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Universal upper bound on the entropy-to-energy ratio for bounded systems

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We present evidence for the existence of a universal upper bound of magnitude $2\pi R/\hbar c$ to the entropy-to-energy ratio S/E of an arbitrary system of effective radius R. For systems with negligible self-gravity, the bound follows from application of the second law of thermodynamics to a gedanken experiment involving a black hole. Direct statistical arguments are also discussed. A microcanonical approach of Gibbons illustrates for simple systems (gravitating and not) the reason behind the bound, and the connection of R with the longest dimension of the system. A more general approach establishes the bound for a relativistic field system contained in a cavity of arbitrary shape, or in a closed universe. Black holes also comply with the bound; in fact they actually attain it. Thus, as long suspected, black holes have the maximum entropy for given mass and size which is allowed by quantum theory and general relativity.

I. INTRODUCTION

Gravitational entropy is one of the most intriguing concepts that have emerged from much recent work on quantum fields in curved space-time and quantum gravity.[1,2] Its most striking manifestation occurs in Hawking's radiation process by black holes,[1] in which it is connected with the area of the event horizon. Even though this area behaves very much like entropy, two obstacles have stood in the way of attempts to understand the still mysterious connection between area, a geometrical quantity, and entropy, a thermodynamic one. First, since its very inception, black-hole entropy has seemed to be numerically much larger than the entropy of any ordinary system of like mass.[3] Thus, a solar-mass black hole has black-hole entropy 10^{20} times the sun's thermal entropy. Is it not preposterous to think there is a common denominator in two quantities so unlike in size? Second, even if the two entropies are of like origin, how can one hope to express black-hole entropy in statistical terms (the logarithm of a number of interior states or configurations) when that task evidently demands a full accounting of all that could possibly happen inside the hole?

In this paper we address only the first difficulty. We point out that it arises from the insistence in comparing black holes with nonrelativistic systems. When compared to relativistic systems of massless particles, black holes do not have inordinately large entropy. Rather, black-hole entropy is revealed as matching the maximal entropy for a given mass of more ordinary systems: There is no gap in magnitude between black-hole entropy and ordinary entropy. This comes about because of the existence of a hitherto unnoticed upper bound to the entropy-to-energy ratio of non-black-hole systems of given effective radius R (see Sec. II for definition):

$$S/E < 2\pi R \tag{1}$$

(in units where $k=\hbar=c=G=1$). Making this bound plausible is our main task in this paper. For systems with negligible self-gravity, inequality (1) keeps S from growing faster than E, a well-known property of ordinary bodies which is responsible for the seeming gap between this entropy and black-hole entropy (which grows as E^2). However, as one compresses a body to its gravitational radius, R becomes of order $2E$, and S can begin growing as E^2 thus "catching up" with black-hole entropy. The closing of the gap shows it is conceivable that black-hole entropy should be calculable in terms of the number of interior black-hole configurations.[3]

The plan of this paper is as follows. Relation (1) is formulated as a conjecture in Sec. II, where we also point out that for systems with negligible self-gravity it follows from the second law. Yet this way of deriving it gives no hint as to what physically limits the entropy. Thus other approaches, based on statistical physics, are imperative in understanding the limit, and in extending it to strongly gravitating systems. One such approach, based on the microcanonical ensemble, has been pioneered by Gibbons.[4] We describe this approach in Sec. II. It shows what keeps S/E from reaching arbitrarily large values. Its main disadvantage is that it can only be applied on a case-by-case basis. It does suggest a more general approach to the problem which we develop in Sec. III, and which is found to lead directly to the canonical ensemble. By this method one goes beyond examples to establish the upper bound on S/E for a broad class of systems with negligible self-gravity. The existence of the bound is found to be intimately connected with the vacuum energy of fields.

II. WHY A BOUND ON S/E?

Classically, entropy is a measure of the phase space available to the system in question. It is then not difficult to see why an upper bound might exist for S/E. Let the system have energy E, or alternatively, let its energy be no more than E. This amounts to a limitation on the momentum space available to the system's components provided the potential energy is bounded from below. If the system is also bounded in space, then its phase space is bounded, and so must its entropy. The bound evidently increases with E. But our simple argument cannot establish that it increases linearly and neither can it say anything about the proportionality constant. In fact, it would seem impossible to write down a concrete bound for S/E without going into details about the system. This may explain why such a bound has gone unnoticed by workers in statistical mechanics (but see Bremermann[5] for an information-theoretic analog).

In fact, black-hole physics yields a specific form for the upper bound on S/E for systems with negligible self-gravity. According to the generalized second law of thermodynamics,[1,3,6] the sum of the thermal entropy outside a black hole and the black-hole entropy ($\frac{1}{4}$ of the horizon's surface area) should never decrease. Now, it has long been known[3,6] that when a stationary hole absorbs a body with negligible self-gravity, energy E and effective radius R (for the precise definition see Ref. 6), the hole's surface area must increase by at least $8\pi ER$. Since one can arrange the absorption process so that this minimal increase can be attained,[3] the second law will be violated unless the body's entropy (what disappears from the hole's exterior) cannot exceed $2\pi ER$. Thus we obtain the bound (1) on S/E as applied to weakly gravitating bodies. Note that although the quoted minimum area increase is derived under the assumption that the body is small compared to the hole, nothing prevents us from choosing a hole as large as needed (this also makes the Hawking radiation entropy negligible). Hence (1) holds regardless of the scale of R.

The intriguing feature of the previous argument is that it uses a law whose very meaning stems from gravitation to derive a bound on S/E for systems in which gravitation is negligible, a bound which has nothing to do with gravity [written out fully, relation (1) would involve $\hbar$ and c, but not G]. This provides a striking illustration of the unity of physics, but it also throws a challenge at the theorist: Provide a proof for the bound independent of gravitational considerations. To attempt this will be the main task before us later on in this paper.

The question suggests itself, how dependent is the bound (1) on the assumption that the self-gravity of the system is negligible? To clarify matters, consider a Kerr black hole, a system with maximal gravitational effects. Let its energy be E; then its surface area will be

$$A = 4\pi\{[E + (E^2 - a^2 - Q^2)^{1/2}]^2 + a^2\}\,, \tag{2}$$

where a is the specific angular momentum and Q the charge. We define the effective radius R by $4\pi R^2 = A$. Then it is clear that $R < 2E$, the equality corresponding to $a = Q = 0$. Now since $S = A/4$, $S/ER = \pi R/E$; it then follows from the inequality that Kerr holes conform to the bound (1); the Schwarzschild hole actually attains the bound.

If systems with negligible self-gravity and black holes both obey the bound (1), it is reasonable to assume that systems in which gravity is of intermediate strength do also. Thus bound (1) appears to be of universal validity so long as R is appropriately defined. In fact, guided by the R for a Kerr black hole and that for an ordinary spherical object (the metric radius), we shall be assuming that R for an arbitrary system is given in terms of the area A of that (quasi) spherical surface which circumscribes the system by

$$R = (A/4\pi)^{1/2}\,. \tag{3}$$

The only exception to (3) will be for closed universes which, of course, lack any "circumscribing sphere". For them we define R in terms of the volume of the universe.

