(m^2+\xi R) - \frac{i\lambda\hat{\phi}^2}{6}\mathrm{Im}\frac{d}{dz}\overline{B}_{2\omega}(m^2+\xi R+z)\bigg|_{z=0}. \tag{3.27}$$

Using (2.20) and[28]

$$\overline{B}_{2\omega}(m^2+\xi R) = i\int_0^\infty d\tau\, e^{-im^2\tau}\overline{K}(x,x;\tau) \tag{3.28}$$

we find

$$\lim_{\omega\to\omega_0} C_{2\omega}(\mu^2;m^2) = i(4\pi)^{-\omega_0}\Bigg(\sum_{l=0}^{\omega_0-1}\sum_{r=2}^{\omega_0-l-1} a_l\frac{(-1)^{\omega_0-l}}{r!\,(\omega_0-l-1-r)!}m^{2(\omega_0-l-1-r)}(\mu^2-m^2)^r\left(\frac{1}{\omega-\omega_0}-\psi(\omega_0-l)\right)$$
$$+\sum_{l=0}^{\omega_0-1} a_l\frac{(-1)^{\omega_0-l}}{(\omega_0-l-1)!}\left\{\mu^{2(\omega_0-l-1)}\ln\frac{\mu^2}{4\pi}+(m^2-\mu^2)m^{2(\omega_0-l-2)}\left[1+(\omega_0-l-1)\ln\frac{|m^2|}{4\pi}\right]\right.$$
$$\left.-m^{2(\omega_0-l-1)}\ln\frac{|m^2|}{4\pi}\right\}$$
$$+\sum_{l=\omega_0}^{\infty} a_l\Gamma(l+1-\omega_0)[\mu^{2(\omega_0-l-1)}+(l+1-\omega_0)(\mu^2-m^2)m^{2(\omega_0-l-2)}-m^{2(\omega_0-l-1)}]\Bigg), \tag{3.29}$$

which is finite provided $\omega_0 \leq 2$, where the theory is renormalizable. By (3.26) the solution $B^R_{2\omega_0}(m^2+\xi R;\frac{1}{6}\lambda\hat{\phi}^2)$ of (3.24) is then finite, provided we do not encounter potential infrared divergences in $B'_{2\omega_0}(\mu^2+\xi R)$.

As (3.20) shows, however, a broken-symmetry state necessarily requires $\mu^2+\xi R=0$ and hence invites such divergences. If the system exists in the disordered phase then $\mu^2+\xi R \geq 0$ and infrared effects only become important as one attempts to make the transition to the ordered phase. As Sec. III C will show, the appearance of such divergences is an indication that the transition cannot be made and the system always exists in the disordered phase.

C. $k=0$ revisited

Generalizing (3.7) to the case of finite-temperature field theory on $R^{2\omega+1-p}\times T^{p-1}$ and making use of (A2) gives

$$B_{2\omega}(\mu^2) = i\int_0^\infty d\tau\,(-4\pi i\tau)^{-1/2}(4\pi i\tau)^{1/2-\omega}e^{-i\mu^2\tau}\theta_{(p)}\left|{}^0_0\right|[(4\pi\tau)^{-1},A]$$
$$= i(4\pi)^{p/2-\omega}(\det A)^{-1/2}\left[\mu^{2(\omega-p/2-1)}\Gamma(p/2+1-\omega)+\sum_{r=0}^\infty\frac{(-1)^r}{r!}\mu^{2r}(4\pi^2)^{\omega-p/2-r-1}\right.$$
$$\left.\times\Gamma(p/2-\omega+r+1)Z_p\left|{}^0_0\right|(p-2\omega+2r+2,A^{-1})\right], \tag{3.30}$$

where $A=\mathrm{diag}(4\pi^2a_1{}^2, 4\pi^2a_2{}^2,\ldots,4\pi^2a_{p-1}{}^2,\beta^2)$ is again a diagonal $p\times p$ matrix with the a_i the tori radii. Substituting this result in (3.24), six possible behaviors are identifiable in the limit $\omega\to\omega_0$:

(i) $\omega_0=$ integer, $l=0$:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2) = i(\det A)^{-1/2}\left[\mu^{-2}+\left(\sum_{r=0}^{\omega_0-2}+\sum_{r=\omega_0}^{\infty}\right)(-1)^r\mu^{2r}(4\pi^2)^{-r-1}Z_p\left|{}^0_0\right|(2r+2,A^{-1})\right]$$
$$+\frac{i(-1)^{\omega_0}(4\pi)^{-\omega_0}}{(\omega_0-1)!}\left\{\mu^{2(\omega_0-1)}\left[\psi(\omega_0)+\psi(1)-\ln\frac{|m^2|}{4}-2Z'_p\left|{}^0_0\right|(0,A)\right]+(m^2-\mu^2)m^{2(\omega_0-2)}\right\}. \tag{3.31a}$$

