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Path-integral evaluation of Feynman propagator in curved spacetime

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We develop an efficient approximation procedure for evaluating the scalar Feynman propagator in arbitrary spacetimes. In the familiar manner we represent it by an integral over the transition amplitude for a Schrödinger-type equation (proper-time method). The amplitude is then represented by a Feynman path integral which is dominated by the contribution of a certain extremal path. The contributions of adjacent paths are then simply expressed by working in Fermi normal coordinates based on the extremal path. In this manner the path integral becomes an ordinary multiple integral over "Fourier coefficients" which represent the various paths. For a conformal field, or for spacetimes with constant scalar curvature, we evaluate the integral in the Gaussian approximation in terms of the curvature along the (geodesic) extremal path. We show the result to be related to the Schwinger-DeWitt expansion for the amplitude, but valid for well-separated end points. In the Einstein universe our expression gives the exact amplitude and propagator. In the de Sitter spacetime it gives a good approximation for the amplitude even for well-separated points. We also evaluate the post-Gaussian corrections to the amplitude, though we do not implement them in a concrete spacetime. For nonconformal fields in spacetimes with varying scalar curvature, we evaluate the amplitude in the Gaussian approximation in terms of the values of the curvature along the extremal (nongeodesic) path. It is very different in form from the one mentioned earlier, which suggests the existence of novel effects arising from variation in the scalar curvature.

I. INTRODUCTION

The Feynman propagator or Green's function is of fundamental importance in any quantum field theory in curved spacetime.[1,2] From its behavior as its two spacetime arguments merge (coincidence limit) one can define a regularization procedure for the theory, and obtain such quantities as the vacuum expectation value of the stress-energy tensor, the vacuum energy, the trace anomaly, etc. In addition, the behavior of the Green's function for separated spacetime arguments contains information about elementary processes such as pair production by the gravitational field. The Green's function is also the basic building block when one generalizes the theory to include nongravitational interactions.

Although each field has its own Green's function, those for higher-spin fields are closely related to the Green's function for a scalar field, and methods applicable to it can be generalized to higher spin by straightforward procedures.[3] The scalar Green's function satisfies the equation (signature +2, $R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} + \cdots$, $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$, $R = R^{\alpha}{}_{\alpha}$)

$$(-\nabla^{\mu}\nabla_{\mu} + m^2 + \xi R)G_F(x,x') = [-g(x)]^{-1/2}\delta(x,x') \tag{1.1}$$

with m the mass of the field, and ξ a constant. The coupling to R is included here both to allow for study of the conformally invariant equation, and because analogous terms appear in the equations used to generate higher-spin Green's functions.[3] The Feynman Green's function is traditionally singled out from among the solutions of (1.1) by the requirement that it propagate positive- (negative-) frequency fields forward (backward) in time. This defining boundary condition is not easily interpreted in curved spacetime, and instead one usually invokes analytic continuation of G_F from the Euclidean signature, or the proper-time (Schwinger-DeWitt) formalism. We shall employ the latter approach, but our technique could be adapted to the former procedure.

The scalar Feynman Green's function is known in flat spacetime, and has been calculated under various boundary conditions for a handful of spacetimes, including the (static) Einstein universe, the de Sitter universe, and a few other examples.[4-10] It has proved difficult to calculate in less special spacetimes, so the information it contains has hardly been tapped. Evidently any new general method for computing G_F, even approximately, would represent a welcome addition to the tools being used to understand quantum processes in curved spacetime.

No better starting point for an approximation scheme offers itself than the Schwinger-DeWitt proper-time formalism.[1-3,11] In this method, one replaces our problem by that of solving the Schrödinger equation[12]

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = (-\nabla^{\mu}\nabla_{\mu} + \xi R)\langle x, s|x', 0\rangle \tag{1.2}$$

subject to the boundary condition

$$\lim_{s\to 0}\langle x, s|x'0\rangle = |g(x)|^{-1/2}\delta(x,x'). \tag{1.3}$$

The kernel $\langle x, s|x', 0\rangle$ is formally the amplitude for a particle coupled to the curvature to propagate from spacetime point x' to x in the course of a *fictitious* proper-time inverval s. The Green's function is recovered by

$$G_F(x, x') = i\int_0^\infty \langle x, s|x', 0\rangle e^{-im^2 s} ds, \qquad (1.4)$$

where one takes m^2 to have an infinitesimal negative imaginary part. With this boundary condition (1.4) represents the Feynman Green's function rather than some other solution of (1.1).[3] The kernel is also useful in calculating the vacuum-to-vacuum amplitude, a fundamental quantity.[1,2,11] Therefore, most of our attention will be devoted to methods for calculating $\langle x, s|x', 0\rangle$ in various contexts.

A direct way to solve for $\langle x, s|x', 0\rangle$ is by a power series in s, the so-called Schwinger-DeWitt expansion. Let $\sigma(x, x') = \sigma(x', x)$ stand for half the proper distance squared (or minus half the proper time squared) along the spacelike (timelike) geodesic between x' and x. One defines the Van Vleck-Morette determinant

$$D_{\rm VM} = \mathrm{Det}(-\partial^2\sigma/\partial x^\mu \partial x^{\nu'}) \qquad (1.5)$$

(determinant of a 4×4 matrix for given x, x'). The quantity

$$\Delta(x, x') = -|g(x)|^{-1/2} D_{\rm VM} |g(x')|^{-1/2} \qquad (1.6)$$

is a biscalar. In terms of it the solution of (1.2) and (1.3) may be written as[1-3,11]

$$\langle x, s|x', 0\rangle = i(4\pi i s)^{-2} \exp(i\sigma/2s)\Delta^{1/2}[1 + f_1 is + f_2(is)^2 + \cdots]. \qquad (1.7)$$

The f_i are functions of x and x'. This expansion is valid for small s and x and x' close to each other.[1] Consequently it has proved very useful in discussing regularization procedures, and especially in calculating the anomalous trace of the stress-energy tensor. For these applications the limits $s\to 0$ and $x\to x'$ are sufficient. For other applications (e.g., particle production, interacting fields) one is interested in x and x' being finitely separated. Then a different method for solving (1.2) and (1.3) is required.

The Feynman path-integral method comes to mind. Originally developed precisely for calculating the amplitude associated with the Schrödinger equation,[13] it is generally acknowledged to provide deep intuitive understanding of the amplitude, and has been used to infer a number of general properties. Yet most would regard it as impractical for concrete calculations. This last view is unduly pessimistic. There exists a growing literature concerned with methods for calculating path integrals for particles moving in various spaces, including Riemannian spacetimes.[14] Many conceptual obstacles have been cleared away. Yet the attention to points of mathematical rigor which characterize many recent contributions obscure the potentialities of the various approaches.

In this paper we explore a new technique for evaluating the path-integral expression for $\langle x, s|x', 0\rangle$ which employs Fermi coordinates as a device for separating neatly those features of the amplitude which depend on spacetime being curved from those which do not. This allows one to evaluate the path integral in an arbitrary curved background by a systematic *covariant* approximation scheme. The emphasis will be on developing a practical approximation scheme, rather than on points of rigor. But we shall nevertheless take up a number of issues of principle, such as the question of normalization of the amplitude for infinitesimally close points. The method developed here should also be applicable to problems outside gravitational physics (e.g., constrained motion, diffusion, etc.).

The plan of the paper is as follows. In Sec. II we write down the path-integral representation for the kernel and recast it into a phase-space form in which all coordinate and momentum dependence is in the action. The path integral is then reexpressed as a multiple integral over the Fourier coefficients of the representation of the paths and momenta in Fermi coordinates; these coordinates are introduced relative to the path which extremizes the action (there may be more than one such path). Because the path integral is dominated by paths extremizing the action, one may obtain an approximation to it good for widely separated end points by working in powers of the Fourier coefficients which effectively measure the deviation of a particular path from the extremizing one. Working to second order in the coefficients, we perform the (Gaussian) integration equivalent to the path integral for geodesic extremizing paths (Sec. III) and for nongeodesic ones (Sec. VII). Since the metric in Fermi coordinates is known in terms of the Riemann tensor, our results are in terms of Fourier transforms of the Riemann tensor along the extremizing path. A check of the Gaussian approximation is performed in Sec. IV where we use it to calculate the Feynman Green's function in the Einstein static universe; we obtain the exact Green's function. In a similar check in de Sitter spacetime (Sec. V) the approximation is found to be valid for finitely separated end points, though it is not exact here. The general expression for the kernel to fourth order in the Fourier coefficients is given in Sec. VI for the case of geodesic extremizing paths.

The Gaussian approximation revolves about a determinant formed from Fourier transforms of the Riemann tensor. In Sec. III we show that the expansion of this determinant in powers of curvature agrees with that for $\Delta(x,x')^{-1}$ as far as the latter expression is known for a general spacetime. For the Einstein and de Sitter spacetimes one can show agreement to all orders. Our determinant is thus apparently identical to Δ^{-1} and it yields a new and useful expression for the Van Vleck-Morette determinant. For example, one can readily expand our determinant to high powers in the curvature.

We conclude that already in the Gaussian approximation our approach yields expressions for $\langle x, s|x', 0\rangle$ and $G_F(x,x')$ valid for well-separated points. These approximations should find numerous applications.

We give here a brief summary of the equation numbers of the main results of the general analysis and of the examples. This should be useful to the general reader who may want to look at the main results and examples without going through the detailed calculations. The phase-space form of the path integral for $\langle x, s|x', 0\rangle$, in which all coordinate and momentum dependence appears in the action, is given in Eq. (2.7) with momenta defined in Eq. (2.4). That expression for $\langle x, s|x', 0\rangle$ is recast as a multiple integral over Fourier coefficients in Eqs. (2.19)–(2.21). For the case when $(\xi-\frac{1}{6})R$ is constant, the Gaussian approximation for $\langle x, s|x', 0\rangle$ is given in Eq. (3.15) with the matrix D defined by Eqs. (3.11) and (3.6). [As noted previously, we present strong evidence that $\mathrm{Det}D=\Delta(x,x')^{-1}$.] The expression for $G_F(x,x')$ in Gaussian approximation is given in Eq. (3.16). In the Einstein static universe, the Gaussian expression for $\langle x, s|x', 0\rangle$ is explicitly evaluated in Eq. (4.11), including the sum indicated after that equation, and G_F is given in Eqs. (4.12) and (4.13). In de Sitter spacetime, Det$D$ is directly evaluated in Eq. (5.5), $\langle x, s|x', 0\rangle$ is given in Eq. (5.11), and G_F in Eq. (5.13). The post-Gaussian expression for $\langle x, s|x', 0\rangle$ when $(\xi-\frac{1}{6})R$ is constant is given in Eqs. (6.18) and (6.19). For the important general case when $(\xi-\frac{1}{6})R$ is not constant, the Gaussian approximation for $\langle x, s|x', 0\rangle$ is given in Eq. (7.28) with $\tilde{D}$ defined in Eqs. (7.27), (7.23), and (7.19)–(7.21). An approximation for Det$\tilde{D}$ is given in Appendix C, Eq. (C9).

II. REPRESENTATION OF $\langle x,s|x',0\rangle$ IN FERMI COORDINATES

It is traditional to begin a discussion of path integrals by writing down the amplitude for infinitesimally close points and then iterating it to obtain the finite amplitude.[15,16] We find it more convenient to start with an expression for the finite amplitude shown by one of us[17,11] to obey the Schrödinger equation (1.2) and the boundary condition (1.3):

$$\langle x, s|x', 0\rangle=\int d[x(s')][\Delta^p]\exp\left[i\int_0^s ds'\left(\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-[\xi+\tfrac{1}{3}(p-1)]R(x)\right)\right], \qquad (2.1)$$

where p is an *arbitrary* dimensionless parameter. (For the meaning of $[\Delta^p]$ see below.) It was DeWitt[15] who first noticed the need to include R in the path integral even when it does not appear in the Schrödinger equation (as when $\xi=0$). He wrote down the cases $\xi=0$, $\xi=\frac{1}{3}$, and $\xi=\frac{1}{6}$ of (2.1) with $p=0$. The existence of infinitely many representations for a given ξ is a rather surprising feature. The interested reader will find a simple explanation of it in Appendix A. In our development we shall choose $p=\frac{1}{2}$ as most suitable for computational purposes.

The path integral (2.1) is given concrete meaning in the following way. Divide the interval $(0,s)$ into $N+1$ equal increments of duration ϵ. We assume N is large and even. Then

$$\langle x, s|x', 0\rangle=\lim_{N\to\infty}\left[\frac{1}{i}\left(\frac{1}{4\pi i\epsilon}\right)^2\right]^{N+1}\int\prod_{n=1}^{N}d^4x_n[-g(x_n)]^{1/2}$$
$$\times\exp\left\{\sum_{m=0}^{N}\left[i\int_{m\epsilon}^{(m+1)\epsilon}\left(\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-[\xi+\tfrac{1}{3}(p-1)]R\right)ds'+p\ln\Delta(x_m,x_{m+1})\right]\right\}, \qquad (2.2)$$

where $g=\det(g_{\alpha\beta})$, $x_0=x'$, and $x_{N+1}=x$. The integral in (2.2) is along the geodesic between x_m and x_{m+1}; similarly, $\Delta(x_m,x_{m+1})$ refers to that same geodesic. If, despite the assumed smallness of $\epsilon=s/(N+1)$, there are several geodesics between x_m and x_{m+1} (compact space), we agree to use the shortest. The repesentation (2.2) is patterned after Feynman's original expression in nonrelativistic quantum mechanics.[13] One describes a generic path by a series of geodesic segments. The amplitude associated with each segment is the imaginary exponential of its action. One introduces a normalization factor $(4\pi i\epsilon)^{-1/2}$ for each coordinate interval of each segment. (The seeming discrepancy by a factor of 2 here is the result of our choice of units.[12]) Since the factor $p\ln\Delta$ can be traded off for a curvature term in the action (see Appendix A), it cannot here be regarded as a normalization factor. This would seem to bring (2.2) in conflict with formalisms which regard $\Delta^{1/2}(4\pi i\epsilon)^{-2}$ as the normalization factor per segment.[15,16,18] But as we shall see in Sec. III, the $\Delta^{1/2}$ automatically arises from the path integration. All formalisms then agree on the presence of $\Delta^{1/2}$, but ours recognizes it as a more fundamental entity connected with the self-consistency of the representation (2.2) at all scales.