How can we begin to see *directly* that bound (1) actually holds for systems with negligible self-gravity? For systems composed of nonrelativistic particles this is easy and the basic idea has long been known.[6] Let the system's mass be M, and let m be the typical mass of the active constituent particles (stars, atoms, nuclei, and nucleons, depending on the kind of system). There are roughly M/m such particles. Thus the system's entropy will also be M/m in order of magnitude and its entropy-to-energy ratio $1/m$. On the other hand, the system's size will exceed, typically by many orders of magnitude, the Compton lengths of its constituent particles, which are of order $1/m$. Thus the bound (1) is obeyed, with orders of magnitude to spare.

Evidently, systems composed of nonrelativistic particles are not very interesting from the point of view of the bound. In them the bulk of the energy is tied up in rest masses, and does not partake of the degrees of freedom which are manifested in the entropy. To put our proposed bound to an interesting test, we must consider assemblies of massless particles. Consider, then, ordinary blackbody radiation confined to a cavity. The familiar formulas[7] tell us that

$$S/E = 4/3T\,, \tag{4}$$

where T is the temperature and E the thermodynamic energy of the radiation. Formula (4) seems to predict that S/E can be made as large as we please by lowering T sufficiently. But in fact the thermodynamic description of radiation on which (4) is based breaks down when T is no longer large compared to the reciprocal of the characteristic size of the system (typical wavelength not small compared to cavity size). Boundary effects make themselves felt. These can be expected to arrest the growth of S/E as T is lowered further. The transition in behavior comes just when S/E is of order of the bound set by (1), so that the present example comes close to challenging (1). This circumstance motivates a shift of interest to the S/E of massless quantum fields confined to a region of prescribed size and shape by a material cavity (or in the case of a closed universe by the topology of the curved spacetime).

Gibbons[4] has developed a microcanonical approach for determining the maximal S/E for such systems. He labels the quantum states of the field by the occupation numbers of the various eigenfrequencies and takes the energy E of each state to be the sum of the eigenfrequencies weighted by the appropriate occupation numbers. He then assumes the entropy at a given E to be given by the Boltzmann formula[7]

$$S_{\rm MC}(E) = \ln N(E)\,, \tag{5}$$

where $N(E)$ is the number of distinct states with energy E. $N(E)$ can always be computed explicitly, at least for low E, if the eigenfrequency spectrum is known. But thus far no general method for doing this has emerged and $N(E)$ must be computed separately and laboriously for each spectrum. In all cases for which this has been done (i.e., scalar field confined to a one-dimensional cavity, to the flat interior of a two-sphere, or to an Einstein universe S^3), $\ln N(E)/E$ has a maximum at some moderately low E. The maxima all lie below the bound (1) (with R interpreted as the radius of the appropriate n-sphere) thus providing support for it.

These results also support our remarks about the relation between the bound on S/E and the breakdown of the thermodynamic (continuum) description of a field at low temperatures (energies). They suggest that the maximal S/E is a phenomenon of the low-excitation states of fields. This may explain why the existence of the bound (1) has not been widely noticed. When cavities having two or three very different scales of length are considered in the Gibbons formalism, it becomes clear that the maximal S/E is determined by the longest scale (that most clearly associated with the lowest eigenfrequencies). This feature is the primary motivation for the definition of effective radius (3): twice R is never less than the longest dimension of the system.

DeWitt has remarked[8] that the question of a universal bound for S/E is quite unique in that it can have no meaning except when a particular choice of the zero of energy is made: obviously (1) cannot hold for every such choice. Gibbons's procedure exemplifies one possible choice of zero: For every field system the zero of energy is the energy of the vacuum (no-particles) state. This choice is not unique; in fact there is an alternative one more in harmony with the lessons of quantum field theory. The (regularized) vacuum energy of a field in flat space confined by some boundaries is, in general, different from the vacuum energy of that field in unconfined Minkowski spacetime.[9] The most famous example of this is, of course, the Casimir effect[10,11] which has been verified experimentally. It thus seems most natural to take, for each type of field, the vacuum energy in unconfined Minkowski spacetime as the zero of energy. It is known that with this choice the energy of a given field state is its gravitating energy (active gravitational mass).[9] It is thus just that energy responsible for the growth in horizon area in the gedanken experiment just discussed. Thus, at least for systems with negligible self-gravity, the present definition of the zero of energy is the appropriate one for stating the bound on S/E in the form (1). By extension of this idea we shall take, for any field system, the energy of the no-quanta state to equal the vacuum energy as computed by an appropriate regularization scheme. Thus in the microcanonical approach just described, a constant (different for each system) must be added to every energy computed from occupation numbers alone.

The big question remaining is, how may the bound (1) be established by the microcanonical method without recourse to a case-by-case analysis which, in any case, could not proceed far because the eigenfrequency spectra of most field systems are not known? We will now replace this question by a more general one which, not only proves more tractable, but also turns out to be a more fundamental statement of the problem being considered. To describe this approach we first recall a few results from quantum statistics.[7] A quantum system is described by its Hamiltonian $\hat{H}$; its state is described by a density operator $\hat{\rho}$ satisfying the normalization condition

$$\mathrm{Tr}\hat{\rho} = 1\,. \tag{6}$$

The mean energy is

$$\bar{E} = \mathrm{Tr}(\hat{\rho}\hat{H}) \tag{7}$$

and the entropy

$$S = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) . \tag{8}$$

It is well known that the microcanonical entropy (5) is just the maximal value which S attains over all $\hat{\rho}$ which satisfy (6) and which assign nonvanishing probabilities only to states with energy E. Thus

$$\ln N(E)/E = \max_{E} (S/\bar{E}) , \tag{9}$$

with the maximization carried out under the same conditions as before. In particular, (9) holds for that E for which $\ln N(E)/E$ is largest. If we now remove the constraint that $\hat{\rho}$ is confined to a given E, $\max(S/\bar{E})$ can only increase. Therefore

$$\max[\ln N(E)/E] < \max(S/\bar{E}) , \tag{10}$$

where the maximization in the left-hand side is over E, while that in the right-hand side is over all $\hat{\rho}$ which fulfill (6). Thus the peak S_{MC}/E value is bounded by the maximum that $S/\bar{E}$ attains for an arbitrary normalized $\hat{\rho}$. If it can be shown that the last quantity is below the bound (1), then one will know that S_{MC}/E fulfills it also. As will become clear in Sec. III, this program can be implemented without recourse to a case-by-case investigation.

Most important, the new statement of the problem is important in its own right. Very often the state of a statistical system is not one of definite energy, but has a well-defined mean energy. The maximum possible value of $S/\bar{E}$ is then of interest and, as (9) shows, it cannot be inferred from microcanonical considerations. Thus from now on we shall concentrate on the question of what upper bound can be set on $S/\bar{E}$ for an arbitrary $\hat{\rho}$.

III. BOUND ON $S/\bar{E}$ FROM QUANTUM STATISTICS

For simplicity we consider only systems composed exclusively of massless fields: scalar, electromagnetic, and neutrino (for technical reasons we do not consider gauge or metric fields). We ignore interactions, except as needed in enforcing boundary conditions at cavity walls. We assume the systems are stationary; this is clearly not a great restriction, for stationarity often implies equilibrium and maximal S, other things being equal. We consider fields in flat spacetime confined by cavities of arbitrary shape, or fields confined in a model (Einstein) universe of constant space curvature. We do not, however, consider fields in flat spacetimes with non-Euclidean topologies (i.e., cube with opposite faces identified to make T^3 topology).