(ii) ω_0 = integer, l = integer > 0:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)=i(\det A)^{-1/2}(4\pi)^{-l}\left\{\frac{(-1)^{l-1}}{(l-1)!}\mu^{2(l-1)}\left[\psi(l)-\psi(1)+2Z_p'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A^{-1})-\ln\frac{\mu^2}{4\pi^2}\right]\right.$$
$$\left.+\left(\sum_{r=l}^{\omega_0-2}+\sum_{r=\omega_0}^{\infty}\right)\frac{(-1)^r}{r!}\mu^{2r}(4\pi^2)^{l-r-1}\Gamma(r+1-l)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r+2-2l,A^{-1})\right\}$$
$$+i\pi^{-\omega_0}\sum_{r=0}^{l-2}\frac{(-1)^r}{r!}2^{-2r-2}\mu^{2r}\Gamma(p/2+l-r-1)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(p+2l-2r-2,A)$$
$$+\frac{i(-1)^{\omega_0}(4\pi)^{-\omega_0}}{(\omega_0-1)!}\left\{\mu^{2(\omega_0-1)}\left[\psi(\omega_0)+\psi(1)-\ln\frac{|m^2|}{4}-2Z_p'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A)\right]+(m^2-\mu^2)m^{2(\omega_0-2)}\right\}. \tag{3.31b}$$

(iii) ω_0 = integer, l = half integer:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)=i(\det A)^{-1/2}(4\pi)^{-l}\left[\mu^{2(l-1)}\Gamma(1-l)+\left(\sum_{r=0}^{\omega_0-2}+\sum_{r=\omega_0}^{\infty}\right)\frac{(-1)^r}{r!}\mu^{2r}(4\pi^2)^{l-r-1}\right.$$
$$\left.\times\Gamma(r+1-l)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r+2-2l,A^{-1})\right]$$
$$+\frac{i(-1)^{\omega_0}}{(\omega_0-1)!}(4\pi)^{-\omega_0}\left\{\mu^{2(\omega_0-1)}\left[\psi(\omega_0)+\psi(1)-\ln\frac{|m^2|}{4}-2Z_p'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A)\right]+(m^2-\mu^2)m^{2(\omega_0-2)}\right\}. \tag{3.31c}$$

(iv) ω_0 = half integer, $l=0$:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)=i(\det A)^{-1/2}\left[\mu^{-2}+\sum_{r=0}^{\infty}(-1)^r\mu^{2r}(4\pi^2)^{-r-1}Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r+2,A^{-1})\right]. \tag{3.31d}$$

(v) ω_0 = half integer, l = integer > 0:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)=i(\det A)^{-1/2}(4\pi)^{-l}\left\{\frac{(-1)^{l}}{(l-1)!}\mu^{2(l-1)}\left[\psi(1)-\psi(l)+\ln\frac{\mu^2}{4\pi^2}-2Z_p'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A^{-1})\right]\right.$$
$$\left.+\sum_{r=l}^{\infty}\frac{(-1)^r}{r!}\mu^{2r}(4\pi^2)^{l-r-1}\Gamma(r+1-l)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r+2-2l,A^{-1})\right\}$$
$$+i\pi^{-\omega_0}\sum_{r=0}^{l-2}\frac{(-1)^r}{r!}\mu^{2r}2^{-2r-2}\Gamma(p/2+l-r-1)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(p+2l-2r-2,A). \tag{3.31e}$$

(vi) ω_0 = half integer, l = half integer:

$$B^R_{2\omega_0}(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)=i(\det A)^{-1/2}(4\pi)^{-l}\left[\mu^{2(l-1)}\Gamma(1-l)+\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\mu^{2r}(4\pi^2)^{l-r-1}\Gamma(r+1-l)Z_p\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r+2-2l,A^{-1})\right], \tag{3.31f}$$

where $2l\equiv 2\omega_0-p$ is the number of noncompact dimensions, and we have assumed $2\omega_0\leq 4$ for the theory to be renormalizable.

Now, as remarked in Sec. III B, an ordered phase can only exist within the large-N approximation if $B^R_{2\omega_0}(m^2;\frac{1}{6}\lambda\hat{\phi}^2)$ has a finite solution for $\mu^2=0$. A study of Eqs. (3.31) shows this to be possible only in the case $\omega_0=2$, $2l=3$, when

$$B^R_4(m^2;\tfrac{1}{6}\lambda\hat{\phi}^2)\big|_{\mu^2=0}=-\frac{i}{2}(\det A)^{-1/2}Z_1\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(-1,A^{-1})+\frac{im^2}{(4\pi)^2}. \tag{3.32}$$

A well-behaved solution $B^R_{2\omega_0}(m^2;0)$ corresponding to the disordered phase, where $\hat{\phi}=0$ and $\mu^2>0$, can be found in all cases. As one attempts to make the transition to the ordered phase by letting $\mu^2\to 0$, however, one generally encounters infrared divergences. These are most severe when $l=0$, corresponding in the Euclidean regime to a manifold which is compact without boundary, and in all cases can only be removed by setting $\det A=\infty$, i.e., by making noncompact one or more of the compact dimensions.