The appearance of $(-g)^{1/2}$ in (2.2) is understandable; it is the natural measure for summing over paths by allowing each of the points which define the geodesic segments to sweep all over spacetime; in conjunction with the Δ^p factors it assures invariance of the path integral.[11] Yet, because it depends on the coordinates, it is an awkward factor when one comes to evaluate the path integral. We therefore adopt the following strategy. First choose $p=\frac{1}{2}$ in Eq. (2.2). Then transform the spacetime integrals in (2.2) to phase-space integrals with subsequent absorption of $(-g)^{1/2}$ into the transformation's Jacobian. To understand the details of this procedure one must be familiar with two alternative descriptions of geodesics in curved spacetime.

Consider the geodesic segment from $x_m{}^\mu$ to $x_{m+1}{}^\mu$ (more briefly x_m and x_{m+1}); it can be represented as $x^\mu(s')$ with $s'=0$ at x_m and $s'=\epsilon$ at x_{m+1} (ϵ need not be small for the following argument). The customary action for the geodesic segment may be expressed as[3]

$$S_0=\int_0^\epsilon \tfrac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}ds'=\epsilon^{-1}\sigma(x,x')\,, \qquad (2.3)$$

where σ is defined in Sec. I. As usual in mechanics, the formal momentum is just the gradient of S_0. Alternatively, from the Lagrangian in Eq. (2.3) it may be written as $g_{\mu\alpha}dx^\alpha/ds'$. Hence at event $x_m{}^\mu$,

$$p_{m\mu}\equiv\left(g_{\mu\alpha}\frac{dx^\alpha}{ds'}\right)_{x_m}=-\epsilon^{-1}\partial\sigma(x_m,x_{m+1})/\partial x_m{}^\mu\,, \qquad (2.4)$$

where the minus sign appears because x_m is the initial point of the segment. One may now construct the 4×4 matrix $\partial p_{m\mu}/\partial x_{m+1}{}^\nu$. It view of Eqs. (1.5), (1.6), and (2.4), we have

$$\mathrm{Det}[\partial p_{m\mu}/\partial x_{m+1}{}^\nu]=-\epsilon^{-4}|g(x_m)|^{1/2}\Delta(x_m,x_{m+1})|g(x_{m+1})|^{1/2}\,. \qquad (2.5)$$

The determinant in Eq. (2.5) is just the Jacobian of the transformation from (x_m, x_{m+1}) to (x_m, p_m): it changes the description from one in which the segment is specified by the positions of the initial and final events to one in which the segment is specified by the initial position and velocity.[3]

Keeping this in mind, one may transform in Eq. (2.2) with $p=\frac{1}{2}$ the integrations at $n=2,4,\ldots,N$ to d^4p integrations at $n=1,3,\ldots,N-1$ (here we are requiring N to be even). The appropriate Jacobians are almost completely supplied by the factors of $\Delta^{1/2}$ and $|g|^{1/2}$. For example, one has

$$\begin{aligned}[\epsilon^{-2}d^4x_1|g(x_1)|^{1/2}]\Delta^{1/2}(x_1,x_2)[\epsilon^{-2}d^4x_2|g(x_2)|^{1/2}]&=\Delta^{-1/2}(x_1,x_2)\,|\,\mathrm{Det}(\partial p_{1\mu}/\partial x_2{}^\nu)|\,d^4x_1d^4x_2\\&=\Delta^{-1/2}(x_1,x_2)d^4x_1d^4p_1\,,\end{aligned}$$

where we used the fact that for the small segment $\Delta(x_1,x_2)$ is positive. We find that

$$\langle x,s|x',0\rangle=\lim_{N\to\infty}\left(\frac{N+1}{s}\right)^2\left[\frac{1}{i}\left(\frac{1}{4\pi i}\right)^2\right]^{N+1}\int\prod_{n=1,3}^{N-1}d^4x_nd^4p_n\Delta^{1/2}(x_0,x_1)\frac{\Delta^{1/2}(x_{n+1},x_{n+2})}{\Delta^{1/2}(x_n,x_{n+1})}\times\exp\left\{\sum_{n=0}^{N}i\int_{m\epsilon}^{(m+1)\epsilon}\left[\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-(\xi-\tfrac{1}{6})R\right]ds'\right\}. \qquad (2.6)$$

Almost all factors of ϵ^{-1} appearing in Eq. (2.2) (with $p=\frac{1}{2}$) have been absorbed in the Jacobians. There remains one factor of $\epsilon^{-2}=(N+1)^2s^{-2}$, which is responsible for giving the amplitude an overall factor of s^{-2}. In Eq. (2.6) one has a sequence of phase-space integrals, one over *every other* point specifying the broken path. This is equivalent to the original form involving spacetime integrations at *every* point specifying the broken path.

As explained in Appendix B, an expression like $\Delta^{1/2}(x_n,x_{n+1})$ has the form $1+O(\epsilon^2)$. For this reason the lone factor $\Delta^{1/2}(x_0,x_1)$ in Eq. (2.6) can be replaced by unity as $N\to\infty$. One cannot similarly deal with the product of $N/2$ ratios of $\Delta^{1/2}$ appearing in Eq. (2.6). It is true that taken over a single path the product is $1+O(\epsilon)$ (recall $N\epsilon$ is finite in the limit). But one is summing over many different paths so that the $O(\epsilon)$ term could build up to a finite quantity in the limit. This is precisely the reason why one cannot replace the product of $\Delta^{1/2}$ in our original expression for the path integral (2.2) by 1. However, as shown in detail in Appendix B, because of a certain cancellation in the ratio of $\Delta^{1/2}$'s the troublesome $O(\epsilon)$ term has just such a form that it averages to zero over all paths rather than building up. For this reason one can simply ignore the $\Delta^{1/2}$'s in Eq. (2.6) from here on. One has

$$\langle x, s\,|x', 0\rangle = \lim_{N\to\infty}\left(\frac{N+1}{s}\right)^2\left[\frac{1}{i}\left(\frac{1}{4\pi i}\right)^2\right]^{N+1}\int\prod_{n=1,3}^{N-1} d^4x_n d^4p_n \exp\left\{\sum_{m=0}^{N} i\int_{m\epsilon}^{(m+1)\epsilon}\left[\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} - (\xi-\tfrac{1}{6})R\right]ds'\right\}. \quad (2.7)$$

The advantage of our transformation is, then, that it eliminates factors of $\Delta^{1/2}$ and g from the path integral.

In the limit $N\to\infty$, the path integral should be dominated by the contribution of that path which extremizes the factor in the exponential (the effective or "classical" action).[13] This "classical path" contribution is known to yield important features of the amplitude. It is then natural to develop an approximation scheme in which one computes in a systematic way the contributions from paths close to the classical one. ("Classical" here does *not* mean "in the limit as $\hbar$ vanishes" because the R term we are including involves $\hbar$.[17]) We shall develop such a scheme by using Fermi normal coordinates built about the classical or extremal path. In such coordinates the metric in regions traversed by the paths of interest will be as close to the Minkowski metric as permitted by the curvature. This results in computational simplification and offers simple ways to resolve various ambiguities by comparison with the known flat spacetime results.

Consider an arbitrary timelike (spacelike) curve in spacetime described by $x^\mu = x_c^\mu(s')$ where s' is a parameter. Fermi coordinates based on it are defined as follows.[19,20] The coordinates of a point on the curve are $x_c^\mu = (\tau, 0, 0, 0)$, where τ is the proper time (distance) measured along $x_c^\mu(s')$ from its origin to the point in question. Let us label by e_0^μ the tangent to x_c^μ: thus $e_0^\mu = dx_c^\mu/d\tau$. Pick a triad of unit orthonormal vectors e_i^μ which are also orthogonal to e_0^μ. We will have $e_i^\mu e_{j\mu} = \mathrm{diag}(\pm 1, 1, 1)$ where upper (lower) sign applies for a timelike (spacelike) curve. The coordinates of a point x^μ off the curve are defined as follows. Consider the geodesic orthogonal to x_c^μ which goes through x^μ. Let its unit tangent at the point where it intersects the curve x_c^μ be expressed as $\alpha^i e_i^\mu$; evidently $\pm(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 = \pm 1$. The sign on the left-hand side depends on the curve x_c^μ, while the sign on the right-hand side tells us whether the geodesic is spacelike or timelike (possible only if the curve x_c^μ is spacelike). Then $x^\mu = (\tau, \alpha^1 z, \alpha^2 z, \alpha^3 z)$ where τ is again the proper time (distance) to the foot of the geodesic, while z is the proper distance (time) along the geodesic from the curve x_c^μ to x^μ. The coordinates are well defined over the region where geodesics do not cross (no caustics).

We come now to specifying the paths which are summed over in Eq. (2.7). The classical path which extremizes the action is just the curve $x_c^\mu(s')$ on which the Fermi coordinates are based. Explicitly $x_c^\mu = (\tau(s'), 0, 0, 0)$. A path off x_c^μ is to be described in terms of its x^μ and p_μ at $s' = \epsilon, 3\epsilon, \ldots, (N-1)\epsilon$, where $\epsilon(N+1) = s$. Half of this information may be represented by

$$x^\mu(s') = x_c^\mu(x') + \delta x^\mu(s') \equiv x_c^\mu(s') + \sum_{p=1}^{N/2} a_p^\mu \sin(p\pi s'/s)\,. \quad (2.8)$$

Here the $4\times N/2$ constants a_p^μ are to be chosen in such a way that for the above mentioned values of s', $x_c^\mu + \delta x^\mu$ produces the required coordinates. To get the a's one has to solve (conceptually only) a system of $2N$ algebraic linear equations in the $2N$ unknowns a_p^μ. The solution will exist in general, except perhaps for rare values of N for which the Cramer determinant vanishes. Whenever it exists, the a's will be unique. We note that $\delta x^\mu = 0$ at $s' = 0$ or $s' = s$. Thus, as required, the path represented by Eq. (2.8) begins and ends at the same x^μ as the classical path, which is just that one with all a_p^μ vanishing. We must stress that for $s' \neq \epsilon, 3\epsilon, \ldots,$ Eq. (2.8) does not in general describe exactly the geodesic segments of which the path is made. But it should give a good average description for large N.

We now turn to the other half of the specification. We write

$$p_\mu = g_{\mu\nu}\frac{dx^\mu}{ds'} + \delta p_\mu \equiv g_{\mu\nu}[x^\mu(s')]\frac{dx^\mu(s')}{ds'} + \sum_{p=1}^{N/2} b_\mu^p \sin(p\pi s'/s)\,, \quad (2.9)$$

where $x^\mu(s')$ is given by Eq. (2.8). The term involving the b's measures the deviation of p_μ from the direction of the curve $x^\mu(s')$. The b_μ^p are to be chosen in such a way that Eq. (2.9) reproduces the values of p_μ at $s' = \epsilon, 3\epsilon, \ldots$ that one has in mind. The procedure is the same as above; one would use in dx^μ/ds' the a's obtained in the previous step. The set a_p^μ, b_μ^p then specify our path just as well as the values of x^μ, p_μ at every other point. We note that when all b's vanish, the p_μ given by Eq. (2.9) are just tangent to the curve $x^\mu(s')$ given by Eq. (2.8); in this case the original broken path lies especially close to the curve $x^\mu(s')$.

We now transform the volume element in Eq. (2.7) to one in ab space. We have

$$\prod_{n=1,3\ldots}^{N-1} d^4x_n d^4p_n = \left|\frac{\partial(x,p)}{\partial(a,b)}\right| \prod_{p=1}^{N/2} d^4a_p d^4b^p . \quad (2.10)$$

The Jacobian above is the determinant of a $4N \times 4N$ matrix. We note that $\partial p_{n\mu}/\partial a_p^\nu$ contains the metric coefficient $g_{\mu\nu}$. However, $\partial x_n^\mu/\partial b_\nu^p = 0$ identically. Because of the way a determinant is built, elements in the $\partial p/\partial a$ block of the matrix will not appear in the Jacobian—they get multiplied by elements of the $\partial x/\partial b$ block which all vanish. Hence the Jacobian is metric independent. In addition, we note that $\partial x_n^\mu/\partial a_p^\mu = \partial p_{n\mu}/\partial b_\mu^p = \sin(p\pi s_n'/s) = \sin[np\pi/(N+1)]$, while all terms not diagonal in the spacetime indices vanish. As a result the Jacobian is just the eighth power (four powers from $\partial x/\partial a$, four from $\partial p/\partial b$) of the determinant

$$Q(N) = \begin{vmatrix} \sin\frac{\pi}{N+1} & \sin\frac{2\pi}{N+1} & \sin\frac{3\pi}{N+1} & \cdots & \sin\frac{N\pi/2}{N+1} \\ \sin\frac{3\pi}{N+1} & \sin\frac{6\pi}{N+1} & & & \\ \sin\frac{5\pi}{N+1} & \sin\frac{10\pi}{N+1} & & & \\ \vdots & & & & \\ \sin\frac{(N-1)\pi}{N+1} & & & & \sin\frac{(N-1)N\pi/2}{N+1} \end{vmatrix} . \quad (2.11)$$

The important thing is that Q depends only on N, so it becomes a multiplicative factor of the path integral.