Let us ask which $\hat{\rho}$ makes $S/\bar{E}$ maximal? It is evidently given by the solution of the variational problem

$$\delta[-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})/\mathrm{Tr}(\hat{\rho}\hat{H}) - \lambda\,\mathrm{Tr}(\hat{\rho})] = 0 , \tag{11}$$

where λ is a Lagrange multiplier used to enforce the normalization condition (6). Varying $\hat{\rho}$, rearranging terms and normalizing, one finds

$$\hat{\rho} = Z(\beta_0)^{-1}\exp(-\beta_0\hat{H}) , \tag{12}$$

$$Z(\beta_0) \equiv \mathrm{Tr}[\exp(-\beta_0\hat{H})] , \tag{13}$$

$$\beta_0 = (S/\bar{E})_{\max} . \tag{14}$$

Thus the wanted distribution is the canonical one whose inverse temperature parameter is just the peak value of $S/\bar{E}$ for the system. To find this value one computes $S/\bar{E}$ with (12) and compares with (14) to get

$$\ln Z(\beta_0) = 0 . \tag{15}$$

The problem is thus superficially simple; the maximal $S/\bar{E}$ is just that β for which the partition function is unity. However, for the field systems we have in mind this prescription is not trivially applied. Each field mode has a ground-state energy $\pm\omega/2$ for bosons or fermions, respectively. If $\ln Z$ is calculated naively, the mode sum of ground energies will make it diverge unless the boson and fermion ground-state energies cancel miraculously. We thus need to regularize $\ln Z$. In principle, one knows how to do this.[12] But technical difficulties have limited explicit calculations of the finite $\ln Z$ to a few systems of high symmetry,[13,14] or to the high-[15,16] and low-[15] temperature limits. Since we are interested in the general β case, and wish to treat systems wholesale rather than on a case-by-case basis, it seems reasonable to adopt a pragmatic approach.

Let ω_i be the eigenfrequencies for our system and let them be g_i-fold degenerate. Then at inverse temperature β the mean energy in all species is given by

$$\bar{E} = E_0 + \sum_i g_i\omega_i(e^{\beta\omega_i} \mp 1)^{-1} , \tag{16}$$

where upper (lower) signs are for boson (fermion) modes. The E_0 is the sum of *regularized* vacuum energies, while the mode sum is just the usual thermal contribution. That the intuitive form (16) follows from detailed regularization of the finite-temperature quantum field theory has been amply demonstrated by Al'taie and Dowker[13] and Dowker and Kennedy.[16] Substituting (16) and (12) into (7) and integrating with respect to β, one obtains the *finite* expression

$$\ln Z = C - \beta E_0 + \sum_i \mp g_i \ln(1 \mp e^{-\beta\omega_i}) , \tag{17}$$

where C is a constant of integration. Now using (12) to calculate S, one finds that at zero temperature ($\beta \to \infty$) $S \to C$ provided no ω_i vanishes (see Appendix C). However, for the fields we have in mind the vacuum state (absence of quanta) is nondegenerate; thus we expect $S \to 0$ at zero temperature (third law of thermodynamics). Hence, we must set $C = 0$.

It is clear from (17) that, provided $E_0 > 0$, $\ln Z$ is a monotonic decreasing function of β. As $\beta \to 0$, $\ln Z \to \infty$, while $\ln Z \to -\infty$ for $\beta \to \infty$. Hence by continuity there exists some β for which (15) is satisfied and there exists a maximal $S/\bar{E}$. This is clearly not the case if $E_0 \leq 0$. Later we shall indicate how one copes with the case $E_0 = 0$ within our formalism, but we shall have to assume that *in nature* no complete system can have a negative vacuum energy. Actually, seeming counterexamples to this assumption can easily be found. The Casimir vacuum energies of the electromagnetic[10] and scalar fields[11] between two infinite parallel plates, or the scalar vacuum energies in some infinitely long rectangular pipes[16] are all negative. But these are infinite systems and for them (1) does not set a bound on S/E. The fact that $E_0 < 0$ and that no β_0 exists is, therefore, irrelevent here.

Fields in flat spaces with non-Euclidean topologies also frequently have $E_0 < 0$ even when the space is finite.[9] But though non-Euclidean topology is important in highly curved spacetimes, the physical relevance of flat non-Euclidean spaces is yet to be demonstrated. Presently they are mere mathematical models in which regularization calculations are tractable. For this reason we do not consider such spaces.[17] By contrast to the above, vacuum energies of massless fields in the Einstein (curved) universe with topology S^3 are all positive. Likewise, the electromagnetic vacuum energy in a Euclidean, flat, spherical space is positive.[18] The known facts are few but they do indicate that $E_0 > 0$ is a physically plausible assumption.

A. Case with $E_0 > 0$

Assume, as is probably the case, that the vacuum energy of every species of field confined to a finite cavity, or to a compact universe is positive; let E_{0k} be the vacuum energy for the kth field. Now solve the equation

$$E_{0k} = \beta_k^{-1} H_k(\beta_k) \equiv \mp \beta_k^{-1} \sum_i^{(k)} g_i \ln(1 \mp e^{-\beta_k \omega_i}) \tag{18}$$

(which involves the sum over the kth field's modes only) for the parameter β_k. Repeat for every other species. One has then a series of distinct β_k. Now $E_0 = \sum_k E_{0k}$. Therefore, substituting the expressions (18) for all k into (17) evaluated at $\beta = \beta_0$ we get

$$\beta_0^{-1} \sum_k H_k(\beta_0) = \sum_k \beta_k^{-1} H_k(\beta_k) . \tag{19}$$

Now, as direct differentiation shows, $\beta^{-1} H_k(\beta)$ is monotonic decreasing in β both for fermions and bosons. It is then fairly clear from (19) that β_0 is bracketed by the smallest and largest β_k. This leads to great simplification: *By treating field species separately one can find upper and lower bounds for β_0 for any conceivable mixture of species.*

1. Upper bound

Introduce $n_k(\omega)$, the number of *modes* (counting degeneracies) of species k with eigenfrequencies up to ω. Clearly at every ω which is an eigenvalue, $n_k(\omega)$ has a step discontinuity of strength $+g_i$. This (18) can be written as

$$E_{0k} = \mp \beta_k \int_0^\infty (dn_k/d\omega) \ln(1 \mp e^{-\beta_k \omega}) d\omega . \tag{20}$$

Integration by parts gives

$$E_{0k} = \int_0^\infty n_k(\omega) (e^{\beta_k \omega} \mp 1)^{-1} d\omega \tag{21}$$

as a condition for β_k. In most cases we know little about the precise form of n_k. However, one can easily obtain from (21) an upper bound for β_k which will suffice for our purpose—establishing that (1) holds generally.

One starts with the obvious inequality[4] (valid for $p > 0$)

$$n_k(\omega) \omega^{-p} < \sum^{\omega} g_i \omega_i^{-p} , \tag{22}$$

where the sum includes all eigenfrequencies up to ω. The inequality holds (for any $p > 0$) since every $\omega_i < \omega$, while $n_k(\omega)$ is just the number of terms in the sum. The sense of the inequality is clearly preserved if the sum is extended to *all* eigenfrequencies in which case it becomes the well-known ζ function for the Hamiltonian of the field k[19,20] (see also Appendix B),

$$n_k(\omega) < \zeta_k(p) \omega^p . \tag{23}$$

Since $\zeta_k(p)$ converges only for $p > 3$, we restrict all our following remarks to that range. We now use (23) to set an upper bound on the integral in (21):

$$E_{0k} < \beta_k^{-p-1} \zeta_k(p) \int_0^\infty x^p (e^x \mp 1)^{-1} dx . \tag{24}$$

Evaluating the integral[21] in terms of Riemann's ζ function $\zeta_R(p)$ and isolating β_k, we get the bound

$$\beta_k < [\Gamma(1+p)\zeta_R(1+p)\zeta_k(p)E_{0k}^{-1}(1-\delta_k 2^{-p})]^{1/(1+p)}, \tag{25}$$

where $\delta_k = 1$ for fermions and $\delta_k = 0$ for bosons.