The appearance of such divergences indicates that, within the large-N approximation, the system can only exist in the disordered phase. As (3.32) shows, at finite temperature in Minkowski space the ordered phase can exist until the temperature rises above a critical value given by

$$(12\beta_c^{\,2})^{-1}+(4\pi)^{-2}m^2=-(N\lambda)^{-1}6m^2, \tag{3.33}$$

which agrees with (3.11) in the small-β_c limit, after which the effect of the (temperature dependent) long-wavelength modes becomes dominant, causing the transition to the disordered phase. If we introduce a further periodicity, so that we are considering finite-temperature field theory on $R^3 \times S^1$, we come under category (ii) above, which exhibits a logarithmic divergence as $\mu^2 \to 0$. We interpret this as meaning that our alteration of the long-wavelength structure has been so dramatic that for *any* finite values of the periodicity parameters the contribution of the long-wavelength modes is sufficient to restore the symmetry. This interpretation may be extended to all the cases treated in Eqs. (3.31) with the conclusion that, within the large-N approximation, an ordered phase cannot exist in four dimensions with two or more periodicities, nor in less than four dimensions with one or more periodicities.

It should be emphasized that the transition to the disordered phase for *any* finite values of the periodicity parameters is a simplification of taking the large-N limit. Had we retained terms of $O(N^{-1})$ we would expect to find the location of the phase transition in terms of a specific set of finite critical global parameters.

D. $k=1$

By corollary 2.7.2 of Wolf[15] the only homogeneous M_3 hypersurfaces are (i) $M_3=S^3$, (ii) $M_3=S^3/Z_n$ $(n\geq 2)$, (iii) $M_3=S^3/D_n^*$ $(n\geq 2)$, (iv) $M_3=S^3/T^*$, (v) $M_3=S^3/0^*$, (vi) $M_3=S^3/I^*$. The discrete subgroups are the cyclic group of order n, Z_n, the binary dihedral group of order $2n$, D_n^*, the binary tetrahedral group, T^*, the binary octahedral group, 0^*, and the binary icosahedral group I^*. Further information on these spaces can be found in Refs. 13, 15, and 19.

We begin with the case $M_3=S^3$. Then $\Gamma=1$, the identity, and for conformal coupling, $\xi=\frac{1}{6}$, the spatial quantum-mechanical propagator is that obtained by Dowker[43]

$$\tilde{K}_3(\vec{x},\vec{x}';\tau)=(4\pi i\tau)^{-3/2}[a\sin(s/a)]^{-1}\sum_{n=-\infty}^{\infty}(s+2\pi na)\exp\left[\frac{i}{4\tau}(s+2\pi na)^2\right], \qquad (3.34)$$

where a is the radius of S^3 and $s(\vec{x},\vec{x}')\equiv a\theta(\vec{x},\vec{x}')$ is the geodesic distance on S^3 between $\vec{x}$ and $\vec{x}'$. In any of the cases $M_3=S^3/\Gamma$, the image sum (3.4) combined with (3.2) then yields for the coincidence limit of the finite-temperature quantum-mechanical propagator

$$K(x,x;\tau)=(-4\pi i\tau)^{-1/2}(4\pi i\tau)^{-3/2}\left[\left(1+a\frac{\partial}{\partial a}\right)\theta_{(2)}\left|{0 \atop 0}\right|((4\pi\tau)^{-1},A)-\frac{2i\tau}{a^2}\sum_{\gamma\neq 1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}\theta_{(2)}\left|{\theta_\gamma/2\pi \atop 0}\;{0 \atop 0}\right|((4\pi\tau)^{-1},A)\right], \qquad (3.35)$$

where $\theta_\gamma\equiv\theta(\gamma,1)$ is the angular separation on S^3 between the preimage $\gamma\in\Gamma$ and the identity element 1. Here $A=4\pi^2a^2\,\mathrm{diag}(1,x^2)$ and $x\equiv\beta(2\pi a)^{-1}$.