The last thing to do is to express the integral in the exponent in Eq. (2.7) in terms of a's and b's. We shall not attempt to calculate the integral along the actual broken paths; rather we shall approximate it by the integral along the interpolating paths defined by Eqs. (2.8) and (2.9). One feels that in the limit $N\to\infty$ the same result will be obtained for $\langle x, s|x', 0\rangle$. The kinetic term is

$$\sum_{m=0}^{N}\int_{m\epsilon}^{(m+1)\epsilon} \tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}ds' \cong \tfrac{1}{4}\epsilon\sum_{m=0}^{N} g_{\alpha\beta}(x_m)\frac{x_{m+1}^\alpha - x_m^\alpha}{\epsilon}\frac{x_{m+1}^\beta - x_m^\beta}{\epsilon} .$$

For m odd we have $x_m^\alpha = x^\alpha(m\epsilon)$. The point x_{m+1}^α lies on the original broken path and not on the curve $x^\mu(s')$ defined in Eq. (2.8). In the definition of $p_{m\mu}$ in Eq. (2.4), the $(dx^\alpha/ds')_{x_m}$ which appears there is the tangent to the original broken curve at x_m pointing along the segment from x_m to x_{m+1}. The function $p_\mu(s')$ in Eq. (2.9) was chosen such that $p_\mu(m\epsilon) = p_{m\mu}$. It follows that $x_{m+1}^\alpha = x_m^\alpha + \epsilon g^{\alpha\beta}(x_m)p_\beta(m\epsilon)$. Thus the above sum takes the form

$$\tfrac{1}{4}\epsilon\Bigg\{g_{\alpha\beta}(x_0)\frac{x_1^\alpha - x_0^\alpha}{\epsilon}\frac{x_1^\beta - x_0^\beta}{\epsilon} + \sum_{m=1,3\ldots}^{N-1}\Bigg[[g^{\alpha\beta}(x_m) + g_{\mu\nu}(x_{m+1})g^{\mu\alpha}(x_m)g^{\nu\beta}(x_m)]p_\alpha(m\epsilon)p_\beta(m\epsilon)$$
$$+ g_{\alpha\beta}(x_{m+1})\frac{x_{m+2}^\alpha - x_m^\alpha}{\epsilon}\left(\frac{x_{m+2}^\beta - x_m^\beta}{\epsilon} - 2g^{\beta\nu}(x_m)p_\nu(m\epsilon)\right)\Bigg]\Bigg\}.$$

Since $g_{\alpha\beta}(x_{m+1}) - g_{\alpha\beta}(x_m)$ is of $O(\epsilon)$ we may in the above replace $g_{\alpha\beta}(x_{m+1})$ by $g_{\alpha\beta}(x_m)$ while preserving the accuracy of the expression to $O(\epsilon)$. Similarly, we may replace $\epsilon^{-1}(x_{m+2}^\alpha - x_m^\alpha)$ by $2dx^\alpha/ds'$ at $s' = m\epsilon$, where dx^α/ds' refers to the curve $x^\mu(s')$ of Eq. (2.8). We then have

$$\lim_{\epsilon\to 0}\tfrac{1}{4}\sum_{m=0}^{N}\int_{m\epsilon}^{(m+1)\epsilon} g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}ds' = \tfrac{1}{4}\int_0^s g^{\alpha\beta}[x(s')]p_\alpha(s')p_\beta(s')ds'$$
$$+ \tfrac{1}{2}\int_0^s\left\{g_{\alpha\beta}[x(s')]\frac{dx^\beta(s')}{ds'} - p_\alpha(s')\right\}\frac{dx^\alpha(s')}{ds'}ds' , \quad (2.12)$$

where we used the fact that the intervals of s' in the previous expression were $\Delta s' = 2\epsilon$. A similar but much simpler argument gives

$$\lim_{\epsilon\to 0}\sum_{m=0}^{N}\int_{m\epsilon}^{(m+1)\epsilon} R\,ds' = \int_0^s R[x(s')]ds' . \qquad (2.13)$$

The functions $x^\alpha(s')$ and $p_\alpha(s')$ appearing on the right-hand side of Eqs. (2.12) and (2.13) are given by Eqs. (2.8) and (2.9). The final term in Eq. (2.12) serves to cancel the linear expression in the b's coming from the previous term.

To write the path integral in a form suitable for evaluation, we must now be more explicit about the form of the metric. In Fermi coordinates

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + h_{\alpha\beta}, \qquad (2.14)$$

where $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ for a timelike curve, and $\gamma_{\alpha\beta} = \text{diag}(1, -1, 1, 1)$ for a spacelike one; $h_{\alpha\beta}$ is calculated from the Riemann tensor at that point on the base curve $x_c^\mu(s')$ having the same time coordinate at which we are evaluating $g_{\alpha\beta}$. In terms of

$$\delta x^i = \alpha^i z = x^i - x_c^i = \sum_{p=1}^{N/2} a_p^\mu \sin(p\pi s'/s),$$

one has[19,20]

$$h_{00} = -2A_{\hat k}\delta x^k - (A_{\hat k}A_{\hat l} + R_{\hat 0\hat k\hat 0\hat l})\delta x^k\delta x^l + O(\delta x^3), \qquad (2.15)$$

$$h_{0i} = -\tfrac{2}{3}R_{\hat 0\hat k\hat i\hat l}\delta x^k\delta x^l + O(\delta x^3), \qquad (2.16)$$

$$h_{ij} = -\tfrac{1}{3}R_{\hat i\hat k\hat j\hat l}\delta x^k\delta x^l + O(\delta x^3), \qquad (2.17)$$

where $A_{\hat l} = e_l^\mu A_\mu$ and $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = e_\alpha^\mu e_\beta^\nu e_\gamma^\sigma e_\delta^\tau R_{\mu\nu\sigma\tau}$. Here $A_\mu = D(dx_{c\mu}/d\tau)/d\tau$ is the four-acceleration of $x_c^\mu(s')$. We also record the inverse metric

$$g^{\alpha\beta} = \gamma^{\alpha\beta} - \gamma^{\alpha\mu}\gamma^{\beta\nu}h_{\mu\nu} - 4\delta_0^\alpha\delta_0^\beta(A_{\hat k}\delta x^k)^2 + O(\delta x^3). \qquad (2.18)$$

Now turning back to Eq. (2.7), substituting Eqs. (2.8) and (2.9) into Eqs. (2.12) and (2.13), and using Eq. (2.10), we obtain the desired representation of $\langle x, s|x', 0\rangle$:

$$\langle x, s|x', 0\rangle = s^{-2}\lim_{N\to\infty} C_1(N)\int\prod_{p=1}^{N/2} d^4a_p d^4b^p \exp\left\{i\int_0^s [U - (\xi - \tfrac{1}{6})R]ds'\right\}, \qquad (2.19)$$

where $C_1(N)$ is a combinatoric factor depending on N alone, and

$$U = \tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} + \tfrac{1}{4}\sum_{p,q} g^{\alpha\beta}b_\alpha^p b_\beta^q \sin\frac{p\pi s'}{s}\sin\frac{q\pi s'}{s}. \qquad (2.20)$$

In this last expression $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are to be calculated from Eqs. (2.14) through (2.18) with $\delta x^i = \sum_p a_p^\mu \sin(p\pi s'/s)$. Similarly, dx^α/ds' in Eq. (2.20) is obtained from Eq. (2.8) as

$$\frac{dx^\alpha}{ds'} = \frac{dx_c^\alpha(s')}{ds'} + \frac{\pi}{s}\sum_{p=1}^{N/2} p\, a_p^\alpha \cos\frac{p\pi s'}{s}. \qquad (2.21)$$

Because $C(N)$ is independent of the curvature, its limiting value can be obtained from flat spacetime. We now proceed with the business of performing the path integral of Eq. (2.19). The case wherein the classical or extremal path from x' to x is geodesic will be treated separately in Sec. III, while the general case of an accelerated classical path will be treated in Sec. VII.

III. CASE WITH GEODESIC EXTREMAL PATH: GAUSSIAN APPROXIMATION

This case covers three interesting situations: (a) any geometry but $\xi = \frac{1}{6}$ (field conformally invariant), (b) any ξ but $R = \text{const}$ (Einstein or de Sitter spacetimes), and (c) any ξ but $R = 0$ (vacuum solutions of Einstein's equations, solutions filled with radiation only, etc.). We thus have $(\xi - \frac{1}{6})R = \text{const}$. Then the path which extremizes the classical action in (2.6) is just the geodesic linking x and x'. It satisfies

$$\frac{D}{ds'}\frac{dx_c^\mu}{ds'} = 0. \qquad (3.1)$$

The first integral of this equation implies that the proper time (distance) τ from the geodesic's starting point is proportional to s'. Thus $\tau = Ts'/s$ where T is the total proper time (distance) from x' to x. Therefore, $A^\mu = D(dx_c^\mu/d\tau)/d\tau = 0$ which much simplifies the expressions for $g_{\alpha\beta}$, Eqs. (2.14)–(2.18). Also, since τ is the only nonzero coordinate for points on $x_c^\mu(s')$,

$$\frac{dx_c^\mu(s')}{ds'} = (T/s, 0, 0, 0). \qquad (3.2)$$

Let us now evaluate (2.19) in the Gaussian approximation, that is, using the expression for U correct to second order in a's and b's. Substituting Eqs. (2.14)–(2.18) into Eq. (2.20) and making use of

$$\int_0^s \cos(p\pi s'/s)\cos(q\pi s'/s)ds' = \int_0^s \sin(p\pi s'/s)\sin(q\pi s'/s)ds' = \tfrac{1}{2}s\delta^{pq}, \qquad (3.3)$$

we can easily show that

$$\int_0^s U\,ds' = \tfrac{1}{2}(\sigma/s) + \tfrac{1}{8}s\gamma^{\alpha\beta}\sum_p b_\alpha^p b_\beta^p + (\pi^2/8s)\gamma_{\alpha\beta}\sum_p p^2 a_p^\alpha a_q^\beta - \tfrac{1}{4}\sum_{pq} A_{ij}^{pq} a_p^i a_q^j\,, \tag{3.4}$$

where $\sigma = \gamma_{00}T^2/2$ (see Sec. II) and where

$$A_{ij}^{pq} = \int_0^s (d\tau/ds')^2 R_{\hat{0}\hat{i}\hat{0}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)ds'\,. \tag{3.5}$$

For completeness we define $A_{0\mu}^{pq} = A_{\mu 0}^{pq} \equiv 0$. Since $\tau = Ts'/s$ we may rewrite Eq. (3.5) as

$$A_{ij}^{pq} = \tfrac{1}{2}(T^2/\pi s)\int_0^\pi R_{\hat{0}\hat{i}\hat{0}\hat{j}}(\tau = \xi T/\pi)[\cos(p-q)\xi - \cos(p+q)\xi]d\xi\,. \tag{3.6}$$

Thus A_{ij}^{pq} is a combination of Fourier transforms of the curvature along the extremal path.

The integrations over the b's in Eq. (2.19) are easily performed using the Gaussian integral

$$\int_{-\infty}^\infty e^{ikx^2}dx = (\pi i/k)^{1/2}\,. \tag{3.7}$$

They give an overall factor $\prod_{p=1}^{N/2}(8\pi i/s)^{3/2}(-8\pi i/s)^{1/2}$ both for timelike and spacelike curves. To perform the integrations over the a's we use the result (valid for a symmetric matrix M)[21]

$$\int_{-\infty}^\infty e^{iM_{nm}x^n x^m}\prod dx^n = [\mathrm{Det}(-iM/\pi)]^{-1/2}\,. \tag{3.8}$$

Let us identify $M_{\alpha\beta}^{pq}$ with $(\pi^2/8s)\gamma_{\alpha\beta}p^2\delta^{pq} - \tfrac{1}{4}A_{\alpha\beta}^{pq}$ where a pair $\binom{p}{\alpha}$ plays the role of an index for M. We may factor M:

$$M_{\alpha\beta}^{pq} = \sum_n C_{\alpha\gamma}^{pn} D_{n\beta}^{\gamma q}, \tag{3.9}$$

where

$$C_{\alpha\gamma}^{pn} = (8s)^{-1}\pi^2 n^2 \delta^{pn}\gamma_{\alpha\gamma} \tag{3.10}$$

and

$$D_{n\beta}^{\gamma q} = \delta_\beta^\gamma\delta_n^q - 2s\pi^{-2}n^{-2}\gamma^{\gamma\epsilon}\sum_m A_{\epsilon\beta}^{mq}\delta_{mn}\,. \tag{3.11}$$

Then since the determinant of a product is the product of determinants, we are able to combine one of the determinants obtained from the a integration with the factor arising from the b integrations. We get

$$\int \exp\left(i\int_0^s U\,ds'\right)\prod_{p=1}^{N/2} d^4a_p d^4b^p = C_2(N)\exp\left(i\frac{\sigma}{2s}\right)(\mathrm{Det}D_{p\beta}^{\alpha q})^{-1/2}\,, \tag{3.12}$$

where C_2 depends only on N. We notice that overall factors of s^N have canceled between the a and b contributions.

Substituting (3.12) into (2.19) we finally get

$$\langle x, s\,|x', 0\rangle = Ks^{-2}\exp\left[i\frac{\sigma(x,x')}{2s} - i(\xi - \tfrac{1}{6})Rs\right](\mathrm{Det}D)^{-1/2} \tag{3.13}$$

where $K = \lim_{N\to\infty} C_1(N)C_2(N)$. We need not evaluate K directly. It is a geometry-independent combinatoric factor. In flat spacetime, $R=0$ and $A_{\alpha\beta}^{pq} = 0$, so that Eq. (3.13) becomes

$$\langle x, s\,|x', 0\rangle = Ks^{-2}\exp\left(i\frac{\sigma}{2s}\right) \text{ (flat spacetime)}\,. \tag{3.14}$$

But it is known that this expression, with $K = -i(4\pi)^{-1}$, is the exact amplitude.[1,3,11] Hence $K = -i(4\pi)^{-1}$ in every geometry, and we have the Gaussian approximation

$$\langle x, s\,|x', 0\rangle_{\mathrm{Gauss}} = \frac{-i}{(4\pi s)^2}(\mathrm{Det}D)^{-1/2}\exp\left[i\frac{\sigma(x,x')}{2s} - i(\xi - \tfrac{1}{6})Rs\right]. \tag{3.15}$$

We now calculate the Feynman Green's function in the Gaussian approximation. First we note that by Eq. (3.6) the quantity sA_{ij}^{pq} is independent of s. Hence when Eq. (3.15) is substituted into the Schwinger-DeWitt formula (1.4), the determinant comes outside the integral. The exponential in R combines trivially with $\exp(-im^2s)$ [recall $(\xi - \tfrac{1}{6})R$ is constant in this case]. Thus, the form of G_F mimics that in flat spacetime,[3] except for an overall factor:

$$G_F(x,x') = -\frac{[m^2 + (\xi - \frac{1}{6})R]}{8\pi}\frac{H_1^{(2)}(\{-2[m^2 + (\xi - \frac{1}{6})R]\sigma\}^{1/2})}{\{-2[m^2 + (\xi-\frac{1}{6})R]\sigma\}^{1/2}}(\mathrm{Det}D)^{-1/2}\,, \tag{3.16}$$

where $H_1^{(2)}$ is the first Hankel function of second order.[22]

The determinant in Eq. (3.15) or Eq. (3.16) is by no means a transparent quantity. We shall now show that it is in fact the reciprocal of the biscalar Δ defined by Eq. (1.6) which is itself simply related to the Van Vleck-Morette determinant. This appears so unlikely at first sight that we shall go through the argument in some detail. Let

$$F \equiv \{\mathrm{Det}D\}^{-1/2}$$
$$= \{\mathrm{Det}[\delta^\alpha_\beta \delta^{pq} - 2(s/\pi^2)p^{-2}\gamma^{\alpha\epsilon}A^{pq}_{\epsilon\beta}]\}^{-1/2} . \qquad (3.17)$$

Then by the well-known identity $\ln \mathrm{Det}D = \mathrm{Tr}\ln D$,

$$F = \exp\{-\tfrac{1}{2}\mathrm{Tr}\ln[\delta^\alpha_\beta\delta^{pq} - 2(s/\pi^2)\gamma^{\alpha\epsilon}p^{-2}A^{pq}_{\epsilon\beta}]\} . \qquad (3.18)$$

Expanding out the logarithm and taking the trace we have

$$F = \exp[(s/\pi^2)p^{-2}\gamma^{\alpha\beta}A^{pp}_{\alpha\beta} + (s/\pi^2)^2p^{-2}q^{-2}\gamma^{\alpha\epsilon}A^{pq}_{\epsilon\delta}\gamma^{\delta\mu}A^{pq}_{\mu\alpha} + \cdots] \quad (3.19)$$

(summation over p and q). We have here exploited the evident symmetry of $A^{pq}_{\alpha\beta}$ in $\alpha\beta$ and pq.