For the scalar, electromagnetic, and neutrino fields in an Einstein universe one knows all quantities in (25).[13] For example, for the scalar field $E_{0k} = (240a)^{-1}$ and $\zeta_R(p) = a^p \zeta_R(p-2)$, where a is the radius of the universe. By trial and error one finds that $p = 6.4$ gives nearly the lowest bound on β_k. Inserting numbers[21] one gets

$$\beta_k(\text{scalar}; S^3) < 5.70a. \tag{26}$$

This may be compared with the exact value $\beta = 4.13a$ which is obtained by finding (numerically) the zero of $\ln Z$, an exact expression for which may be inferred from the work of Dowker and Critchley[14] [Eqs. (30)–(33)]. In like manner, and still using $p = 6.4$, one finds

$$\beta_k(\text{neutrino}; S^3) < 4.25a, \tag{27}$$

which again compares favorably with the exact result $\beta = 2.61a$ inferred from the $\ln Z$ implicit in Dowker and Al'taie.[12] Finally, for the electromagnetic field one has (also with $p = 6.4$)

$$\beta_k(\text{electromagnetic}; S^3) < 2.70a. \tag{28}$$

In this case there exists no exact result for comparison. Evidently, if we identify a with the effective radius of the universe R, then all our β_k's are below $2\pi R$. Consequently, an Einstein universe filled with any mixture of photons, neutrinos and scalar particles obeys (1).

For fields in flat space confined to a cavity of arbitrary shape one does not generally know E_{0k}, or the spectrum out of which $\zeta_k(p)$ is constructed. To minimize the effect of our ignorance we work with large p; in the limit of large p, $\zeta_k(p) \to g_1 \omega_1^{-p}$, so we need know only the first eigenfrequency and its degeneracy. Also, because of the exponent in (25), the dependence on E_{0k} is weak. Actually, we cannot take $p \to \infty$; because of the $\Gamma(1+p)$ the bound on β_k would blow up. But it will be sufficient to restrict attention to $p > 10$. Appendix B shows that for such p, approximating ζ_k by $g_1\omega_1^{-p}$ leads to fractional errors of order 10^{-3}–10^{-1} in the right-hand side (RHS) of (25) for all cases in which one knows ζ_k accurately. The replacement is thus a good approximation. Also for $p > 10$ one may neglect the 2^{-p} in (25) and replace $\zeta_R(1+p)$ by unity. The fractional errors incurred are only 10^{-4}. We thus get the bound

$$\beta_k \omega_i < [\Gamma(1+p)\, g_1 \omega_1 / E_{0k}]^{1/(1+p)}. \tag{29}$$

Calculations of E_{0k} are proverbial for their difficulty. Only for the electromagnetic field in a sphere,[18] and for all three fields in the Einstein universe,[13,14] are precise results known (and for the Casimir energies of some infinitely long systems not of relevance here[15,16]). In all these cases $\xi \equiv E_{0k}/g_1\omega_1$ is in the range 10^{-3}–10^{-2}. In fact, since ω_1 sets the energy scale in each case, and g_1 is of order unity, one expects, on order-of-magnitude grounds, that ξ will be within a few orders of magnitude of unity. We shall allow it the generous range 10^{-6}–10^{6}. Although one can conceive of E_{0k} being exactly zero (see below), it seems unlikely that it can fall below the range indicated and yet be nonvanishing.

Let us now find the tightest bound on $\beta_k\omega_1$ by differentiating the RHS of (29) with respect to p; it occurs for $p = p_c$ where p_c is determined by

$$\xi = \Gamma(1+p_c)\exp[-(1+p_c)\psi(1+p_c)] \tag{30}$$

or its alternative form

$$[\Gamma(1+p_c)\xi^{-1}]^{1/(1+p_c)} = \exp[\psi(1+p_c)], \tag{31}$$

where

$$\psi(p) \equiv d\ln\Gamma(p)/dp. \tag{32}$$

By inserting in (30) a trial $p_c \geq 10$, one finds the ξ for which it gives the best upper bound for $\beta_k\omega_1$; that bound is obtained by simply evaluating the RHS of (31). Inserting numerical values,[21] one finds that p_c increases monotonically from 10 to 12.8 as ξ goes from 2.1×10^{-5} to 1.1×10^{-6}; simultaneously the bound increases from 10.50 to 13.30. For $\xi > 2.1\times10^{-5}$ the p_c would be below 10. In order to preserve the accuracy of our approximation, we simply use $p = 10$ in (29) for $\xi > 2.1\times10^{-5}$. Evidently the bound on $\beta_k\omega_1$ decreases as ξ increases beyond 2.1×10^{-5}. Hence for $10^{-6} < \xi < 10^{6}$ (or even larger)

$$\beta_k\omega_1 < 13.30. \tag{33}$$

The only question now is what is the first eigenfrequency ω_1. As shown in Appendix A, for fields in a sphere of radius R, $\omega_1 = \pi/R$ for the scalar and neutrino fields and $\omega_1 = 2.082/R$ for the electromagnetic field. As shown in Appendix C, for a nonspherical cavity we have $\omega_1 > \pi/R$ for scalar and neutrino fields and $\omega_1 > 2.082/R$ for the electromagnetic field, where R is the radius of the sphere which circumscribes the cavity; this is just the cavity's effective radius by (3). Combining these values with (33) we finally get

$$\beta_k(\text{scalar or neutrino; cavity}) < 4.233R \tag{34}$$

and

$$\beta_k(\text{electromagnetic; cavity}) < 6.388R. \tag{35}$$

Thus for scalar and neutrino fields, $\beta_k < 2\pi R$. The RHS of (35) exceeds $2\pi R$ by only one percent. But one can be quite confident that for the electromagnetic field β_k is also below $2\pi R$. To have it otherwise would require an unlikely "conspiracy"[22]: $\xi \simeq 10^{-6}$, $\omega_1 \simeq 2.082/R$, and $n_k(\omega) \simeq \zeta_k(12.8)\omega^{12.8}$. The last condition cannot possibly hold for arbitrarily large ω. The second would be true only for a quasispherical cavity, but for a spherical cavity one knows[18] that $\xi = 7.39 \times 10^{-3}$, not 10^{-6}. Hence, we can see that an arbitrary mixture of scalar, electromagnetic, and neutrino fields would satisfy bound (1).