It is now a straightforward matter to substitute (3.35) in (2.14) to obtain the effective potential in one-loop order. Renormalization is carried out according to (2.32), the spatial quantum-mechanical propagator $\overline{K}_3(\vec{x},\vec{x}';\tau)$ which governs field theory on the hypersurface $\overline{M}_3$, locally isometric to S^3 but devoid of global features, being conveniently obtained by retaining only the $n=0$ direct geodesic term in (3.34).[43,44] Omitting the details of a calculation which parallels that leading to (3.9) and makes use of the functional properties listed in the Appendix, we find, for zero temperature,

$$\begin{aligned}V_R(\hat{\phi})=&\frac{1}{2}\left(m^2+\frac{R}{6}\right)\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4+\frac{N}{16\pi^2a^4}\left[\tfrac{1}{30}-2\sum_{\gamma\neq 1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_1\left|{0 \atop -\theta_\gamma/2\pi}\right|(-1,1)\right]\\&+\frac{1}{32\pi^2a^2}[M_{1R}{}^2+(N-1)M_{2R}{}^2]\left[-\tfrac{1}{3}-2\sum_{\gamma\neq 1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_1\left|{0 \atop -\theta_\gamma/2\pi}\right|(1,1)\right]\\&-\frac{3}{128\pi^2}\left[(M_{1R}{}^2-\tfrac{2}{3}m^2)^2+(N-1)(M_{2R}{}^2-\tfrac{2}{3}m^2)^2-\frac{Nm^4}{9}\right]\\&+\frac{1}{64\pi^2}[M_{1R}{}^4+(N-1)M_{2R}{}^4]\left[2\psi(1)-\tfrac{1}{2}-\ln\frac{|m^2|a^2}{4}+\sum_{\gamma\neq 1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_1\left|{0 \atop -\theta_\gamma/2\pi}\right|(3,1)\right]\\&-(4\pi)^{-5/2}\sum_{r=3}^{\infty}\frac{(-1)^r}{r!}[M_{1R}{}^{2r}+(N-1)M_{2R}{}^{2r}]\Gamma(r-\tfrac{1}{2})a^{2r-4}\left[4\zeta(2r-3)-2\sum_{\gamma\neq 1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_1\left|{0 \atop -\theta_\gamma/2\pi}\right|(2r-1,1)\right]\end{aligned} \qquad (3.36a)$$

and, for finite temperature,

$$\begin{aligned}V_R(\hat\phi)=&\frac{1}{2}\left(m^2+\frac{R}{6}\right)\hat\phi^2+\frac{\lambda}{4!}\hat\phi^4-\frac{N}{4\pi^2}\left[2\frac{\partial}{\partial a}\left(aZ_2\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(4,A)\right)-a^{-3}\beta^{-1}\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_2'\left|\begin{smallmatrix}0 & 0\\-\theta_\gamma/2\pi & 0\end{smallmatrix}\right|(0,A^{-1})\right]\\
&+\frac{[M_{1R}^2+(N-1)M_{2R}^2]}{8\pi^2 a\beta}\left[a\frac{\partial}{\partial a}Z_2'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A^{-1})-(4\pi^2)^{-1}a^{-2}\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_2\left|\begin{smallmatrix}0 & 0\\-\theta_\gamma/2\pi & 0\end{smallmatrix}\right|(2,A^{-1})\right]\\
&-\frac{3}{128\pi^2}\left[(M_{1R}^2-\tfrac{2}{3}m^2)^2+(N-1)(M_{2R}^2-\tfrac{2}{3}m^2)^2-\frac{Nm^4}{9}\right]+\frac{1}{64\pi^2}[M_{1R}^4+(N-1)M_{2R}^4]\\
&\times\left[2\psi(1)+\tfrac{3}{2}-\ln\frac{|m^2|}{4}-2\frac{\partial}{\partial a}\left(aZ_2'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A)\right)+(4\pi^4)^{-1}\beta^{-1}a^{-3}\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_2\left|\begin{smallmatrix}0 & 0\\-\theta_\gamma/2\pi & 0\end{smallmatrix}\right|(4,A^{-1})\right]\\
&-(4\pi)^{-2}\sum_{r=3}^{\infty}\frac{(-1)^r}{r!}[M_{1R}^{2r}+(N-1)M_{2R}^{2r}]\left[\Gamma(r-1)(4\pi^2)^{1-r}\beta^{-1}\frac{\partial}{\partial a}Z_2\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(2r-2,A^{-1})\right.\\
&\left.-2\Gamma(r)(4\pi^2)^{-r}\beta^{-1}a^{-3}\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_2\left|\begin{smallmatrix}0 & 0\\-\theta_\gamma/2\pi & 0\end{smallmatrix}\right|(2r,A^{-1})\right].\qquad(3.36\text{b})\end{aligned}$$

These expressions contain previous results as special cases; e.g., if $m^2=\hat\phi=0$, the first term in the first square brackets in (3.36b) gives a compact form for the free energy density of a massless conformal scalar gas in an Einstein universe found by Dowker and Critchley,[38] the second term accounts for the correction due to possible multiple connectedness.[19] (3.36b) can, of course, be used to provide expressions for various other thermodynamic quantities of interest, a pursuit which we leave to the reader.