To evaluate the A^{pq}_{ij} we shall expand the $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ in Eq. (3.6) about $\tau = 0$ in powers of τ:

$$R_{\hat{0}\hat{i}\hat{0}\hat{j}}(\tau = \xi T/\pi) = R_{\hat{0}\hat{i}\hat{0}\hat{j}}(0) + R_{\hat{0}\hat{i}\hat{0}\hat{j},\hat{0}}(0)\xi T/\pi + \tfrac{1}{2}R_{\hat{0}\hat{i}\hat{0}\hat{j},\hat{0}\hat{0}}(0)(\xi T/\pi)^2 + \cdots . \qquad (3.20)$$

Because A^{pq}_{ij} has an overall factor of T^2, the terms retained in (3.20) are sufficient to give us F to $O(T^4)$. In fact, to calculate A^{pq}_{ij} for the second term in the exponential of Eq. (3.19), one needs only the first term in Eq. (3.20) which is constant. The integral in Eq. (3.6) is then trivial and nonzero only for $p = q$. Also, in the first term in the exponential one only needs A^{pq}_{ij} with $p = q$. The relevant integrals are

$$\int_0^\pi \xi^n(1-\cos 2p\xi)d\xi = \begin{cases}\pi, & n=0\\ \tfrac{1}{2}\pi^2, & n=1\\ \tfrac{1}{3}\pi^3 - \tfrac{1}{2}\pi/p^2, & n=2 .\end{cases} \qquad (3.21)$$

We get

$$A^{pp}_{ij} = \tfrac{1}{2}(T^2/s)\{R_{\hat{0}\hat{i}\hat{0}\hat{j}}(0) + \tfrac{1}{2}R_{\hat{0}\hat{i}\hat{0}\hat{j},\hat{0}}(0)T + \tfrac{1}{6}[1 - 3/(2\pi^2p^2)]R_{\hat{0}\hat{i}\hat{0}\hat{j},\hat{0}\hat{0}}(0)T^2 + \cdots\} . \qquad (3.22)$$

We now carry out the sum over p in Eq. (3.19) using[23]

$$\sum_{p=1}^\infty p^{-2} = \pi^2/6 , \quad \sum_{p=1}^\infty p^{-4} = \pi^4/90 . \qquad (3.23)$$

In carrying out the sum over the spacetime indices we recall that we have defined $A^{pq}_{0\mu} \equiv 0$; this is in accordance with $R_{\hat{0}\hat{0}\hat{0}\mu} = 0$ (by antisymmetry). The vanishing of the Christoffel symbols all along $x^\mu_c(s')$ allows us to replace $d/d\tau$ and $d^2/d\tau^2$ by $D/d\tau$ and $D^2/d\tau^2$ in Eq. (3.22). Then, since $\gamma^{\alpha\beta}$ is just $g^{\alpha\beta}$ on x^μ_c, one may interpret every sum over spacetime indices in Eq. (3.19) as a spacetime trace on the Riemann tensor. When all this is done, and the exponential in Eq. (3.19) is expanded to second order, we get

$$F = 1 + \tfrac{1}{12}R_{\hat{0}\hat{0}}T^2 + \tfrac{1}{24}R_{\hat{0}\hat{0};\hat{0}}T^3 + \tfrac{1}{8}[\tfrac{1}{10}R_{\hat{0}\hat{0};\hat{0}\hat{0}} + \tfrac{1}{36}(R_{\hat{0}\hat{0}})^2 + \tfrac{1}{45}R_{\hat{0}\alpha\hat{0}\beta}R_{\hat{0}}{}^{\alpha}{}_{\hat{0}}{}^{\beta}]T^4 + \cdots . \qquad (3.24)$$

In this expression all quantities are evaluated at $\tau = 0$ (point x') and $R_{\hat{0}\hat{0};\hat{0}} = R_{\alpha\beta;\delta}e^\alpha_0 e^\beta_0 e^\gamma_0$, etc., where $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$.

Now for comparison we compute the expansion of $\Delta(x,x')^{1/2}$ to $O(T^4)$. We shall not attempt to do this directly (an intricate calculation), but shall instead exploit a useful fact pointed out by one of us[11]: In *Riemann* normal coordinates centered at x', $\Delta(x,x') = [-g(x)]^{-1/2}$. Now an expression for $(-g)$ in these coordinates has been worked out by Petrov[24] (several misprints in that expression are corrected in Ref. 11):

$$-g = 1 - \tfrac{1}{3}R_{\alpha\beta}y^\alpha y^\beta - \frac{1}{3!}R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma - \frac{1}{4!}(-\tfrac{4}{3}R_{\alpha\beta}R_{\gamma\delta} + \tfrac{4}{15}R_{\mu\alpha\beta}{}^{\nu}R^\mu{}_{\delta\gamma\nu} + \tfrac{6}{5}R_{\alpha\beta;\gamma\delta})y^\alpha y^\beta y^\gamma y^\delta . \qquad (3.25)$$

Now the Riemann coordinates of x are defined by $y^\alpha = t^\alpha T$, where t^α is the unit tangent at x' of that geodesic which links x' to x, and T is the proper time (distance) along it. Thus in our notation $y^\alpha = e^\alpha_0 T$. For example, $R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma = R_{\hat{0}\hat{0};\hat{0}}T^3$, etc. Making the necessary substitutions in Eq. (3.25) and calculating $(-g)^{-1/4}$ to $O(T^4)$ we get exactly the expression (3.24) for F.

Thus the expansions of F and $\Delta(x,x')^{1/2}$ in powers of T agree exactly, at least to $O(T^4)$. Further, as we shall see in Secs. IV and V, for both the Einstein universe and de Sitter spacetime $F = \Delta(x,x')^{1/2}$ exactly. Thus although we have not succeeded in proving the equality to all orders in a general metric, there are good reasons to believe it holds true, i.e., that $\mathrm{Det}D = \Delta^{-1}$. This means, for example, that our result (3.15) can be written as

$$\langle x, s\,|x', 0\rangle_{\mathrm{Gauss}} = i(4\pi i s)^{-2}\Delta(x,x')^{1/2}e^{i\sigma/2s}e^{-i[\xi-(1/6)]Rs} , \qquad (3.26)$$

an expression which invites several remarks. We first note that, apart from the constant phase factor $e^{-i(\xi-1/6)Rs}$, it coincides with the first term in the Schwinger-DeWitt expansion for the amplitude, Eq. (1.7). Thus in the Gaussian approximation, the path-integral representation of $\langle x, s\,|x', 0\rangle$ agrees for small s with the "short-time" amplitude inferred directly from the Schrödinger equation by DeWitt,[15]

$$\langle x, s|x', 0\rangle = i(4\pi i s)^{-2}\Delta(x,x')^{1/2}e^{i\sigma/2s}. \quad (3.27)$$

Since we did not assume small T in arriving at Eq. (3.15), the agreement suggests that, for small s, Eq. (3.27) is approximately valid even for well-separated points, and not just for close x and x' as usually assumed (see, for example, Refs. 1 and 15). With the factor $\exp[-i(\xi-\frac{1}{6})Rs]$ in Eq. (3.26), our approximation should also be good when s is not very small. That factor, when expanded out, gives

$$e^{-i[\xi-(1/6)]Rs} = 1 - i(\xi-\tfrac{1}{6})Rs - \tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2s^2 + \cdots. \quad (3.28)$$

The terms appearing here agree with all those which involve R in the correction terms mentioned in Eq. (1.7) when they are computed for $x = x'$.[3,11] Again, since we did not assume $x\approx x'$, the implication is that for $R = \text{const} \neq 0$ and $\xi \neq \frac{1}{6}$, those terms in the f_i in Eq. (1.7) which involve R must be independent of x and x'.[25] There are other terms in the f_i not proportional to powers of R,[3,11,25] which are not obtained from Eq. (3.26). Presumably these would come from post-Gaussian contributions.

Feynman[13] was the first to note, in the context of the nonrelativistic Schrödinger equation, that the exponential in the action must carry a normalization factor $(4\pi i s)^{-1/2}$ per coordinate.[12] This is necessary both for the short-time amplitude to obey the equation, and for self-consistency in combining many short-time amplitudes to get the amplitude for finite time. Following him there has been a tendency[15,16] to regard the $\Delta^{1/2}$ in Eq. (3.27) as the extra normalization required in curved spacetime over the Feynman value $(4\pi i s)^{-2}$ (recall, in flat spacetime $\Delta = 1$). This point of view rather obscures an important feature. We have seen that the $\Delta^{1/2}$ is nothing but the contribution in Gaussian order of the sum over paths apart from the classical path (which gives $e^{i\sigma/2s}$). This will be just as true in the limit $s\to 0$. Thus the factor $\Delta^{1/2}$ is an expression of the fact that even in curved spacetime each individual short-time (ϵ) amplitude may itself be regarded as composed of many elementary amplitudes corresponding to a finer division of the "short time" one is starting with. Were one to fear that by representing a typical path by the superposition of sines of Eq. (2.8), one is giving up the chance to include in the amplitude the contribution of paths which "zig-zag" in an arbitrarily fine scale (these are reputed to make an important contribution to the path integral[13,18]), one's fears should be allayed by the realization that the formalism is self-consistent (i.e., even if there are paths which in the limit of infinite fineness cannot be represented as the limit of such Fourier series, their contributions are evidently included). The contribution of paths on a scale finer than, say, ϵ is implicitly contained in the amplitude $\langle x, \epsilon|x', 0\rangle$ because this has just the form that one would get by summing all paths represented by (2.8) and (2.9) with $s=\epsilon$. As $N\to\infty$ the contributions of zig-zag paths get included.

The most dramatic demonstration of what we have just discussed was provided by Feynman.[13] Having argued for the importance of paths which zig-zag in an arbitrarily fine scale, he proceeded to ignore them by representing paths by a Fourier series like Eq. (2.8). Nevertheless, the path integral gave him *exactly* the amplitude for the harmonic oscillator. Obviously the formalism took care of self-consistency. We now present an example in curved spacetime of this notable feature of the formalism.

IV. EXAMPLE: PROPAGATOR IN THE EINSTEIN UNIVERSE

Here we shall calculate the Feynman Green's function in the Einstein universe which may be represented by the line element

$$g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)], \quad (4.1)$$

where a is the (constant) radius of the universe, and χ is an angular variable: $0\le\chi\le\pi$. The first order of business is to find the geodesics so that Eq. (3.15) may be employed. Because the metric is static, a free particle conserves its energy. Hence $dt/d\tau = E = \text{const}$ along geodesic paths. Let us consider a radial goedesic with $\theta = \text{const}$ and $\phi = \text{const}$. Because of the isotropy of the space this does not restrict generality. The radial component of $dx^\mu/d\tau$ may be inferred from $g_{\alpha\beta}(dx^\alpha/d\tau)(dx^\beta/d\tau) = \mp 1$ where upper (lower) sign applies to a timelike (spacelike) geodesic: $d\chi/d\tau = a^{-1}(E^2\mp 1)^{1/2}$. The velocity vector is thus

$$e_0{}^\mu = [E, a^{-1}(E^2\mp 1)^{1/2}, 0, 0]. \quad (4.2)$$

One may immediately construct a triad of parallel transported unit vectors orthogonal to $e_0{}^\mu$:

$$\begin{aligned} e_1{}^\mu &= [(E^2\mp 1)^{1/2}, Ea^{-1}, 0, 0],\\ e_2{}^\mu &= [0, 0, (a\sin\chi)^{-1}, 0],\\ e_3{}^\mu &= [0, 0, 0, (a\sin\chi\sin\theta)^{-1}]. \end{aligned} \quad (4.3)$$

Here $e_1{}^\mu$ is spacelike (timelike) for a timelike (spacelike) geodesic. Finally, if a geodesic starts from $\chi = 0$, $t = 0$ at $\tau = 0$ and terminates at χ, t at proper time (distance) T, then by the constancy of $d\chi/d\tau = e_0{}^\chi$ we have

$$T = a\chi(E^2\mp 1)^{-1/2}. \quad (4.4)$$

The curvature tensor in the Einstein universe is given by

$$R^l{}_{ijk} = \delta^l{}_j H_{ik} - \delta^l{}_k H_{ij}\,, \tag{4.5}$$

where H_{ij} may be inferred from the line element of the three-sphere:

$$H_{ij}dx^i dx^j = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{4.6}$$

All other elements of $R^\alpha{}_{\beta\gamma\delta}$ vanish. We can see that

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} \equiv R_{\alpha\beta\gamma\delta}e_0^\alpha e_1^\beta e_0^\gamma e_1^\delta \tag{4.7}$$

is a linear combination of $R_{\chi\chi\chi\chi}$ and components of $R_{\alpha\beta\gamma\delta}$ with at least one t index; all these vanish, so $R_{\hat{0}\hat{1}\hat{0}\hat{1}} = 0$. By contrast,

$$\begin{aligned} R_{\hat{0}\hat{2}\hat{0}\hat{2}} &= R_{\alpha\beta\gamma\delta}e_0^\alpha e_2^\beta e_0^\gamma e_2^\delta = R_{\chi\theta\chi\theta}(e_0^\chi)^2(e_2^\theta)^2 \\ &= a^{-2}(E^2 \mp 1)\,. \end{aligned} \tag{4.8}$$

Similarly $R_{\hat{0}\hat{3}\hat{0}\hat{3}} = a^{-2}(E^2 \mp 1)$. All other $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ vanish. It is trivial to compute $A^{pq}_{\alpha\beta}$ from Eq. (3.6) because the $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are constant:

$$A^{pq}_{22} = A^{pq}_{33} = \tfrac{1}{2}T^2 s^{-1} a^{-2}(E^2 \mp 1)\delta^{pq}\,. \tag{4.9}$$

All other $A^{pq}_{\alpha\beta}$ vanish. The matrix $D = I - (2s/\pi^2)\gamma^{\alpha\epsilon}p^{-2}A^{pq}_{\epsilon\beta}$ is thus diagonal in pq and $\alpha\beta$. Half of its elements are unity (those with $\alpha = 0, 1$). The rest are $1 - (T/ap\pi)^2(E^2 \mp 1)$ for $\alpha = 2, 3$. The determinant is easily computed by means of Euler's identity

$$\prod_{p=1}^{\infty}\left(1 - \frac{w^2}{p^2\pi^2}\right) = \frac{\sin w}{w} \tag{4.10}$$

applied twice (to $\alpha = 2$ and $\alpha = 3$). Substituting from Eq. (4.4) we get $F \equiv \{\mathrm{Det}D\}^{-1/2} = \chi(\sin\chi)^{-1}$.