2. Lower bound

Thus far we have no inkling of how much our bounds for β_0 overestimate it. It is important to know this in order to support our claim in Sec. I that ordinary systems can approach bound (1) and thus close the gap between ordinary matter and black holes. There is no easier way to settle this than to obtain lower bounds for the β_k. Let us return to the defining relation (18). If one discards in the sum the terms beyond that with the lowest ω_i and solves

$$\overline{\beta}_k E_{0k} = \mp g_1 \ln(1 \mp e^{-\overline{\beta}_k \omega_1}) \tag{36}$$

for $\overline{\beta}_k$, one will evidently get an underestimate for β_k. We only consider the case for which $E_{0k} \ll \omega_1$; examples[13,16,18] for which E_{0k} is known are of this type. Then (36) can be consistent only if $\exp(-\overline{\beta}_k\omega_1) \ll 1$ so that

$$\overline{\beta}_k E_{0k} \simeq g_1 e^{-\overline{\beta}_k \omega_1} \tag{37}$$

or, in our previous notation,

$$\xi \overline{\beta}_k \omega_1 \simeq e^{-\overline{\beta}_k \omega_1}. \tag{38}$$

Numerically we find that as ξ decreases from 10^{-2} to 10^{-4} to 10^{-6}, $\overline{\beta}_k\omega_1$ increases from 3.4 to 7.2 to 11.4. These values are consistent with (33). As we mentioned, $\omega_1 > 2.082/R$ for the electromagnetic field and $\omega_1 > \pi/R$ for the other two. Thus there could exist a cavity in which $\beta_k > 2.3R$ for a neutrino field if only the ξ in question had a rather modest value $\lesssim 10^{-4}$. Likewise, for the electromagnetic field in a spherical cavity (see Ref. 18 and Appendix A) $\xi = 7.39 \times 10^{-3}$, so that we know with confidence that $\overline{\beta}_k\omega_1 \simeq 3.62$; thus $\beta_k > 1.74R$.[23] There are, therefore, fields in cavities which are only a factor of 3 below the universal bound (1). As we mentioned earlier, a scalar-filled Einstein universe reaches $\frac{2}{3}$ of the bound (1). Hence there is no great gap between matter and black holes as far as $S/\overline{E}$ is concerned.

B. Case with $E_0 = 0$

If one or more E_{0k} vanish but still $E_0 > 0$, no change is required in our previous analysis. We may simply define fictitious positive E_{0k} for the fields which have none at the expense of those which have in such a way that the sum of E_{0k} is still the physical E_0. The fictitious E_{0k} are unlikely to have to be smaller than the minimum we require (corresponding to $\xi = 10^{-6}$). Thus our arguments go through. A problem of principle arises, however, if (and we emphasize if) *all* vacuum energies vanish. We may then argue that a vacuum (no particles) state with vanishing energy is impossible to detect and could well be discarded from $\hat{\rho}$ and Z without ill effect for the sort of question we have in mind. Since $Z = \sum e^{-\beta_0 E}$,[24] this amounts to subtracting unity from Z. Then the condition that the *new* Z is unity at β_0 implies [see (17)] that

$$\ln 2 = \sum_i \mp g_i \ln(1 \mp e^{-\beta_0 \omega_i}). \tag{39}$$

Let there be ν species of massless fields present (including several neutrinos, etc.). The role of (18) is now taken over by

$$(\ln 2)/\nu = H_k(\beta_k). \tag{40}$$

We can still conclude that β_0 is bracketed by the extreme β_k's and we can carry out an analysis analogous to the one leading to (29) to obtain

$$\beta_k \omega_1 < [\Gamma(1+p)\, g_1 \nu/\ln 2]^{1/p}. \tag{41}$$

The slightly different power of p here has its root in the absence of a factor β_0 in (39) as compared with (17) at $\beta = \beta_0$. Implicit in (41) is the condition $p > 10$ introduced for the same reasons as before. In general, g_1 is a small integer while ν is evidently not large. We can with confidence assume that $g_1\nu/\ln 2 < 100$. Then with $p = 10$, (41) gives $\beta_k\omega_1 < 7.18$. In view of the above-mentioned lower bounds on ω_1, we see that no β_k may exceed $3.45R$. Thus an arbitrary mixture of scalar, electromagnetic, and neutrino fields with vanishing vacuum energy would also satisfy bound (1).

IV. ASSESSMENT

Resting as they do at three points on plausibility rather than on rigor, our arguments do not constitute an airtight proof that $S/\overline{E}$ complies with bound (1), but they do make this very plausible for mixtures of scalar, neutrino, and electromagnetic fields in cavities of arbitrary shapes. With the same degree of plausibility, they indicate that the microcanonical S_{MC}/E satisfies (1) for those same situations. These partial successes in "explaining" (1) underscore the need for deepening the rigor and broadening the scope of the arguments.

In particular, one would like to encompass in the scheme Dirac fields in interaction with gauge fields. Not only is this a case of great physical interest (are there thermodynamic constraints on the possible structure of hadrons?), but it also poses a great technical challenge: how to handle the zero-frequency Dirac modes that may appear in this context? One would also like to make bound (1) plausible for self-gravitating systems, not merely for fields in a prescribed gravitational background. This is perhaps the greatest challenge in connection with (1), relating as it does to the question of whether black-hole entropy can be calculated in terms of the same concepts used for calculating matter entropy.

Note added in proof. Recently K. A. Milton [Phys. Rev. D 22, 1444 (1980)] has demonstrated the positivity of the vacuum energy of a confined system of gauge fields and fermions in interaction.

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APPENDIX A

Here we summarize the eigenfrequency spectra for scalar, electromagnetic, and neutrino fields in a spherical cavity of radius R and make some remarks about the choice of boundary conditions. For the neutrino case the well-known problem with prescribing boundary conditions is resolved. We regard spacetime as flat here and in Appendix C.

Solutions of the scalar equation which are harmonic in time may be found only for discrete eigenfrequencies ω_i which arise from the eigenvalue problem defined by

$$\nabla^2\phi = -\omega^2\phi\,, \tag{A1}$$

together with suitable boundary conditions for ϕ. We shall follow DeWitt in demanding $\phi=0$ on the boundary.[11] One advantage of this (Dirichlet) boundary condition is that it is conformally invariant (whereas the Neumann condition is not). Thus one can treat the conformal scalar field. The solutions of (A1) which are regular at the origin are $j_l(\omega r)Y_{lm}(\theta,\phi)e^{-i\omega t}$, where j_l is the standard spherical Bessel function of order l. The boundary condition then demands that ωR be a positive zero of j_l. Hence the spectrum is

$$\omega_{nl} = j_{n,l}R^{-1}\,,\quad n=1,2,\ldots;\quad l=0,1,\ldots\,, \tag{A2}$$

where $j_{n,l}$ is the nth positive zero of $j_l(x)$; the degeneracy is $2l+1$. The lowest eigenfrequency is $\omega_{10}=\pi/R$.