Approximate results regarding the criteria for symmetry restoration can be obtained by substituting (3.36b) in (2.17), equality holding in the small $\det A$ regime when

$$\begin{aligned}-(m^2+a^{-2})&=\frac{\lambda}{8\pi^2 a\beta}\left[1+\frac{(N-1)}{3}\right]\left[a\frac{\partial}{\partial a}Z_2'\left|\begin{smallmatrix}0\\0\end{smallmatrix}\right|(0,A^{-1})-(4\pi^2)^{-1}a^{-2}\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\partial}{\partial\theta_\gamma}Z_2\left|\begin{smallmatrix}0 & 0\\-\theta_\gamma/2\pi & 0\end{smallmatrix}\right|(2,A^{-1})\right]\\
&=\frac{\lambda}{96\pi^2a^2}\left[1+\frac{(N-1)}{3}\right]\left[\frac{\theta_1'''(0,\tilde q)}{\theta_1'(0,\tilde q)}+6\sum_{\gamma\neq1}\csc\theta_\gamma\frac{\theta_1'\left(\frac{\theta_\gamma}{2},\tilde q\right)}{\theta_1\left(\frac{\theta_\gamma}{2},\tilde q\right)}\right]\\
&\equiv\frac{\lambda}{96\pi^2a^2}\left[1+\frac{(N-1)}{3}\right]I_\Gamma(x)\,,\qquad(3.37)\end{aligned}$$

where $\tilde q\equiv e^{-\pi x}$ and we have made use of (A11) and (A12) to express the result in terms of Jacobi θ functions.[45] By assumption the symmetry is initially broken in the tree approximation on $\bar M$ and so the left-hand side of (3.37) is positive. A necessary condition, therefore, for restoration through one-loop global quantum effects is $I_\Gamma(x)>0$.

A plot of $I_\Gamma(x)$ is shown in Fig. 1 for $\Gamma=1, Z_2, Z_3$. If $\Gamma\neq1$, $I_\Gamma(x)$ remains positive in the large-x (zero-temperature) limit. This follows from the expansions[45]

$$\frac{\theta_1'''(0,\tilde q)}{\theta_1'(0,\tilde q)}=6\sum_{n=1}^{\infty}\operatorname{csch}^2(n\pi x)-1\,,\qquad(3.38\text{a})$$

$$\frac{\theta_1'(\theta_\gamma/2,\tilde q)}{\theta_1(\theta_\gamma/2,\tilde q)}=\cot\frac{\theta_\gamma}{2}+4\sum_{n=1}^{\infty}\frac{\tilde q^{2n}}{1-\tilde q^{2n}}\sin n\theta_\gamma\,,\qquad(3.38\text{b})$$

which show that

$$I_\Gamma(\infty)=3\sum_{\gamma\neq1}\csc^2\frac{\theta_\gamma}{2}-1\,,\qquad(3.39)$$

a result which can also be obtained on substituting (3.36a) in (2.17). Particular examples are (using the θ_γ values given in Ref. 19)

$$I_{Z_m}(\infty)=m^2-2\,,\qquad(3.40\text{a})$$

$$I_{D_2^*}(\infty)=38.\qquad(3.40\text{b})$$

(3.37) and (3.39) lead one to conclude that at zero-temperature one-loop global quantum effects cannot restore a broken symmetry when the spatial section M_3 is simply connected, but can for sufficiently small a when M_3 is multiply connected. At finite temperature the additional alteration of the infrared structure serves in general to enhance the possibility of symmetry restoration. This is apparent from Fig. 1: in the case $\Gamma=1$, the thermal fluctuations dominate if $x<x^*=0.52$, resulting in $I_1(x)>0$. Gibbons[46] has also examined the simply connected $k=1$ case using the ζ-function method of regularization[47,48]: (3.37) gives the exact evaluation of his ζ function.

We cannot place too much confidence in the above conclusions, however, since strictly speaking we ought not to neglect terms in (3.36b) of $O(M^4)$ and higher in passing to (3.37). One cannot allow

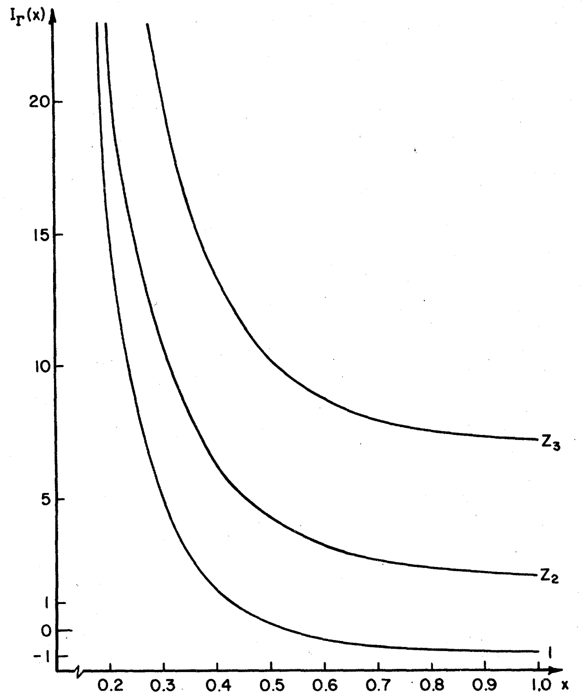

FIG. 1. A plot of $I_\Gamma(x)$ for the multiply connected $k=1$ static Clifford-Klein Robertson-Walker universes with spatial section S^3/Γ for $\Gamma=1, Z_2, Z_3$.