Finally, according to Eq. (3.15) we must compute $\sigma(x, x')$. From Eq. (4.2) we have $t = ET$ and $\chi = a^{-1}(E^2 \mp 1)^{1/2}T$ from which it follows that $\sigma = \frac{1}{2}T^2 = \frac{1}{2}(a^2\chi^2 - t^2)$, where t, χ are the coordinates of point x. Substituting all this into Eq. (3.15) we get

$$\begin{aligned} \langle t, \chi, s|0,0,0\rangle_{\mathrm{Gauss}} = -i(4\pi s)^{-2}\chi(\sin\chi)^{-1}\exp[i\tfrac{1}{4}(a^2\chi^2 - t^2)s^{-1} \\ - i(\xi - \tfrac{1}{6})Rs]\,. \end{aligned} \tag{4.11}$$

This is the amplitude associated with the *direct* geodesic from $t = 0$, $\chi = 0$ to t, χ. Because of the compact nature of the Einstein universe, there are other such geodesics; each circles the universe one or more times in the direction of $+\chi$ or $-\chi$ before arriving at the point t, χ. The true amplitude must evidently be the sum of contributions like (4.11) for all such geodesics. In each such contribution $\chi + 2\pi n$ plays the role of χ in Eq. (4.11); n is an integer (can be 0 or negative) which indicates how many times the geodesic in question circumnavigates the universe.

The typical amplitude Eq. (4.11) has an s dependence entirely analogous to that of the amplitude in flat spacetime. Thus the integral (1.4) for G_F can be taken over from standard references. The result, taking account of all geodesics, is

$$G_F(\chi, t; 0, 0) = -\frac{m^2 + (\xi - \frac{1}{6})R}{8\pi}\sum_{n=-\infty}^{\infty}\frac{H_1^{(2)}(u_n)}{u_n}\,\frac{\chi + 2\pi n}{\sin\chi}\,, \tag{4.12}$$

where

$$u_n = \{-[m^2 + (\xi - \tfrac{1}{6})R][a^2(\chi + 2\pi n)^2 - t^2]\}^{1/2}\,. \tag{4.13}$$

How good is this Gaussian approximation? To check we first note that because R is constant in the Einstein universe, G_F can only depend on the parameter $m^2 + \xi R$, not on m^2 and ξ separately [see Eq. (1.1)]. Thus it suffices to consider the G_F for $\xi = \frac{1}{6}$ (conformally invariant) because the G_F for any other ξ may be obtained from it by replacing $m^2 \to m^2 + (\xi - \frac{1}{6})R$. Our expression (4.12) obeys this essential rescaling relation. For $\xi = \frac{1}{6}$ an exact expression for G_F has been given by Dowker[26] (see also Dowker and Critchley[4]) and Eq. (4.12) with $\xi = \frac{1}{6}$ agrees exactly with it. Thus for the Einstein universe our *Gaussian approximation* to the path integral for $\langle x, s|x', 0\rangle$ is *exact*.

Dowker[26] also established that for $\xi = \frac{1}{6}$ the exact amplitude coincides with the first term of the Schwinger-DeWitt expansion (1.7), the so called WKB approximation. He referred to this as the "exactness of the sum over classical paths." This observation proves that the factor F which we computed here directly in fact equals $\Delta^{1/2}$ in the Einstein universe. This further supports our identification of F with $\Delta^{1/2}$ in general spacetimes. Consider now the amplitude $\langle x, s|x', 0\rangle_{\mathrm{Gauss}}$ for $\xi \neq \frac{1}{6}$ as given by Eq. (4.11). Expansion of the R-dependent phase factor now gives the R-dependent terms in the f_i of the Schwinger-DeWitt expansion. This confirms our expectation in Sec. III that when R is constant those parts of the f_i must be independent of x and x'.

V. EXAMPLE: PROPAGATOR IN de SITTER SPACETIME

A second example in which the quantities appearing in Eq. (3.15) can be evaluated in closed form is de Sitter spacetime. The determinant appearing in the Gaussian approximation is easily evaluated in this case, and is shown to yield again the exact expression for $\Delta(x, x')$. We show that the expression for $\langle x, s|x', 0\rangle$, although not exact, is a good approximation for well-separated points. The factor $\exp[-i(\xi - \frac{1}{6})Rs]$ appearing in Eq. (3.15) is found to be essential for the approximation to be good when s is not very small.

A standard form of the de Sitter line element is

$$g_{\mu\nu}dx^\mu dx^\nu = -(1-\kappa r^2)dt^2 + (1-\kappa r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (5.1)$$

which in Cartesian form is

$$g_{\mu\nu}dx^\mu dx^\nu = -(1-\kappa r^2)dt^2 + [\delta_{ij} + \kappa(1-\kappa r^2)^{-1}x_i x_j]dx^i dx^j, \quad (5.2)$$

where $x_i = x^i$ and $r^2 = \delta_{ij}x^i x^j$. The nonvanishing components of the Riemann tensor are

$$R_{\mu\nu\lambda\sigma} = \kappa(\eta_{\mu\lambda}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\lambda}). \quad (5.3)$$

Consider the geodesic at $r=0$. Because of the maximal symmetry it is equivalent to any timelike geodesic in de Sitter spacetime. The maximal symmetry also implies that the biscalar $\langle x, s\,|x', 0\rangle$ will depend on x and x' only through the total proper time T along the geodesic. (The corresponding results for spacelike geodesics will be inferred by analytic continuation in the variable T.) The orthonormal tetrad $e^\mu{}_\alpha = \delta^\mu{}_\alpha$ satisfies $g_{\mu\nu}e^\mu{}_\alpha e^\nu{}_\beta = \eta_{\alpha\beta}$ at $r=0$ and is parallel transported along the geodesic. Therefore, the nonvanishing components of $R_{\hat\mu\hat\nu\hat\lambda\hat\sigma}$ in Fermi normal coordinates on the geodesic at $r=0$ are the same as in Eq. (5.3). Then Eq. (3.6) yields, for the nonvanishing components of $A^{pq}_{\alpha\beta}$,

$$A^{pq}_{ij} = -\tfrac{1}{2}\kappa T^2 s^{-1}\delta_{ij}\delta^{pq}. \quad (5.4)$$

This matrix is diagonal, so that

$$\begin{aligned}\mathrm{Det}D &\equiv \mathrm{Det}[\delta^\alpha_\beta\delta^{pq} - (2s/\pi^2)\eta^{\alpha\epsilon}p^{-2}A^{pq}_{\epsilon\beta}] \\ &= \prod_{p=1}^{\infty}(1+\pi^{-2}p^{-2}\kappa T^2)^3 \\ &= (\kappa^{1/2}T)^{-3}\sinh^3(\kappa^{1/2}T). \end{aligned} \quad (5.5)$$

Assuming that Det$D$ is an analytic function of T it follows by analytic continuation that for any timelike or spacelike geodesic,

$$\begin{aligned}(\mathrm{Det}D)^{-1} &= (2\kappa\sigma)^{3/2}\sin^{-3}[(2\kappa\sigma)^{1/2}] \\ &= \begin{cases}(\kappa^{1/2}T)^3\sinh^{-3}(\kappa^{1/2}T) & \text{(timelike)},\\ (\kappa^{1/2}T)^3\sin^{-3}(\kappa^{1/2}T) & \text{(spacelike)}.\end{cases}\end{aligned} \quad (5.6)$$

For spacelike separations, the range of T is $0 \leqslant T < 2\pi\kappa^{-1/2}$.

To verify that $(\mathrm{Det}D)^{-1}$ is just $\Delta(x,x')$, it is convenient to introduce Riemann normal coordinates centered at the point x'. The Riemann normal coordinates y^α of a point x are $y^\alpha = t^\alpha T$, where t^α is the unit tangent at x' of the geodesic from x' to x, and T is the proper time (distance) along that geodesic. By summing the infinite series for $g_{\alpha\beta}$ given by Petrov (Ref. 24, p. 37) for spaces of constant curvature (being careful of several misprints), we find that the metric of de Sitter spacetime in Riemann normal coordinates is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \left\{1 - \frac{\sin^2[(\kappa y^2)^{1/2}]}{\kappa y^2}\right\}\left(\frac{y_\alpha y_\beta}{y^2} - \eta_{\alpha\beta}\right). \quad (5.7)$$

In these coordinates, one has $y_\alpha = g_{\alpha\beta}y^\beta = \eta_{\alpha\beta}y^\beta$ (Ref. 24, p. 36), and

$$y^2 = y_\alpha y^\alpha = 2\sigma(x,x'). \quad (5.8)$$

We again make use of the relation (Ref. 11) that $\Delta(x,x') = [-g(y)]^{-1/2}$, where y denotes the Riemann normal coordinates of x. Writing $g = \epsilon^{\alpha\beta\gamma\delta}g_{0\alpha}g_{1\beta}g_{2\gamma}g_{3\delta}$ (where $\epsilon^{0123}=1$), and noting that terms involving products like $(y_0 y_\alpha)(y_1 y_\beta)$ do not contribute because of the antisymmetry of $\epsilon^{\alpha\beta\gamma\delta}$, we find that

$$g = -\{\sin[(\kappa y^2)^{1/2}](\kappa y^2)^{-1/2}\}^6, \quad (5.9)$$

and in view of Eq. (5.8) that

$$\Delta(x,x') = (2\kappa\sigma)^{3/2}\sin^{-3}[(2\kappa\sigma)^{1/2}]. \quad (5.10)$$

Thus, the expression in Eq. (5.6) obtained from our determinant is the exact expression for $\Delta(x,x')$.

Making use of Eq. (5.6), the Gaussian approximation for the kernel given by Eq. (3.15) becomes

$$\langle x, s\,|x', 0\rangle_{\mathrm{Gauss}} = -i(4\pi s)^{-2}\exp[i\sigma/2s - i(\xi - \tfrac{1}{6})Rs](2\kappa\sigma)^{3/4}\sin^{-3/2}[(2\kappa\sigma)^{1/2}], \quad (5.11)$$

where the scalar curvature is

$$R = 12\kappa. \quad (5.12)$$

The Feynman Green's function in this approximation is now given by Eq. (3.16) as

$$G_F(x,x') = -\frac{[m^2 + (\xi-\frac{1}{6})R]}{8\pi}\,\frac{H_1^{(2)}(\{-2[m^2+(\xi-\frac{1}{6})R]\sigma\}^{1/2})}{\{-2[m^2+(\xi-\frac{1}{6})R]\sigma\}^{1/2}}\,\frac{(2\kappa\sigma)^{3/4}}{\sin^{3/2}[(2\kappa\sigma)^{1/2}]}. \quad (5.13)$$

This gives the contribution of a single geodesic. When x and x' can be joined by a timelike geodesic, there is only one such geodesic. On the other hand, when x and x' can be joined by a spacelike geodesic there are in general two such geodesics. Together they form a great circle of total proper length $2\pi\kappa^{-1/2}$ on the de Sitter hypersphere. When x is at the antipode of x', there are infinitely many spacelike geodesics between the points. We find below that the approximation in Eq. (5.11) is good for $T \lesssim \frac{1}{2}\pi\kappa^{-1/2}$, so that the contribution of the longer spacelike geodesic cannot be accurately represented in this way. Assuming that the shorter spacelike geodesic makes the main contribution for T in the above range, we include only the contribution of the shorter geodesic in the above approximation.