Sourceless electromagnetic fields which are harmonic in time obey the time-independent Maxwell equations

$$\vec\nabla\times\vec{\mathrm{E}} = i\omega\vec{\mathrm{B}}\,, \tag{A3}$$

$$\vec\nabla\times\vec{\mathrm{B}} = -i\omega\vec{\mathrm{E}}\,. \tag{A4}$$

The boundary conditions depend on the physical characteristics of the cavity. Intuitively one would like the fields to be confined to the cavity. Then one must assume it is a perfect conductor which implies that the tangential component of $\vec{\mathrm{E}}$, $\vec{\mathrm{E}}_t$ vanishes on the boundary. The "electric" solutions of (A3) and (A4) have[25] $\vec{\mathrm{E}} = j_l(\omega r)\vec{\mathrm{X}}_{lm}e^{-i\omega t}$, where $\vec{\mathrm{X}}_{lm}$ are the vector spherical harmonics obtained by operating on the Y_{lm} with the angular momentum differential operator. The $\vec{\mathrm{B}}$ may be inferred from (A3). Evidently $\vec{\mathrm{E}}_t$ vanishes on the surface only if j_l vanishes. Hence the spectrum of the electric modes is the same as the one for the scalar field (A2), except $l=1,2,\ldots$ because $\vec{\mathrm{X}}_{00}\equiv 0$. The magnetic modes have[25] $\vec{\mathrm{B}} = j_l(\omega r) \times\vec{\mathrm{X}}_{lm}e^{-i\omega t}$. By (A4) the $\vec{\mathrm{E}}$ field is proportional to the curl of this expression. Because $\vec{\mathrm{X}}_{lm}$ has only θ and ϕ components, $\vec{\mathrm{E}}_t$ can vanish on the surface only if the radial derivative of j_l vanishes. Thus the spectrum for the magnetic modes is

$$\omega_{nl} = j'_{n,l}R^{-1}\,,\quad n=1,2,\ldots;\quad l=1,2,\ldots\,, \tag{A5}$$

where $j'_{n,l}$ is the nth positive root of $j'_l(x)$; the degeneracy is $2l+1$. The lowest eigenfrequency is $\omega_{11}=2.082/R$.

The free neutrino field is described by the Weyl equation.[26] Since the neutrino is purely a negative-helicity particle, for a field with time dependence $\exp(-i\omega t)$ the equation reads

$$-\vec\sigma\cdot\vec{\mathrm{p}}\psi = \omega\psi\,, \tag{A6}$$

where $\vec\sigma$ is the triplet of Pauli matrices, ψ is the neutrino bispinor, and $\vec{\mathrm{p}}$ is the momentum operator $-i\vec\nabla$. One may introduce the operators for orbital angular momentum L^2 and total angular momentum J^2 and J^z. Their joint eigenfunctions are the spinorial spherical harmonics ${}_J\mathcal{Y}_L^M$ with eigenvalues $L(L+1)$, $J(J+1)$, and M for the three operators, respectively. One then has[27]

$$\vec\sigma\cdot\vec{\mathrm{p}} = -i\sigma^r r^{-1}[(\partial/\partial r)r - (J^2+\tfrac{1}{4}-L^2)]\,, \tag{A7}$$

where $\sigma^r = \vec\sigma\cdot\vec{\mathrm{r}}$; σ^r has the property[27]

$$\sigma^r\,{}_J\mathcal{Y}^M_{J\pm1/2} = -\,{}_J\mathcal{Y}^M_{J\mp1/2}\,. \tag{A8}$$

With the help of (A7) and (A8), as well as the relations[21]

$$j'_{J+1/2} = j_{J-1/2} - (J+\tfrac{3}{2})x^{-1}j_{J+1/2}\,, \tag{A9}$$

$$j'_{J-1/2} = -j_{J+1/2} + (J - \tfrac{1}{2})x^{-1} j_{J-1/2} \tag{A10}$$

satisfied by the spherical Bessel function $j_l(x)$, one verifies that

$$\psi = [j_{J+1/2}(\omega r)\, {}_J\mathcal{Y}^M_{J+1/2} - i j_{J-1/2}(\omega r)\, {}_J\mathcal{Y}^M_{J-1/2}] e^{-i\omega t} \tag{A11}$$

is a solution of (A6) (evidently $J \geq \frac{1}{2}$). Since (A11) is an eigenfunction of J^2, J^z, and the helicity $\omega^{-1}\vec{\sigma}\cdot p$ [eigenvalues $J(J+1)$, M, and -1, respectively], it is just the right mode function for the problem. The only thing missing is the spectrum of ω. To get it one must prescribe boundary conditions, and this is where the difficulty lies. One cannot demand $\psi = 0$ or $\hat{n}\cdot(\vec{\nabla}\psi) = 0$ on the boundary; the solution (A11) does not satisfy either condition essentially because the two spinorial harmonics are idependent. Thus traditional boundary conditions do not work. This is the well-known difficulty in posing the boundary conditions required to define a discrete spectrum for neutrino eigenfrequencies in a cavity.

We argue that the difficulty has its root in the fermionic nature of ψ; one should not impose boundary conditions on ψ as one does for a bosonic field (i.e., scalar). Rather, one should impose conditions on a bilinear expression in ψ, for only such a quantity is an observable. For example, one might require the normal component of the neutrino number current J^μ to vanish at the boundary (cavity material impermeable to neutrinos). In standard notation[26]

$$J^\mu = \psi^\dagger \tilde{\sigma} s^\mu \psi\,, \tag{A12}$$

where $\psi^\dagger$ is the Hermitian conjugate to ψ, s^μ is a four vector of 2×2 matrices whose anticommutator is twice the metric and $\tilde{\sigma}$ is a "Hermitizing" matrix. One may choose $\tilde{\sigma}$ and s^μ in such a way[26] that the space part of $\tilde{\sigma}s^\mu$ is $\vec{\sigma}$ while the time part is +1. Thus the radial component we require is just $\psi^\dagger\sigma^r\psi$. Substituting (A11) we have

$$\begin{aligned} J^r = &-[j_{J+1/2}{}^2\, {}_J\mathcal{Y}^{M\dagger}_{J+1/2}\, {}_J\mathcal{Y}^M_{J-1/2} + (J+\tfrac{1}{2} \leftrightarrow J-\tfrac{1}{2})] \\ &+ i j_{J+1/2} j_{J-1/2} [{}_J\mathcal{Y}^{M\dagger}_{J+1/2}\, {}_J\mathcal{Y}^M_{J+1/2} - (J+\tfrac{1}{2} \to J-\tfrac{1}{2})]\,. \end{aligned} \tag{A13}$$

Evidently J^r must be real. It is obvious that the two angular terms in the last square brackets are real. Hence the second line in (A13) is pure imaginary. Its radial dependence is different from that of the first line, so it cannot be canceled by it. Hence the second square brackets in (A13) must vanish identically. By the same token, the two angular factors in the first square brackets (each other's complex conjugates) must be real (hence equal) since they are multiplied by different radial functions. Hence

$$J^r = -(j_{J+1/2}{}^2 + j_{J-1/2}{}^2)\, {}_J\mathcal{Y}^{M\dagger}_{J+1/2}\, {}_J\mathcal{Y}^M_{J-1/2}\,. \tag{A14}$$

Since the zeros of different j_l do not coincide, we see that J^r cannot vanish on a spherical boundary (although its angular integral does). Thus our presumed boundary condition is also inadequate.

According to Unruh,[28] the fact that J^r cannot be made to vanish at the boundary reflects a Klein-type paradox: If the cavity material is impermeable to neutrinos, it will create neutrino-antineutrino pairs and so defeat the boundary condition. Apparently for the same reasons, the condition that the normal component of the energy current vanish on the boundary fails also. However, a related, somewhat mysterious condition works.