m^2a^2 to become too small without restoring the symmetry at the tree level,[46] and so the justification in using an expansion such as (3.36b) about the massless conformally coupled result is at best dubious. It is not difficult to see that one ought, instead, to expand about the massless minimally coupled result, but such an approach would lead to much more difficult summations and to the abandonment of a simple closed-form expression such as (3.37). We should therefore regard (3.37) as a first approximation which allows some analytic insight into the workings of one-loop global quantum effects. We postpone to the near future improvement of this approximation and refinement of its conclusions by studying the problem from the standpoint of the renormalization group.[49]

E. $k=-1$

This case is trivial since by theorem 2.7.1. of Wolf[15] the only homogenous M_3 hypersurface locally isometric to H^3 is in fact H^3 itself, thereby prohibiting any exotic effects due to multiple connectedness of the spatial section. The spatial quantum-mechanical propagator for conformal coupling on H^3 $(R=-6a^{-2})$ is[37]

$$K_3(\vec{x},\vec{x}';\tau)=(4\pi i\tau)^{-3/2}\,\frac{s(\vec{x},\vec{x}')}{a}\,\operatorname{csch}\frac{s(\vec{x},\vec{x}')}{a}\times\exp\left[\frac{i}{4\tau}\,s^2(\vec{x},\vec{x}')\right], \qquad (3.41)$$

which can be obtained by analytic continuation of (3.34), there being no sum over indirect geodesics since H^3 is noncompact. (3.2) then yields for the coincidence limit of the finite-temperature quantum-mechanical propagator on a static open universe,

$$K(x,x;\tau)=(-4\pi i\tau)^{-1/2}(4\pi i\tau)^{-3/2}\theta_{(1)}\left|{}^0_0\right|((4\pi\tau)^{-1},\beta^2). \qquad (3.42)$$

The program of substitution in (2.14) and renormalization according to (2.32) may now be carried out as before to yield the renormalized one-loop effective potential. It is not difficult to see that, apart from the inclusion of an additional term $(R/12)\hat{\phi}^2=-(a^{-2}/2)\hat{\phi}^2$ in the tree approximation, the result is identical to the $p=1$, $k=0$ result (3.10). In the high-temperature regime one therefore finds symmetry restoration when

$$\beta_c{}^2=-\frac{\lambda}{24(m^2-a^{-2})}\left[1+\frac{(N-1)}{3}\right]. \qquad (3.43)$$

Gibbons[46] also treated the $k=-1$ case, but unfortunately included a sum over imaginary indirect geodesics in (3.41). The correct propagator was given in a paper by Bunch.[37]

IV. DISCUSSION

The calculations of Sec. III demonstrate how nontrivial topological features can influence the occurrence of a phase transition in a theory with a broken-symmetry ground state. Spatial periodicities act in a manner similar to the finite-temperature periodicity in imaginary time (although not always with the same sign, as we saw in the case of S^3). Increasing the number of periodicities in flat space has a cumulative effect on the occurrence of a phase transition: Indeed within the large-N approximation we found that only the disordered phase could exist in four dimensions with two or more periodicities, or in less than four dimensions with one or more periodicities, for *any* finite values of the periodicity parameters. A similar dependence of the existence of the ordered phase on the number of noncompact dimensions is familiar from the realm of critical phenomena.[49]

The particular models investigated were chosen for their calculational simplicity. The restriction that $\hat{\phi}^i$ be constant would need to be removed if one wished to study symmetry restoration of twisted scalar fields[50] in these models. This would also be true if one had in mind an analysis on nonstatic manifolds or of symmetry restoration near boundaries. (The effects of periodicities are, after all, just an example of the Casimir effect and so one might expect similar behavior in, say, the standard configuration of spacetime sandwiched between two parallel plates. In fact a preliminary analysis[51] shows that a field satisfying Neumann boundary conditions on a single plane boundary undergoes symmetry restoration at a critical length perpendicular to the boundary, the symmetry remaining broken outside this critical length. The extension to curved boundaries, a subject of some recent interest,[52,53] is being investigated.) Brown and Duff[36] have considered the nonconstant $\hat{\phi}^i(x)$ case, and their work holds hope for generalization to these and more realistic situations of interest. We also note that Banach[54] has recently given a general approach to the effective potential for twisted fields.