To find out how well the expression in Eq. (5.11) approximates $\langle x, s | x', 0\rangle$, we substitute it into the Schrödinger equation (1.2). By comparison with Eqs. (3.19) and (3.21) of Ref. 11 one finds that Eq. (5.11) will be an exact solution of the Schrödinger equation if and only if

$$i\frac{\partial}{\partial s}e^{-i[\xi-(1/6)]Rs}=[-\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2})+\xi R]e^{-i[\xi-(1/6)]Rs}$$

or

$$\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2})=\tfrac{1}{6}R=2\kappa, \tag{5.14}$$

where Δ is given by Eq. (5.10). Working in Riemann normal coordinates and making use of Eq. (3.9) of Ref. 11 with $U=1$, one finds that $\Delta^{1/2}$ actually satisfies the equation

$$\Delta^{-1/2}\nabla^\mu\nabla_\mu(\Delta^{1/2})=\tfrac{3}{4}\kappa\{(2\kappa\sigma)^{-1}-\sin^{-2}[(2\kappa\sigma)^{1/2}]+3\}. \tag{5.15}$$

The series for the right-hand side is $2\kappa[1-(\kappa\sigma/20)+O(\kappa^2\sigma^2)]$, so that the Schrödinger equation is satisfied to excellent approximation when κT^2 is small. For larger values of κT^2, we find that for spacelike geodesics the right-hand side of Eq. (5.15) remains within 10% of 2κ for $\kappa T^2 \leq 2.6$, corresponding to a geodesic stretching about one quarter of the way around the hypersphere. Therefore, when x and x' have spacelike separation the contribution of the shorter geodesic in Eq. (5.11) and (5.13) generally gives a good approximation, while the (smaller) contribution of the longer geodesic is not accurately represented in that form, and should not appear in Eqs. (5.11) and (5.13). The approximation should then be good if T is not much larger than $\frac{1}{2}\pi\kappa^{-1/2}$ and does not approach $\pi\kappa^{-1/2}$ (the antipodal point). When the separation of x and x' is timelike, the right-hand side of Eq. (5.15) is $\frac{3}{4}\kappa[-(\kappa T^2)^{-1}+\sinh^{-2}(\kappa^{1/2}T)+3]$. This remains between 2κ and 2.25κ for all values of T, indicating that the approximation may be better in the timelike directions. For timelike separation, the factor of $\sinh^{-3/2}(\kappa^{1/2}T)$ in Eq. (5.11) causes $\langle x, s | x', 0\rangle$ to vanish exponentially for $T > \kappa^{-1/2}$. The sharp contrast between spacelike and timelike directions can be attributed to the existence of a focus for the spacelike geodesics emanating from a point. Because the approximation to $\langle x, s | x', 0\rangle$ in Eq. (5.11) satisfies the δ-function boundary condition when s approaches zero, and satisfies nearly the exact Schrödinger equation in both timelike and spacelike directions for $T \lesssim \frac{1}{2}\pi\kappa^{-1/2}$, it should be close to the exact $\langle x, s | x', 0\rangle$ for T in that range.

If the factor of $\exp[-i(\xi-\frac{1}{6})Rs]$ were not present in Eq. (5.11), or if $(\xi-\frac{1}{6})$ were replaced by a different number in that factor, then the right-hand side of Eq. (5.14) would be changed, while that of Eq. (5.15) would of course remain the same. Thus, the factor of $\exp[-i(\xi-\frac{1}{6})Rs]$ appearing in the Gaussian approximation, Eq. (3.15), plays an important role in the above considerations.

As our main concern here was to evaluate and check our approximation, we do not wish to go into peripheral questions. We only note that our method of evaluating the path integral works directly with quantities like the Riemann tensor which are entirely insensitive to coordinate-induced boundaries. For example, it is immediately obvious that in flat spacetime our method gives the usual Minkowski-space Green's function whether one starts with Minkowski or Rindler coordinates.[27] The paths entering into the evaluation of the path integral recognize no coordinate boundaries. Thus, one would expect that in de Sitter spacetime this method would approximate the global Green's function.[5,6] There is a further complication in de Sitter spacetime, namely, there are pairs of points which cannot be connected by any geodesic (i.e., when x lies in the light cones of the point antipodal to x'). The path integral and hence G_F between such points should be small compared to the case when a geodesic path is dominant, and may be presumed to vanish in our approximation.

In the next section, we extend the work of Sec. III to fourth order in the Fourier coefficents. The reader interested in the Gaussian approximation in the important case when $(\xi-\frac{1}{6})R$ is not constant may skip directly to Sec. VII.

VI. BEYOND THE GAUSSIAN APPROXIMATION

Continuing to consider the case when $(\xi-\frac{1}{6})R$ is a constant, let us find $\langle x, s | x', 0\rangle$ working to fourth order in the Fourier coefficients $a^\mu{}_p$ and $b^p{}_\mu$. The extremum path is geodesic, so that no acceleration terms appear in the metric. The necessary expansion of the metric in Fermi normal coordinates has been worked out by Li and Ni.[20] With the $h_{\alpha\beta}$ defined by Eq. (2.14), one has to order $(\delta x)^4$

$$h_{00}=-R_{\hat{0}\hat{k}\hat{0}\hat{l}}\delta x^k\delta x^l-\tfrac{1}{3}R_{\hat{0}\hat{k}\hat{0}\hat{l};\hat{j}}\delta x^k\delta x^l\delta x^j-\tfrac{1}{12}R_{\hat{0}\hat{k}\hat{0}\hat{l};\hat{i}\hat{j}}\delta x^k\delta x^l\delta x^i\delta x^j+\tfrac{1}{3}R_{\hat{0}\hat{k}\hat{\mu}\hat{l}}R^{\hat{\mu}}{}_{\hat{i}\hat{0}\hat{j}}\delta x^k\delta x^l\delta x^i\delta x^j, \tag{6.1}$$

$$h_{0i} = -\tfrac{2}{3}R_{\hat{0}\hat{k}\hat{i}\hat{l}}\delta x^k \delta x^l - \tfrac{1}{4}R_{\hat{0}\hat{k}\hat{i}\hat{l};\hat{j}}\delta x^k \delta x^l \delta x^j - \tfrac{1}{15}R_{\hat{0}\hat{k}\hat{i}\hat{l};\hat{j}\hat{h}}\delta x^k \delta x^l \delta x^j \delta x^h + \tfrac{2}{15}R^{\hat{\mu}}{}_{\hat{k}\hat{i}\hat{l}}R_{\hat{0}\hat{j}\hat{\mu}\hat{h}}\delta x^k \delta x^l \delta x^j \delta x^h, \tag{6.2}$$

$$h_{ij} = -\tfrac{1}{3}R_{\hat{i}\hat{k}\hat{j}\hat{l}}\delta x^k \delta x^l - \tfrac{1}{6}R_{\hat{i}\hat{k}\hat{j}\hat{l};\hat{h}}\delta x^k \delta x^l \delta x^h - \tfrac{1}{20}R_{\hat{i}\hat{k}\hat{j}\hat{l};\hat{g}\hat{h}}\delta x^k \delta x^l \delta x^g \delta x^h + \tfrac{2}{45}R^{\hat{\mu}}{}_{\hat{k}\hat{j}\hat{l}}R_{\hat{\mu}\hat{g}\hat{i}\hat{h}}\delta x^k \delta x^l \delta x^g \delta x^h. \tag{6.3}$$

Using (2.20) we may now calculate $i\int U ds'$ to *fourth* order in a's *and* b's After taking the exponential we expand the exponential of the third- and fourth-order terms to obtain

$$\begin{aligned}\exp\left(i\int_0^s U ds'\right) = {}&\exp\left(\frac{i\sigma}{2s} + \frac{i}{8}s\gamma^{\alpha\beta}\sum_p b_\alpha^p b_\beta^p + \frac{i\pi^2}{8s}\gamma_{\alpha\beta}\sum_p p^2 a_p^\alpha a_p^\beta - \frac{i}{4}\sum_{pq} A_{ij}^{pq} a_p^i a_q^j\right)\\
&\times\Bigg(1 - \frac{i\pi T}{2s^2}\sum_{pqm} m A_{ij0}^{pqm} a_p^i a_q^j a_m^0 - \frac{iT^2}{12s^2}\sum_{pqm} A_{ijk}^{pqm} a_p^i a_q^j a_m^k - \frac{i\pi^2}{4s^2}\sum_{pqmn} mn A_{ij00}^{pqmn} a_p^i a_q^j a_m^0 a_n^0\\
&+ \frac{iT^2}{12s^2}\sum_{pqmn} A_{ijkl}^{pqmn} a_p^i a_q^j a_m^k a_n^l - \frac{i\pi T}{6s^2}\sum_{pqmn} n A_{ijk0}^{pqmn} a_p^i a_q^j a_m^k a_n^0 - \frac{i\pi T}{3s^2}\sum_{pqm} m B_{ijk}^{pqm} a_p^i a_q^j a_m^k\\
&- \frac{i\pi^2}{3s^2}\sum_{pqmn} mn B_{ijk0}^{pqmn} a_p^i a_q^j a_m^k a_n^0 - \frac{i\pi T}{8s^2}\sum_{pqmn} n B_{ijkl}^{pqmn} a_p^i a_q^j a_m^k a_n^l\\
&- \frac{i\pi^2}{12s^2}\sum_{pqmn} mn C_{ijkl}^{pqmn} a_p^i a_q^j a_m^k a_n^l - \frac{i}{4}\sum_{pqmn} D_{kl\,mn}^{pq\alpha\beta} a_p^k a_q^l b_\alpha^m b_\beta^n\Bigg),\end{aligned} \tag{6.4}$$

where A_{ij}^{pq} was defined in Eq. (3.5), and

$$A_{ij0}^{pqm} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{0}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)\cos(m\pi s'/s), \tag{6.5}$$

$$A_{ijk}^{pqm} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{0}\hat{j};\hat{k}} \sin(p\pi s'/s)\sin(q\pi s'/s)\sin(m\pi s'/s), \tag{6.6}$$

$$A_{ij00}^{pqmn} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{0}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)\cos(m\pi s'/s)\cos(n\pi s'/s), \tag{6.7}$$

$$A_{ijkl}^{pqmn} = \int_0^s ds' (R_{\hat{0}\hat{k}\hat{\mu}\hat{i}}R^{\hat{\mu}}{}_{\hat{l}\hat{0}\hat{j}} - \tfrac{1}{4}R_{\hat{0}\hat{k}\hat{0}\hat{l};\hat{i}\hat{j}}) \sin(p\pi s'/s)\sin(q\pi s'/s)\sin(m\pi s'/s)\sin(n\pi s'/s), \tag{6.8}$$

$$A_{ijk0}^{pqmn} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{0}\hat{j};\hat{k}} \sin(p\pi s'/s)\sin(q\pi s'/s)\sin(m\pi s'/s)\cos(n\pi s'/s), \tag{6.9}$$

$$B_{ijk}^{pqm} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{k}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)\cos(m\pi s'/s), \tag{6.10}$$

$$B_{ijk0}^{pqmn} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{k}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)\cos(m\pi s'/s)\cos(n\pi s'/s), \tag{6.11}$$

$$B_{ijkl}^{pqmn} = \int_0^s ds' R_{\hat{0}\hat{i}\hat{l}\hat{j};\hat{k}} \sin(p\pi s'/s)\sin(q\pi s'/s)\sin(m\pi s'/s)\cos(n\pi s'/s), \tag{6.12}$$

$$C_{ijkl}^{pqmn} = \int_0^s ds' R_{\hat{k}\hat{i}\hat{l}\hat{j}} \sin(p\pi s'/s)\sin(q\pi s'/s)\cos(m\pi s'/s)\cos(n\pi s'/s), \tag{6.13}$$

$$D_{kl\,mn}^{pq\alpha\beta} = \int_0^s ds' (\gamma^{\alpha 0}\gamma^{\beta 0}R_{\hat{0}\hat{k}\hat{0}\hat{l}} + \tfrac{4}{3}\gamma^{\alpha 0}\gamma^{\beta i}R_{\hat{0}\hat{k}\hat{i}\hat{l}} + \tfrac{1}{3}\gamma^{\alpha i}\gamma^{\beta j}R_{\hat{i}\hat{k}\hat{j}\hat{l}}) \sin(p\pi s'/s)\sin(q\pi s'/s)\sin(m\pi s'/s)\sin(n\pi s'/s). \tag{6.14}$$

All these integrals can easily be evaluated if $R_{\mu\nu\alpha\beta}$ is known as a function of proper time τ along the geodesic. We only need to recall that $s' = \tau s/T$.

Now from Eq. (2.19) we have

$$\langle x, s | x', 0\rangle = Cs^{-2}\exp[-i(\xi - \tfrac{1}{6})Rs]\int \prod_p d^4a_p d^4b^p \exp\left(i\int_0^s U ds'\right), \tag{6.15}$$

where C depends only on N. We notice that because the exponential in (6.4) is a quadratic form, those terms in the post-Gaussian corrections involving three a's will integrate out to zero in (6.15). The terms involving four a's can be integrated with help of the functional

$$Z(s,\lambda,\mu)\equiv\int\prod_p d^4a_p d^4b^p \exp\Big(i\frac{\pi^2}{8s}\gamma_{\alpha\beta}\sum_p p^2 a_p^\alpha a_p^\beta-\frac{i}{4}\sum_{pq}A^{pq}_{jk}a_p^j a_q^k+\frac{is}{8}\gamma^{\alpha\beta}\sum_p b^p_\alpha b^p_\beta+i\sum_p \lambda^p_\alpha a^\alpha_p+i\sum_p \mu^\alpha_p b^p_\alpha\Big), \tag{6.16}$$

where λ^p_α and μ^α_p are parameters. By differentiating Z an appropriate number of times with respect to λ's or μ's and setting $\lambda=\mu=0$ one can produce the integral over a's and b's of any one term figuring in (6.4). Evaluating the Gaussian integral in (6.16) we have

$$Z(s,\lambda,\mu)=Z(s,0,0)\exp\Big(-\tfrac{1}{4}i\sum_{pq}\lambda^p_\alpha L^{\alpha\beta}_{pq}\lambda^q_\beta-2is^{-1}\sum_p \mu^\alpha_p\gamma_{\alpha\beta}\mu^\beta_p\Big), \tag{6.17}$$

where $L^{\alpha\beta}_{pq}$ is the matrix inverse of $M^{pq}_{\alpha\beta}$ [see (3.9)]. Performing now the appropriate differentiations of Z to evaluate the post-Gaussian corrections in the integral (6.15) as specified by Eq. (6.4) we find

$$\begin{aligned}\langle x,s|x',0\rangle=\langle x,s|x',0\rangle_{\text{Gauss}}\Big(1&+\frac{i\pi^2}{16s^2}\sum_{pqmn} mnA^{pqmn}_{ij00}E^{ij00}_{pqmn}-\frac{iT^2}{48s^2}\sum_{pqmn}A^{pqmn}_{ijkl}E^{ijkl}_{pqmn}+\frac{i\pi T}{24s^2}\sum_{pqmn} nA^{pqmn}_{ijk0}E^{ijk0}_{pqmn}\\&+\frac{i\pi^2}{12s^2}\sum_{pqmn} mnB^{pqmn}_{ijk0}E^{ijk0}_{pqmn}+\frac{i\pi T}{32s^2}\sum_{pqmn} nB^{pqmn}_{ijkl}E^{ijkl}_{pqmn}\\&+\frac{i\pi^2}{48s^2}\sum_{pqmn} mnC^{pqmn}_{ijkl}E^{ijkl}_{pqmn}+\frac{i}{2s}\sum_{pqmn}D^{pq\alpha\beta}_{kl\,mn}L^{kl}_{pq}\delta^{mn}\gamma_{\alpha\beta}\Big),\end{aligned} \tag{6.18}$$

where

$$E^{\alpha\beta\gamma\delta}_{pqmn}=L^{\alpha\beta}_{pq}L^{\gamma\delta}_{mn}+L^{\alpha\gamma}_{pm}L^{\beta\delta}_{qn}+L^{\alpha\delta}_{pn}L^{\beta\gamma}_{qm}, \tag{6.19}$$

and the second-order contributions have been factored out and displayed as $\langle x,s|x',0\rangle_{\text{Gauss}}$.