From the four-spinor expression for the Dirac stress-energy tensor,[26] one may easily compute the energy flux for a fictitious *right*-handed neutrino field in terms of the appropriate bispinor ϕ:

$$\vec{P} = -\tfrac{1}{2}\omega\phi^\dagger\vec{\sigma}\phi + \tfrac{1}{4}i(\phi^\dagger\vec{\nabla}\phi - \vec{\nabla}\phi^\dagger\phi)\,. \tag{A15}$$

The boundary condition requires us to formally replace ϕ by the *left*-handed ψ and set the normal component of $\vec{P}$ to zero on the boundary:

$$-P^r = \tfrac{1}{2}\omega J^r - \tfrac{1}{4}i(\psi^\dagger\psi_{,r} - \psi^\dagger_{,r}\psi) = 0\,. \tag{A16}$$

Putting (A11) and (A14) into (A16), canceling terms, and taking account of the realness of the angular factor appearing in (A14) gives as boundary condition that at $r=R$,

$$j_{J+1/2}{}^2 + j_{J-1/2}{}^2 + j'_{J-1/2} j_{J+1/2} - j'_{J+1/2} j_{J-1/2} = 0\,. \tag{A17}$$

In view of (A9) and (A10) it now follows that

$$(J+1) j_{J+1/2}(\omega R) j_{J-1/2}(\omega R) = 0\,. \tag{A18}$$

Since $J \geq \frac{1}{2}$, we see that the allowed ωR are the positive zeros of either of the j_l in (A18). The neutrino eigenfrequency spectrum is then

$$\omega_{n\epsilon J} = j_{n,\,J+\epsilon} R^{-1}\,, \tag{A19}$$

where $\epsilon = \pm\frac{1}{2}$. These eigenvalues are $2J+1$ degenerate. In addition, half of the eigenvalues for given J ($\epsilon = +\frac{1}{2}$) are degenerate with half of the eigenvalues for $J+1$ ($\epsilon = -\frac{1}{2}$).

Thus far we have ignored the possibility of zero eigenfrequencies. We now show there are none in our problem. For $\omega = 0$ the scalar equation (A11) reduces to Laplace's equation. The nonsingular solutions have the form $\phi = r^l Y_{lm}(\theta,\phi)$; none vanish at $r=R$. Thus $\omega = 0$ is not an eigenfrequency. For $\omega = 0$, Maxwell's Eqs. (A3) and (A4) imply E and B are each the gradient of a different potential. The divergence equations then imply that the potentials solve Laplace's equation. The tangential derivative of the potential $r^l Y_{lm}$ cannot vanish at $r=R$. Hence no E field satisfying the boundary conditions for a perfectly conducting cavity exists for $\omega = 0$. The $\vec{B}$ field is uncoupled to $\vec{E}$; hence one must (for $\omega = 0$ only) specify boundary conditions for $\vec{B}$. It is simplest to imagine the cavity material as superconducting. It then excludes the $\vec{B}$ field; thus the normal component of $\vec{B}$ must vanish at the cavity boundary. But the radial derivative of $r^l Y_{lm}$ cannot vanish at $r=R$ unless $\vec{B}$ vanishes identically. Hence $\omega=0$ is not an electromagnetic eigenfrequency. An analysis of the Weyl equation along the lines of the one given above shows the regular solutions for $\omega=0$ to be of the form $\psi=r^{J-1/2}\,{}_J\mathcal{Y}^M_{J-1/2}$. Only that with $J=\frac{1}{2}$ satisfies the boundary condition and this only because P^r vanishes identically. Further, such a solution does not have negative helicity: $\vec{\sigma}\cdot\vec{p}$ operating on it gives zero. Hence we would not include the associated eigenfrequency in the mode sum (16). Thus, in the context of this paper, $\omega=0$ is not an eigenfrequency.

APPENDIX B

Here we show by example that for large p various ζ functions of interest are well approximated by their ground-state parts. For a spectrum ω_i with degeneracy g_i,

$$\zeta(p)=\sum_{i=1}^{\infty} g_i \omega_i^{-p}. \tag{B1}$$

The ground-state contribution is $g_1\omega_1^{-p}$. Evidently as $p\to\infty$, $\zeta(p)\to g_1\omega_1^{-p}$. But as we shall see, already for $p=10$ the approximation is adequate for the purposes of Sec. V. We first consider fields in an Einstein universe of radius a.

For the conformal scalar field[13] $\omega_i a=i$, $g_i=i^2$ with $i=1,2,\ldots$. Therefore $\zeta(p)=a^p\zeta_R(p-2)$, where ζ_R denotes the Riemann ζ function. Thus $\zeta(10)=1.004\omega_1^{-10}$. For the electromagnetic field,[12,29] $\omega_i a=i+1$, $g_i=2i(i+1)$ with $i=1,2,\ldots$. Thus $\zeta(p)=2a^p[\zeta_R(p-2)-\zeta_R(p)]$ and $\zeta(10)=1.05\times 6\omega_1^{-10}$. For the two-component neutrino field[13,30] $\omega_i a=i+\frac{1}{2}$, $g_i=2i(i+1)$ with $i=1,2,\ldots$. Thus[21]

$$\zeta(p)=\tfrac{1}{2}a^p[(2^p-4)\zeta_R(p-2)-(2^p-1)\zeta_R(p)],$$

so that $\zeta(10)=1.02\times 4\omega_1^{-10}$. Of interest in Sec. V is $\zeta(p)^{1/(1+p)}$. For $p=10$ the fractional errors incurred in this quantity by replacing any of the above ζ by $g_1\omega_1^{-p}$ are less than $\frac{1}{2}$ percent.

Let us now consider the various fields in a sphere of radius R. According to Appendix A, for the scalar field

$$\zeta(p)=R^p\sum_{l=0}^{\infty}(2l+1)\sum_{n=1}^{\infty} j_{n,l}^{-p}. \tag{B2}$$

For the neutrino field ζ is just twice this expression. A good approximation to these $\zeta(p)$ is obtained by summing only over the eigenvalues up to five times larger than $\omega_1=\pi/R$ (about 220 ω's). One finds $\zeta(10)\simeq 1.10\omega_1^{-10}$ for both the scalar and neutrino fields. For the electromagnetic field, $j_{n,l}+j'_{n,l}$ replaces $j_{n,l}$ in (B2). A good approximation to this $\zeta(p)$ may be obtained by summing over the eigenvalues up to five times larger than $\omega_1=2.082/R$ (about 140 ω's). One finds $\zeta(10)\simeq 3.33\times 3\omega_1^{-10}$. In all these cases the fractional errors incurred in $\zeta(p)^{1/(1+p)}$ by replacing $\zeta(p)$ by $g_1\omega_1^{-p}$ are 1–10% for $p=10$.

APPENDIX C

Here we establish lower bounds for the lowest eigenfrequencies of the scalar, electromagnetic, and neutrino fields in an odd-shaped cavity.[31] Let C denote the cavity's interior, and ∂C its boundary. Imagine C is circumscribed by a sphere of radius R. Let the sphere's interior be denoted by S, and its boundary by ∂S. Finally, let $\overline{C}$ denote the part of S outside C.