As stated in the Introduction, our interest in symmetry restoration arises from a desire to construct a comprehensive model describing gauge theories coupled to Higgs fields in the early universe. Whether or not the topological investigations treated here will be of any relevance in this regime remains to be seen, but one might well expect them to have some importance in the vicinity of an initial singularity. If so, the gauge bosons would propagate with an anisotropic mass decided by the topology, in analogy to the Debye screening in a plasma[25] (the short-range force being equivalent to a temperature—and hence periodicity—generated mass). This has the interesting corollary that since such masses contribute to the energy-momentum tensor, Einstein's equations contain, via the Higgs mechanism, information about the global spacetime structure. The back reaction may therefore promote dynamical changes in the topology. An attack on gauge fields in the early universe is presently underway.

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APPENDIX

We compile here some useful properties of the multidimensional θ functions and Epstein ζ functions[55,56] used extensively in Sec. III.

The p-dimensional θ function[57] is defined by [we use the parenthesis (p) to distinguish this from the Jacobi-type θ functions[45]]

$$\theta_{(p)}\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(z,A)=\sum_{\{m\}}\exp[2\pi i\tilde{m}h+i\pi z(\tilde{m}+\tilde{g})A(m+g)], \tag{A1}$$

where g and h are p-dimensional column vectors, A is a $p\times p$ matrix, and the tilde indicates transpose. It satisfies the fundamental functional relation

$$\theta_{(p)}\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(z,A)=\exp[-2\pi i\tilde{g}h](-iz)^{-p/2}\times(\det A)^{-1/2}\theta_{(p)}\begin{vmatrix}\tilde{h}\\ -\tilde{g}\end{vmatrix}(-z^{-1},A^{-1}). \tag{A2}$$

The p-dimensional Epstein ζ function is defined by (we follow Epstein's original notation[55])

$$Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(s,A)=\sum_{\{m\}}{}'\exp[2\pi i\tilde{m}h][(\tilde{m}+\tilde{g})A(m+g)]^{-s/2},\quad \operatorname{Re}s>p, \tag{A3}$$

where, in the case that some g_i are integers, the prime indicates the omission of terms which cause $(\tilde{m}+\tilde{g})A(m+g)$ to vanish. It obeys the functional relation[55,56]

$$Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(s,A)=\exp[-2\pi i\tilde{g}h](\det A)^{-1/2}\pi^{s-p/2}\times\frac{\Gamma(p/2-s/2)}{\Gamma(s/2)}Z_p\begin{vmatrix}\tilde{h}\\ -\tilde{g}\end{vmatrix}(p-s,A^{-1}) \tag{A4}$$

and, if none of the g_i are integral, is simply related to $\theta_{(p)}$ by

$$Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(s,A)=\frac{(-i\pi)^{s/2}}{\Gamma(s/2)}\int_0^\infty dz\, z^{s/2-1}\theta_{(p)}\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(z,A). \tag{A5}$$

$Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(s,A)$ is an entire function of s provided all h_i are not integers. If all h_i are integers then it is analytic in s except for a simple pole at $s=p$ of residue $2\pi^{p/2}(\det A)^{-1/2}[\Gamma(p/2)]^{-1}$. $Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(-2m,A)=0$ for m a positive integer. Using these properties and (A4) we further note $Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(0,A)=0$ except if all g_i are integers, in which case $Z_p\begin{vmatrix}\tilde{g}\\ \tilde{h}\end{vmatrix}(0,A)=-\exp(-2\pi i\tilde{g}h)$.

For particular p values Z_p is sometimes expressible in terms of more elementary functions.

$$p=1:\ Z_1\left|{}^0_0\right|(s,A)=2A^{-s/2}\zeta(s), \tag{A6}$$

$p=2$:

$$x=1:\ Z_2\left|{}^0_0\right|(s,A)=2^{2-s}\pi^{-s}a^{-s}\zeta(s/2)\beta(s/2), \tag{A7}$$

$$x=\tfrac{1}{2}:\ Z_2\left|{}^0_0\right|(s,A)=2(1-2^{-s/2}+2^{1-s})\pi^{-s}a^{-s}\zeta(s/2)\beta(s/2), \tag{A8}$$

$$x=\tfrac{1}{4}:\ Z_2\left|{}^0_0\right|(s,A)=2^s\pi^{-s}a^{-s}\{[2^{2-2s}+(1+2^{1-s})(1-2^{-s/2})]\zeta(s/2)\beta(s/2)$$
$$+[1-2^{-s/2}\beta(s/2,3/2)]^2-[3^{-s/2}-2^{-s/2}\beta(s/2,5/2)]^2\}, \tag{A9}$$