In any concrete geometry, calculation of the post-Gaussian corrections will evidently be a laborious task. Thus it is not clear under which circumstances is the above result an improvement on the Gaussian approximation. We give it for completeness, and will not go into such questions here. Rather, we turn to the more interesting problem of evaluating the Gaussian approximation when $(\xi-\frac{1}{6})R$ is not constant.

VII. CASE WITH ACCELERATED EXTREMAL PATHS: GAUSSIAN APPROXIMATION

The main contribution to the path integral in Eq. (2.7) comes from paths between x' and x which extremize the action $\tilde{S}$ appearing in the exponential, where

$$\tilde{S}=\int_0^s\Big[\tfrac{1}{4}g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}-(\xi-\tfrac{1}{6})R\Big]ds'. \tag{7.1}$$

In a general curved spacetime the term $(\xi-\frac{1}{6})R$ acts as a potential which causes the extremal path to be different from the geodesic joining x' to x. Furthermore, we will find that the "proper-time" parameter s' is not linearly related to the physical proper time or proper length along the extremal path. These complications, which are not present in the case when $(\xi-\frac{1}{6})R$ is constant, result in the appearance of interesting new features. For example, in the Gaussian approximation to $\langle x,s|x',0\rangle$ one finds terms like $\int_0^s R\,ds'$ which are not evident from the proper-time expansion (1.7). Such "history-dependent" terms are perhaps related to such phenomena as particle creation by the geometry.

Varying the action $\tilde{S}$ gives the equation satisfied by the extremal path,

$$\frac{D}{ds'}\Big(\frac{dx^\mu}{ds'}\Big)+2(\xi-\tfrac{1}{6})g^{\mu\nu}\frac{\partial}{\partial x^\nu}R=0. \tag{7.2}$$

To find the relationship between s' and τ, the physical proper time (length) along the timelike (spacelike) extremal path, let

$$f(s')\equiv g_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}=\gamma_{00}\Big(\frac{d\tau}{ds'}\Big)^2, \tag{7.3}$$

where $x^\alpha(s')$ is the extremal path (recall that γ_{00} is -1 or $+1$ for timelike or spacelike paths, respectively). One has

$$\frac{d}{ds'}f(s')+4(\xi-\tfrac{1}{6})\frac{dx^\alpha}{ds'}\frac{\partial}{\partial x^\alpha}R=0, \tag{7.4}$$

from which follows that

$$f(s')=4K(x,x',s)-4(\xi-\tfrac{1}{6})R(x(s')), \tag{7.5}$$

where the constant of integration K is independent of s'. Then

$$\tau(s')=2\int_0^{s'}[\gamma_{00}K-\gamma_{00}(\xi-\tfrac{1}{6})R]^{1/2}ds''. \tag{7.6}$$

To find the dependence of K on s, the total parameter change along the extremal path from x' to x, we use the Hamilton-Jacobi equation

$$\Big(\frac{\partial\tilde{S}}{\partial s}\Big)_{x,x'}+\tilde{H}=0, \tag{7.7}$$

where $\tilde{H}$ is the Hamiltonian obtained from the action $\tilde{S}$. One finds with the aid of Eq. (7.5) that

$$\tilde{H}=K(x,x',s) \tag{7.8}$$

at any point along the extremal path.

Evaluating $\tilde{S}$ of Eq. (7.1) along the extremal path gives

$$\tilde{S}=Ks-2(\xi-\tfrac{1}{6})\int_0^s R\,ds'. \tag{7.9}$$

The Hamilton-Jacobi equation is

$$s\left(\frac{\partial K}{\partial s}\right)_{x,x'}+2K-2(\xi-\tfrac{1}{6})R(x)=0, \tag{7.10}$$

where $R(x)$ is evaluated at the end point of the path. It follows that

$$K(x,x',s)=V(x,x')s^{-2}+(\xi-\tfrac{1}{6})R(x), \tag{7.11}$$

where V is independent of s. Therefore, in the limit that s approaches zero the action of Eq. (7.9) approaches $V(x,x')s^{-1}$. However, as s approaches zero the first term in Eq. (7.2) is dominant and the extremal path approaches the geodesic from x' to x. Thus, $\tilde{S}$ should approach $\frac{1}{2}\sigma(x,x')s^{-1}$. Therefore,

$$V(x,x')=\tfrac{1}{2}\sigma(x,x'). \tag{7.12}$$

Thus the action along the extremal path from x' to x is

$$\tilde{S}=\tfrac{1}{2}\sigma(x,x')s^{-1}+(\xi-\tfrac{1}{6})R(x)s-2(\xi-\tfrac{1}{6})\int_0^s R\,ds'. \tag{7.13}$$

Finally, the proper time or proper length in Eq. (7.6) takes the form

$$\tau(s')=2\int_0^{s'}\{\tfrac{1}{2}\gamma_{00}\sigma(x,x')s^{-2}+\gamma_{00}(\xi-\tfrac{1}{6})[R(x)-R(x(s''))]\}^{1/2}ds''. \tag{7.14}$$

One easily sees that these expressions reduce to the known result $\tau(s')=Ts'/s$ when R is constant.

We are now in a position to evaluate the path integral of Eq. (2.19). The acceleration $A^\mu=D(dx^\mu/d\tau)/d\tau$ appearing in Eqs. (2.15)–(2.18) for the metric in Fermi coordinates is readily evaluated from Eq. (7.2). Recalling that in Fermi coordinates $x^i=0$ and $x^0=\tau$ along the path, one finds that A^0 vanishes, and

$$A_{\hat{i}}=-2(d\tau/ds')^{-2}(\xi-\tfrac{1}{6})\partial_i R. \tag{7.15}$$

Also in these coordinates, Eq. (2.8) is

$$x^\mu(s')=\tau(s')\delta^{\mu 0}+\sum_{p=1}^{N/2}a_p^\mu\sin(p\pi s'/s). \tag{7.16}$$

Expanding the scalar curvature in a Taylor series about the extremal path, we obtain to second order

$$\left(\int_0^s R\,ds'\right)_{x^\mu(s')}=\int_0^s R\,ds'+\sum_p a_p^\mu\int_0^s\partial_\mu R\sin(p\pi s'/s)ds'+\tfrac{1}{2}\sum_p\sum_q a_p^\mu a_q^\nu\int_0^s\partial_\mu\partial_\nu R\sin(p\pi s'/s)\sin(q\pi s'/s)ds', \tag{7.17}$$

where the integrals on the right-hand side are evaluated along the extremal path. Substituting Eqs. (2.21) and (2.14)–(2.18) [with $\delta x^k=\sum a_p^k\sin(p\pi s'/s)$] into Eq. (2.20), and using Eq. (7.17), one obtains to second order in the a's and b's,

$$\int_0^s[U-(\xi-\tfrac{1}{6})R]ds'=\tilde{S}+\tfrac{1}{8}s\gamma^{\alpha\beta}\sum_p b_\alpha^p b_\beta^p+\frac{\pi^2}{8s}\gamma_{\alpha\beta}\sum_p p^2a_p^\alpha a_p^\beta-\tfrac{1}{4}\sum_{pq}\tilde{A}_{ij}^{pq}a_p^i a_q^j-\frac{\pi}{s}\sum_{pq}qa_p^j a_q^0A_j^{pq}-\tfrac{1}{2}(\xi-\tfrac{1}{6})\sum_{pq}a_p^\mu a_q^\nu B_{\mu\nu}^{pq}, \tag{7.18}$$

where $\tilde{S}$ is given by Eq. (7.13), and

$$\tilde{A}_{ij}^{pq}=\int_0^s(A_{\hat{i}}A_{\hat{j}}+R_{\hat{0}\hat{i}\hat{0}\hat{j}})\sin\left(\frac{p\pi s'}{s}\right)\sin\left(\frac{q\pi s'}{s}\right)\left(\frac{d\tau}{ds'}\right)^2ds', \tag{7.19}$$

$$A_j^{pq}=\int_0^s A_{\hat{j}}\sin\left(\frac{p\pi s'}{s}\right)\cos\left(\frac{q\pi s'}{s}\right)\left(\frac{d\tau}{ds'}\right)ds', \tag{7.20}$$

$$B_{\mu\nu}^{pq}=\int_0^s(\partial_\mu\partial_\nu R)\sin\left(\frac{p\pi s'}{s}\right)\sin\left(\frac{q\pi s'}{s}\right)ds', \tag{7.21}$$

with the integrals evaluated along the extremal paths. In these expressions $A_{\hat{i}}$ is given by Eq. (7.15) with $(d\tau/ds')^2$ given by [see Eq. (7.14)]

$$(d\tau/ds')^2=2\gamma_{00}\sigma(x,x')s^{-2}+4\gamma_{00}(\xi-\tfrac{1}{6})[R(x)-R(x(s'))]. \tag{7.22}$$

The term in Eq. (7.18) involving A_j^{pq} can be written in symmetric form by defining the matrix $Q_{\alpha\beta}^{pq}$ such that

$$Q_{j0}^{pq}=Q_{0j}^{qp}=\tfrac{1}{2}qA_j^{pq},\quad Q_{00}^{pq}=0,\quad Q_{jk}^{pq}=0. \tag{7.23}$$

Then $\sum_{pq}qa_p^j a_q^0A_j^{pq}=\sum_{pq}a_p^\alpha Q_{\alpha\beta}^{pq}a_q^\beta$, and we can rewrite

Eq. (7.18) as

$$\int_0^s [U-(\xi-\tfrac{1}{6})R]ds' = \tilde{S} + \tfrac{1}{8}s\gamma^{\alpha\beta}\sum_p b_\alpha^p b_\beta^p + \sum_{pq} a_p M_{\alpha\beta}^{pq} a_q^\beta\,, \qquad (7.24)$$

where $M_{\alpha\beta}^{pq}$ is the symmetric matrix

$$M_{\alpha\beta}^{pq} = \sum_n C_{\alpha\gamma}^{pn}\tilde{D}_{n\beta}^{\gamma q}, \qquad (7.25)$$

with

$$C_{\alpha\gamma}^{pn} = (8s)^{-1}\pi^2 n^2 \delta^{pn}\gamma_{\alpha\gamma} \qquad (7.26)$$

and

$$\tilde{D}_{n\beta}^{\gamma q} = \delta_\beta^\gamma \delta_n^q - 2s\pi^{-2}n^{-2}\gamma^{\gamma\epsilon}\sum_m \tilde{A}_{\epsilon\beta}^{mq}\delta_{mn} - 4s\pi^{-2}(\xi-\tfrac{1}{6})n^{-2}\gamma^{\gamma\epsilon}\sum_m B_{\epsilon\beta}^{mq}\delta_{mn} - 8\pi^{-1}n^{-2}\gamma^{\gamma\epsilon}\sum_m Q_{\epsilon\beta}^{mq}\delta_{mn}\,. \qquad (7.27)$$

Here we defined $\tilde{A}_{\mu 0}^{pq} = \tilde{A}_{0\mu}^{pq} = 0$. The matrix $M_{\alpha\beta}^{pq}$ reduces to that of Eq. (3.9) when $(\xi-\frac{1}{6})R$ is constant.

Now Eq. (2.19) has the form

$$\langle x, s\,|x', 0\rangle = s^{-2}Ce^{i\tilde{S}}\int \prod_{p=1}^{N/2} d^4a_p d^4b^p \exp\left(\frac{is}{8}\gamma^{\alpha\beta}\sum_p b_\alpha^p b_\beta^p + i\sum_{pq} a_p^\alpha M_{\alpha\beta}^{pq} a_q^\beta\right),$$

where C depends only on N. The integrations are the same as in Sec. III. The integration over the b's yields an overall factor proportional to s^{-N}. The integration over the a's gives $(\text{Det}M)^{-1/2} = (\text{Det}C_{\alpha\beta}^{pq})^{-1/2}(\text{Det}\tilde{D}_{p\beta}^{\alpha q})^{-1/2}$, and $(\text{Det}C_{\alpha\beta}^{pq})^{-1/2}$ is proportional to s^N. Noting that the N-dependent factors must have the same value as in flat spacetime, we finally have the Gaussian approximation

$$\langle x, s\,|x', 0\rangle_{\text{Gauss}} = \frac{-i}{(4\pi s)^2}(\text{Det}\tilde{D}_{p\beta}^{\alpha q})^{-1/2} \times \exp\left[\frac{i\sigma(x,x')}{2s} + i(\xi-\tfrac{1}{6})R(x)s - 2i(\xi-\tfrac{1}{6})\int_0^s R\,ds'\right], \qquad (7.28)$$

where $\tilde{D}_{p\beta}^{\alpha q}$ is defined in Eq. (7.27) and $\int_0^s R\,ds'$ is evaluated along the extremal path, while $\sigma(x,x')$ refers as usual to the geodesic between x and x'. (A simple approximation for Det$\tilde{D}$ and Δ when $R_{\alpha\beta\gamma\delta}$ is slowly varying between x' and x is given in Appendix C.)

This result clearly reduces to Eq. (3.15) when $(\xi-\frac{1}{6})R$ is constant, but it involves several new features. First note that $\{\text{Det}\tilde{D}\}^{-1/2}$ depends on s and is not the same as $\Delta^{1/2}(x,x')$ in general. Recall from Sec. III that (even when R is not constant)

$$\Delta(x,x') = [\text{Det}(\delta_\beta^\alpha\delta^{pq} - 2\pi^{-2}s\gamma^{\alpha\epsilon}p^{-2}A_{\epsilon\beta}^{pq})]^{-1} \qquad (7.29)$$

with

$$A_{ij}^{pq} = T_g{}^2 s^{-2}\int_0^s R_{\hat{0}\hat{i}\hat{0}\hat{j}}\sin(p\pi s'/s)\sin(q\pi s'/s)ds'\,, \qquad (7.30)$$

the integral being evaluated along the geodesic from x' to x, and T_g being the proper time (or distance) along the geodesic. The determinant in (7.29) is not Det$\tilde{D}$. A second new feature of interest is the appearance in the exponential of the integral of R. The Feynman Green's function is obtained by substituting Eq. (7.28) into Eq. (1.4). Because of the s dependence of Det$\tilde{D}$, its form will be very different from that of G_F for the case that $(\xi-\frac{1}{6})R=\text{const}$, namely, Eq. (3.16). This may signal the existence of novel quantum effects resulting from variations in R.