Let ϕ_1 be the normalized eigenfunction of the scalar equation in C which corresponds to the lowest eigenfrequency ω_1. We assume the boundary condition $\phi=0$ on ∂C. We now consider a test function $\tilde{\phi}$ which equals ϕ_1 in C and vanishes in $\overline{C}$. Evidently $\tilde{\phi}$ is normalized in S, obeys the boundary condition on ∂S, and is continuous throughout S. Thus by the Rayleigh-Ritz principle, as applied to Eq. (A1), we can set an upper bound on the lowest eigenfrequency for the scalar field in S, which we know (Appendix A) to be $\omega=\pi/R$:

$$-\int_S \tilde{\phi}\nabla^2\tilde{\phi}\,dV>\pi^2/R^2. \tag{C1}$$

Now the normal component of $\vec{\nabla}\tilde{\phi}$ has a step discontinuity at ∂C, so $\nabla^2\tilde{\phi}$ has a δ-function singularity there. However, $\nabla^2\tilde{\phi}$ in (C1) appears multiplied by $\tilde{\phi}$ which vanishes at ∂C. Hence the singularity does not contribute to the integral. Neither does the region $\overline{C}$ where $\tilde{\phi}\equiv 0$. The contribution from C follows because by (A1) $\nabla^2\tilde{\phi}=-\omega_1^2\tilde{\phi}$ in C. Hence

$$\omega_1>\pi/R. \tag{C2}$$

In dealing with the electromagnetic field it proves convenient to write Maxwell's Eqs. (A3) and (A4) as

$$\omega F=\omega\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}=\begin{pmatrix}0 & i\vec{\nabla}\times\\ -i\vec{\nabla}\times & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}=MF. \tag{C3}$$

Let F_1 denote a solution of (C3) in C which obeys the boundary condition $\vec{E}_t=0$ on ∂C (conducting cavity) and corresponds to the lowest eigenfrequency ω_1. We assume it is normalized: $\int_C F_1^T\cdot F_1\,dV=\int_C(E^2+B^2)dV=1$. As before, we define a test field $\tilde{F}$ as coinciding with F_1 in C and vanishing in $\overline{C}$. Evidently $\tilde{F}$ is normalized in $\mathring{S}$ and obeys the boundary condition on ∂S. But in general it has step discontinuities in some of its components across ∂C. However, being piecewise continuous in S, it can be expanded in the eigensolutions of (C3) for S, and so a straightforward generalization of the Rayleigh-Ritz principle applies:

$$\int_S \tilde{F}^T \cdot M\tilde{F} dV > j_{1,1}/R\,, \tag{C4}$$

where $j_{1,1}/R$ is the lowest eigenfrequency in S (see Appendix A).

Now

$$\tilde{F}^T \cdot M\tilde{F} = i\vec{\tilde{E}} \cdot \vec{\nabla} \times \vec{\tilde{B}} - i\vec{\tilde{B}} \cdot \vec{\nabla} \times \vec{\tilde{E}}\,. \tag{C5}$$

Due to discontinuities in $\tilde{F}$ across ∂C there could appear δ-function contributions to (C5) which would then contribute to its integral in (C4). We show this does not occur. Evidently only derivatives of $\vec{\tilde{E}}$ or $\vec{\tilde{B}}$ normal to ∂C can generate δ functions. Since $\vec{\tilde{E}}_t$ is continuous across ∂C, $\vec{\nabla} \times \vec{\tilde{E}}$ does not have a singularity there. Now $\vec{\nabla} \times \vec{\tilde{B}}$ could have one, but it would occur only in the components tangential to ∂C. However, $\vec{\tilde{E}}_t = 0$ there, so $\vec{\tilde{E}} \cdot \vec{\nabla} \times \vec{\tilde{B}}$ cannot contribute to the integral right at ∂C. Neither can $\vec{\tilde{B}} \cdot \vec{\nabla} \times \vec{\tilde{E}}$ which, as we saw, does not have singularities. Thus only the integral over C contributes in (C4); it can be evaluated from (C3) and we get immediately

$$\omega_1 > j_{1,1}/R\,. \tag{C6}$$

We now turn to the neutrino field. Let ψ_1 be a solution of (A6) in C which obeys the boundary condition $P^n = 0$ (n for normal) at ∂C and which correspond to the lowest eigenfrequency ω_1 for the region C. Again we define a test field $\tilde{\psi}$ which coincides with ψ_1 in C and vanishes in $\bar{C}$. Evidently $\tilde{\psi}$ obeys the boundary condition at ∂S rather trivially. We normalize ψ_1 by $\int_C \psi_1^\dagger \psi_1 dV = 1$; then $\tilde{\psi}$ is normalized in S. Since $\tilde{\psi}$ will have only a step discontinuity across ∂C, it can be expanded in eigensolutions for S and one can generalize the Rayleigh-Ritz principle to apply to the neutrino equation (A6). It gives

$$i \int_S \tilde{\psi}^\dagger \vec{\sigma} \cdot \vec{\nabla} \tilde{\psi}\, dV > \pi/R\,, \tag{C7}$$

where π/R is the lowest eigenfrequency for the neutrino in S (see Appendix A).

Of interest here is whether the δ-function singularity that arises in $\vec{\nabla}\tilde{\psi}$ from the jump of $\tilde{\psi}$ across ∂C contributes to the integral. We show it does not. If $\vec{n}$ denotes the outward normal to ∂C, then clearly

$$\vec{\nabla}\tilde{\psi} = -\psi_{1B}\delta(\partial C)\vec{n} + \cdots\,, \tag{C8}$$

where $\delta(\partial C)$ denotes a δ function in the coordinate running normal to ∂C, ψ_{1B} is ψ_1 at ∂C, and the additional terms $(\cdots)$ are tangential components with no δ functions. Hence the contribution from ∂C to the integral is

$$K = -i \int_{\partial C} \psi_{1B}^\dagger \sigma^n \psi_{1B}\, d\Sigma\,, \tag{C9}$$

where $\sigma^n = \vec{\sigma} \cdot \vec{n}$ and $d\Sigma$ is the element of area on ∂C. We shall now write down explicitly the boundary condition for ψ_1, namely $P^n = 0$, at ∂C by making use of (A15):

$$-i\psi_{1B}^\dagger \sigma^n \psi_{1B} + \tfrac{1}{2}\omega_1^{-1}(\vec{\nabla}\psi_{1B}^\dagger \psi_{1B} - \psi_{1B}^\dagger \vec{\nabla}\psi_{1B}) \cdot \vec{n} = 0\,. \tag{C10}$$

Substituting this into (C9) and making use of Gauss's theorem it follows that

$$K = -\tfrac{1}{2}\omega_1^{-1} \sum_C (\nabla^2 \psi_1^\dagger \psi_1 - \psi_1^\dagger \nabla^2 \psi_1) dV\,. \tag{C11}$$

However, by iterating the neutrino equation (A6) one finds $\nabla^2\psi_1 = -\omega_1^2\psi_1$ and a like equation for $\psi_1^\dagger$. Hence the integrand in (C11) vanishes identically.

Thus only C contributes to the integral in (C7). By applying (A6) one has $i\vec{\sigma} \cdot \vec{\nabla}\tilde{\psi} = \omega_1 \tilde{\psi}$ in C. Thus for neutrinos the bound for ω_1 is again given by (C2). We thus have lower bounds for the lowest eigenfrequencies of the three fields in terms of the effective radius R of the cavity (radius of circumscribing sphere). One can prove exactly analogous *upper* bounds for ω_1 in terms of the radius of the sphere *inscribed* in C.

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[22]By "conspiracy" we mean that we are close to the equality limits of *all* inequalities used to get (35). The exponent 12.8 is just the p_c corresponding to $\xi = 10^{-6}$.

[23]In fact, by using (29) with $p = 10$ we have for the electromagnetic field in a spherical cavity $1.74\,R < (S/\overline{E})_{max} < 2.96\,R$.

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