where $\zeta(s)$ is the usual Riemann ζ function and for comparison with Sec. III we have in the $p=2$ case set $A=4\pi^2a^2\,\mathrm{diag}(1,x^2)$ with $x=\beta(2\pi a)^{-1}$. These and other lattice sums may be found in the recent papers of Glasser[58-60] who also details the properties of the functions $\beta(s,\omega)\equiv\sum_{k=0}^{\infty}(-1)^k(2k+1+\omega)^{-s}$ and $\beta(s)\equiv\beta(s,0)$. (Unfortunately some numerical errors exist in Glasser's listing of these properties,[58] e.g., the correct value of $\beta'(0)$ is $\ln\Gamma(\frac{1}{4})-\ln[2\Gamma(\frac{3}{4})]$, as recently pointed out by Campbell and Ziff.[61]) Following Glasser's initial investigations, Zucker and Robertson[62,63] have achieved amazing success in the simplification of a wide variety of two-dimensional lattice sums. A severe restriction, of course, in using any of these two-dimensional results is the loss of freedom to continuously alter x. Happily, however, we can retain this freedom if we choose to evaluate $Z_2\left|{}^0_0\right|(s,A)$ at a particular value of s.

Using the one-dimensional equivalent of (A2) (a Jacobi imaginary transformation[45]),

$$Z_2\left|{}^0_0\right|(s,A)=[\Gamma(s/2)]^{-1}\int_0^\infty dt\,t^{s/2-1}\left[\sum_{n=-\infty}^{\infty}{}' e^{-4\pi^2a^2x^2n^2t}+(4\pi a^2x^2t)^{-1/2}\sum_{m=-\infty}^{\infty}{}'\sum_{n=-\infty}^{\infty}\exp\left(-4\pi^2a^2m^2t-\frac{n^2}{4a^2x^2t}\right)\right]$$
$$=2\zeta(s)(2\pi ax)^{-s}+2\pi^{1/2}x^{-1}\frac{\Gamma(s/2-1/2)}{\Gamma(s/2)}\zeta(s-1)(2\pi a)^{-s}$$
$$+2^{3/2}\pi^{-1/2}a^{-3/2}x^{-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-1}(2\pi nmx^{-1})^{1/2}K_{1/2}(2\pi nmx^{-1})+O(s-2)$$
$$=2\zeta(s)(2\pi ax)^{-s}+2\pi^{1/2}x^{-1}\frac{\Gamma(s/2-1/2)}{\Gamma(s/2)}\zeta(s-1)(2\pi a)^{-s}-(\pi xa^2)^{-1}\ln\left[\prod_{n=1}^{\infty}(1-q^{2n})\right]+O(s-2)$$
$$=(2\pi a^2x)^{-1}[(s-2)^{-1}+\gamma-\tfrac{2}{3}\ln(4\pi^{3/2}a^{3/2}\theta_1'(0,q))]+O(s-2), \tag{A10}$$

where the definition of the Jacobi elliptic function is that of Ref. 45. Epstein[55] also arrived at (A10) via a different route. Combining (A10) with (A4) we readily obtain the exact result for arbitrary x

$$Z_2'\left|{}^0_0\right|(0,A^{-1})=-\tfrac{2}{3}\ln[4\pi^3a^{3/2}\theta_1'(0,q)] \tag{A11a}$$
$$=-\tfrac{2}{3}\ln[2^{1/2}\pi^{3/2}\beta^{3/2}\theta_1'(0,\tilde q)], \tag{A11b}$$

where $q\equiv e^{-\pi x^{-1}}=e^{2\pi^2a/\beta}$, $\tilde q\equiv e^{-\pi x}=e^{-\beta/2a}$, and the second equality above follows from the symmetry properties of $Z_2\left|{}^0_0\right|(s,A)$. It can be checked that these general formulas are consistent with (A7)–(A9). The logarithmic structure ensures that the results of Sec. III are dimensionally correct.

To discuss the multiply connected $k=1$ CKRW universes in Sec. III D we need the evaluation of a more general two-dimensional ζ function, $Z_2\left|{}^{\ 0}_{-\tilde g}\right|(2,A^{-1})$. Fortunately this has already been evaluated by Glasser,[60] although we find that the coefficient of ln2 in his Eq. (20) ought to be divided by 4. Doing so his work gives

$$Z_2\left|{}^{\ 0}_{-\tilde g}\right|(2,A^{-1})=8\pi^3a^2\left[\pi g_1{}^2-\frac{x}{3}\ln 2-x\ln\left|\frac{\theta_1(\pi\alpha,q)}{[\theta_1'(0,q)]^{1/3}}\right|\right] \tag{A12a}$$
$$=8\pi^3a^2\left[\pi g_2{}^2x^2-\frac{x}{3}\ln 2-x\ln\left|\frac{\theta_1(\pi\tilde\alpha,\tilde q)}{[\theta_1'(0,\tilde q)]^{1/3}}\right|\right], \tag{A12b}$$

where $\tilde g=(g_1,g_2)$ and $\alpha\equiv g_2+ix^{-1}g_1$, $\tilde\alpha\equiv g_1+ixg_2$.

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