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APPENDIX A

Here we show that the expression (2.2) for the amplitude is, in fact, invariant under changes of the parameter p. The argument is independent of the original one[11] used to prove (2.2). First we write the obvious composition law

$$\int \langle x, s\,|x_N, s-\epsilon\rangle\langle x_N, s-\epsilon\,|x', 0\rangle(-g)^{1/2}d^4x_N = \langle x, s\,|x', 0\rangle\,, \qquad (\text{A}1)$$

where we regard ϵ as infinitesimal ($x=x_{N+1}$ in the notation of Sec. II). For $\langle x, s\,|x_N, s-\epsilon\rangle$ we take the expression

$$\langle x, s\,|x_N, s-\epsilon\rangle = [\Delta(x,x_N)]^p\exp\{i\sigma(x,x_N)/2\epsilon - i\epsilon[\xi+\tfrac{1}{3}(p-1)]R\}, \qquad (\text{A}2)$$

which is just the infinitesimal version of (2.2); use has been made of (2.3). From the discussion preceding (3.25) we now infer that

$$[\Delta(x,x_N)]^p = 1+\tfrac{1}{6}pR_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}\epsilon^2+O(\epsilon^3)\,, \qquad (A3)$$

where $dx^\alpha/ds' = (x^\alpha - x_N^\alpha)/\epsilon$. But from the relation between $\sigma_{,\alpha}$ and dx^α/ds' (see Sec. II) it follows that

$$[\Delta(x,x_N)]^p = 1+\tfrac{1}{6}pR^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}+O(\epsilon^3)\,, \qquad (A4)$$

where $\sigma_{,\alpha}\equiv\partial\sigma(x,x_N)/\partial x^\alpha$. Now denoting the exponential in (A2) by E we see that

$$\langle x, s\,|x_N, s-\epsilon\rangle = \{1-\tfrac{2}{3}pR^{\alpha\beta}\epsilon^2[\nabla_\alpha\nabla_\beta - \tfrac{1}{2}i\epsilon^{-1}\sigma_{,\alpha;\beta}-\tfrac{1}{2}(\xi-\tfrac{1}{3}+\tfrac{1}{3}p)(\sigma_{,\alpha}R_{,\beta}+\sigma_{,\beta}R_{,\alpha})]\}+O(\epsilon^3)\,. \qquad (A5)$$

Now, it might be thought that the operator $\nabla_\alpha\nabla_\beta$ when applied to E produces terms of $O(\epsilon^{-2})$; in fact this is not so. For when one multiplies E by $\langle x_N, s-\epsilon\,|x',0\rangle$ in (A1), it combines with the exponential of the action for any path from x' to x_N to give the exponential of the action from x' to x via x_N. This last action is well behaved as $\epsilon\to 0$. Thus when (A5) is substituted in (A1) $\nabla_\alpha\nabla_\beta$ does not give rise to terms singular in ϵ. In fact,

$$\int [1+\tfrac{1}{3}i\epsilon pR^{\alpha\beta}\sigma_{\alpha;\beta}+O(\epsilon^2)]E\langle x_N, s-\epsilon\,|x',0\rangle(-g)^{1/2}d^4x_N = \langle x, s\,|x',0\rangle\,. \qquad (A6)$$

Now in the limit $N\to\infty$, x and x_N are arbitrarily close for the average path. Under these circumstances[3] $\sigma_{\alpha;\beta}\to g_{\alpha\beta}$; hence $R^{\alpha\beta}\sigma_{\alpha;\beta}\to R$. Thus

$$\int \exp[i\sigma(x,x_N)/2\epsilon - i\epsilon(\xi-\tfrac{1}{3})R]\langle x_N, s-\epsilon\,|x',0\rangle(-g)^{1/2}d^4x_N = \langle x, s\,|x',0\rangle\,. \qquad (A7)$$

But p no longer appears in this expression which corresponds to the expression for $\langle x, s\,|x',0\rangle$ given by (2.2) with $p=0$: Whatever value of p we start with gives the same amplitude as $p=0$. Hence the expression (2.2) for $\langle x, s\,|x',0\rangle$ is p invariant.

The fact that there exists an independent derivation of the p invariance[11] which was never concerned with the expansion (A3) for Δ^p has an important consequence: Only terms to $O(\epsilon^2)$ in the expansion for Δ^p enter into the final value of the path integral. We shall use this feature next.

APPENDIX B

We shall show here that one can ignore the $\Delta^{1/2}$ factors in the path integral (2.6). The first result we need comes from the discussion immediately preceding and following (3.25) in Sec. III. We infer from it that

$$\Delta^{1/2}(x_1,x_2) = 1+\tfrac{1}{12}R_{\alpha\beta}(x_1)(x_2^\alpha - x_1^\alpha)(x_2^\beta - x_1^\beta) + O((x_2-x_1)^3)\,. \qquad (B1)$$

Since $x_2-x_1 = O(\epsilon)$, $\Delta^{1/2}=1+O(\epsilon^2)$. This is the reason for replacing $\Delta^{1/2}(x_0,x_1)$ by 1 in the limit $\epsilon\to 0$.

Consider now

$$\frac{\Delta^{1/2}(x_2,x_3)}{\Delta^{1/2}(x_1,x_2)} = 1+\tfrac{1}{12}R_{\alpha\beta}(x_1)[(x_3^\alpha - x_2^\alpha)(x_3^\beta - x_2^\beta) - (x_2^\alpha - x_1^\alpha)(x_2^\beta - x_1^\beta)]+O(\epsilon^3)\,. \qquad (B2)$$

In obtaining this result we expanded the denominator and also identified $R_{\alpha\beta}$ at x_1 and x_2 since the error thus incurred in expression (B1) is of $O(\epsilon^3)$. In the spirit of Sec. II we now replace x_2^α by $x_1^\alpha + \epsilon g^{\alpha\beta}(x_1)p_\beta(\epsilon)$ in the above expression obtaining

$$\frac{\Delta^{1/2}(x_2,x_3)}{\Delta^{1/2}(x_1,x_2)} = 1+\tfrac{1}{12}R_{\alpha\beta}(x_1)(x_3^\alpha - x_1^\alpha)[(x_3^\beta - x_1^\beta) - 2\epsilon g^{\beta\mu}(x_1)p_\mu(\epsilon)] + O(\epsilon^3)\,. \qquad (B3)$$

For small ϵ we may now interpret $x_3^\alpha - x_1^\alpha$ as $2\epsilon\, dx^\alpha/ds'$ at $s'=\epsilon$ where $x^\alpha(s')$ is the function given by (2.8). Upon substituting from (2.9) we find that

$$\frac{\Delta^{1/2}(x_2,x_3)}{\Delta^{1/2}(x_1,x_2)} = 1-\tfrac{1}{3}\epsilon^2 R_\alpha^\beta(x_1)\frac{dx^\alpha}{ds'}(\epsilon)\sum_{p=1}^{N/2} b_\beta^p \sin\frac{p\pi}{N+1} + O(\epsilon^3)\,. \qquad (B4)$$

Other ratios in (2.6) will have analogous form.

To finish our argument we note that the functional $\int_0^s U\,ds'$ defined by (2.20) which enters into the final form of the path integral is *exactly* quadratic in the b's (this has nothing to do with the Gaussian approximation—it is a general statement). Thus the terms in expressions like (B4) linear in b's will, when put into the path integral (2.19), average out to zero. Of course the product of ratios of $\Delta^{1/2}$'s in (2.6) also contains terms quadratic, quartic, etc., in b's These do not average to zero. But all these contributions are of $O(\epsilon^4)$ or higher. From what has been said in Appendix A they should vanish as $\epsilon\to 0$ ($N\to\infty$). Thus one may simply ignore the $\Delta^{1/2}$ factors in (2.6). One cannot ignore $\Delta^{1/2}$'s that do not appear in ratios, for example, the $\Delta^{1/2}$'s in (2.2) for $p=\frac{1}{2}$ because a discussion analogous to the preceding one shows the $O(\epsilon^2)$ term in the counterpart of

(B3) to be b independent—it will not average out to zero.

APPENDIX C

In this appendix, we give an approximate expression for Det$\tilde{D}$ in the case wherein $R_{\alpha\beta\gamma\delta}$ is slowly varying along the extremizing path from x' to x. This will also yield an approximation for Det$D$ and by implication for $\Delta(x, x')$ and the Van Vleck-Morette determinant.

When we say that $R_{\alpha\beta\gamma\delta}$ is slowly varying, we mean that one can neglect certain Fourier components of it and of quantities, such as the "acceleration." formed from it. In particular, suppose that (with $n=1, 2, \ldots$)

$$\left|\int_0^s (A_{\hat{i}}A_{\hat{j}} + R_{\hat{0}\hat{i}\hat{0}\hat{j}})\cos\left(\frac{n\pi s'}{s}\right)\left(\frac{d\tau}{ds'}\right)^2 ds'\right| \ll \left|\int_0^s (A_{\hat{i}}A_{\hat{j}} + R_{\hat{0}\hat{i}\hat{0}\hat{j}})\left(\frac{d\tau}{ds'}\right)^2 ds'\right|, \quad (C1)$$

$$\left|\int_0^s \partial_\mu\partial_\nu R\cos\left(\frac{n\pi s'}{s}\right)ds'\right| \ll \left|\int_0^s \partial_\mu\partial_\nu R\, ds'\right|, \quad (C2)$$

and that

$$\left|\int_0^s A_{\hat{j}}\sin\left(\frac{n\pi s'}{s}\right)\frac{d\tau}{ds'}ds'\right| \ll n^{-1}, \quad (C3)$$

where $A_{\hat{j}}$ is defined in Eq. (7.15). Condition (C3) is reasonable because $\int_0^s A_{\hat{j}}(d\tau/ds')ds'$ is the velocity change along the geodesic, which is generally less than unity in magnitude. Then the matrix $\tilde{D}^{\gamma q}_{n\beta}$ of Eq. (7.27) will be diagonal in the indices n and q. This follows if one considers Eqs. (7.19)–(7.21), uses the elementary identities

$$\sin n\theta\sin m\theta = \tfrac{1}{2}\cos(n-m)\theta - \tfrac{1}{2}\cos(n+m)\theta,$$

$$\sin n\theta\cos m\theta = \tfrac{1}{2}\sin(n-m)\theta - \tfrac{1}{2}\sin(n+m)\theta,$$

and then imposes the conditions of Eqs. (C1)–(C3) which permit one to neglect the off-diagonal terms in the matrix $\tilde{D}$. The result is

$$\tilde{D}^{\gamma q}_{n\beta} = \delta^q_n(\delta^\gamma_\beta - n^{-2}\pi^{-2}X^\gamma_\beta), \quad (C4)$$

where

$$X^\gamma_\beta = 2s\gamma^{\gamma\epsilon}\tilde{A}_{\epsilon\beta} + 4s(\xi - \tfrac{1}{6})\gamma^{\gamma\epsilon}B_{\epsilon\beta}, \quad (C5)$$

with

$$\tilde{A}_{\epsilon\beta} = \tfrac{1}{2}\int_0^s (A_{\hat{\epsilon}}A_{\hat{\beta}} + R_{\hat{0}\hat{\epsilon}\hat{0}\hat{\beta}})\left(\frac{d\tau}{ds'}\right)^2 ds' \quad (C6)$$

and

$$B_{\epsilon\beta} = \tfrac{1}{2}\int_0^s \partial_\epsilon\partial_\beta R\, ds'. \quad (C7)$$

The matrix $X_{\alpha\beta} \equiv \gamma_{\alpha\gamma}X^\gamma_\beta$ is symmetric and can be diagonalized, from which it follows that X^γ_β can be diagonalized. Let the eigenvalues of X^γ_β be denoted by λ_β. Then

$$\mathrm{Det}\tilde{D}^{\gamma q}_{n\beta} = \prod_{\beta=0}^{3}\prod_{n=1}^{\infty}(1 - n^{-2}\pi^{-2}\lambda_\beta) = \prod_{\beta=0}^{3}\lambda_\beta^{-1}\sin\lambda_\beta, \quad (C8)$$

or in terms of the original matrix X,

$$\mathrm{Det}\tilde{D} = \mathrm{Det}(\sin X)/\mathrm{Det}X, \quad (C9)$$

where the matrix $\sin X$ can be defined by similarity transformation from the matrix of eigenvalues $\sin\lambda_\beta$ or by matrix power-series expansion. Equation (C8) or (C9) gives the approximation for Det$\tilde{D}(x, x')$ when $R_{\alpha\beta\gamma\delta}$ is slowly varying along the extremizing path from x' to x.

In a similar way, one can obtain an approximation for $\Delta(x, x')$. Suppose that $R_{\alpha\beta\gamma\delta}$ is slowly varying along the geodesic from x' to x, in the sense that

$$\left|\int_0^s R_{\hat{0}\hat{i}\hat{0}\hat{j}}\cos\left(\frac{n\pi s'}{s}\right)ds'\right| \ll \left|\int_0^s R_{\hat{0}\hat{i}\hat{0}\hat{j}}ds'\right| \quad (n=1, 2, \ldots). \quad (C10)$$

Then the matrix A^{pq}_{ij} is diagonal in (p, q), and one obtains from Eq. (7.29) the approximation

$$\Delta(x, x') \approx \mathrm{Det}Y/\mathrm{Det}(\sin Y) \quad (C11)$$

with

$$Y^\gamma_\beta = T_g{}^2\gamma^{\gamma\epsilon}\int_0^1 R_{\hat{0}\hat{\epsilon}\hat{0}\hat{\beta}}du, \quad (C12)$$

where $u = s'/s$ and $\gamma^{\gamma\epsilon} = \gamma_{\gamma\epsilon}$ was defined after Eq. (2.14).

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