Models of evaporating black holes. I

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Classical spacetimes which contain evaporating black holes (EBH's) are constructed using the Vaidya metric. These model EBH spacetimes are spherically symmetric, asymptotically flat, and contain event horizons which terminate after finite duration. These model EBH spacetimes are then used as fixed backgrounds for particle-creation calculations, in an attempt to learn something of the dynamics of real, semiclassical EBH spacetimes while bypassing the difficulties associated with the back-reaction problem. The stress-energy tensor of a quantized massless scalar field is studied on the two-dimensional EBH spacetimes obtained by setting $d\theta = d\phi = 0$. If $\lim(dM/dv) \neq 0$ as $M \to 0$ (v the usual ingoing null coordinate), then an infinite flux of outgoing radiation is produced along the Cauchy horizon of the EBH. This behavior suggests that a correct, self-consistent semiclassical EBH spacetime must have $\lim(dM/dv) = 0$ as $M \to 0$, contrary to the behavior deduced by naively extrapolating Hawking's result all the way down to $M = 0$, which gives $dM/dv \sim M^{-2}$. It also shows that not all zero-mass naked singularities left at the end point of an evaporating black hole are benign; they may produce a divergent flux of created particles.

I. INTRODUCTION

The discovery by Hawking that a black hole emits particles like a blackbody with temperature proportional to its surface gravity[1,2] has acted as a great stimulus to research on the quantum theory of gravity and the theory of quantized matter fields on curved-space backgrounds.[3] Despite a truly impressive amount of effort, little progress has been made at extending Hawking's work to find a self-consistent, quantum mechanically correct, dynamically evaporating black-hole spacetime (hereafter referred to as an "EBH spacetime").

The primary approach to the back-reaction problem has been through the semiclassical theory of gravity, wherein one has a classical gravitational field coupled to the expectation value of the stress-energy tensor of quantized matter fields via the semiclassical Einstein equations $G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle$. Although solutions to these equations representing EBH spacetimes have not yet been found, substantial progress has been made in studying the semiclassical theory. In particular, solutions within the theory (such as Minkowski space) seem to be violently unstable owing to fourth derivatives of the metric appearing in the field equations.[4,5]

Much effort has gone into regularizing and calculating $\langle T_{\mu\nu} \rangle$ on a fixed background spacetime such as the Schwarzschild geometry. Thanks to the efforts of many people,[6,7] it appears that it is now feasible, at least in principle, to calculate $\langle T_{\mu\nu} \rangle$ for the Schwarzchild black-hole spacetime.[8] The result, however, is likely to be a stack of computer output because of the complexity of the mode sums involved. The extreme difficulty of calculating $\langle T_{\mu\nu} \rangle$ for the fixed Schwarzschild background illustrates how much more trying it will be to find self-consistent EBH spacetimes within the semiclassical theory.

In this paper I will study particle creation in EBH spacetimes by using a rather *ad hoc* procedure. The procedure is to construct a classical spherically symmetric, asymptotically flat spacetime which contains an evaporating black hole, and then do a particle-creation calculation on that fixed background spacetime. The procedure is motivated by the observation that while we have extreme difficulty performing back-reaction calculations, there are by now several different methods available to calculate particle-creation effects in a fixed background spacetime relatively easily. One could hope, in the best of all possible worlds, to fortuitously guess a classical EBH metric which would turn out to have its calculated $\langle T_{\mu\nu} \rangle$ equal to its preordained classical $T_{\mu\nu}$. Such a metric would in fact be a correct, self-consistent, semiclassical EBH spacetime. Unfortunately, nothing so grandiose will happen in this paper. We will see that it is possible to make some fairly general statements about how the mass of the black hole decreases as $M = 0$ is approached, in particular, showing that not all zero-mass naked singularities at the end point of event horizons are "benign". Some mass-decrease functions, including the naive one obtained by extrapolating Hawking's fixed background result all the way down to $M = 0$, have naked singularities which produce a diverging flux of outgoing radiation along the Cauchy horizon.

Section II of this paper details the construction of the classical model EBH spacetimes. Section

III contains the particle-creation calculation. Discussion of the results and conclusions are in Sec. IV. Two appendices detail models for which one can explicitly calculate $\langle T_{\mu\nu}\rangle$ (in two dimensions) everywhere in the EBH spacetime.

The sign conventions and notational practices of Misner, Thorne, and Wheeler[9] are used throughout the paper. Units are chosen so that $G=c=\hbar=1$.

II. VAIDYA MODEL OF AN EVAPORATING BLACK HOLE

I will model the Hawking process evaporation of a spherically symmetric black hole with the Vaidya metric[10,11] which represents imploding null fluid. The metric of the model spacetime is

$$ds^2=-\left(1-\frac{2M(v)}{r}\right)dv^2+2dv\,dr + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{1}$$

where

$$M(v)=0\,,\quad v<0\,,$$

and

$$M(v)=0\,,\quad v>v_0\,, \tag{2}$$

and in the region $v_0>v>0$, $M(v)$ is an arbitrary decreasing function of v, which approaches a positive finite value m as $v\to 0$ and approaches zero as $v\to v_0$ (see Fig. 1).

The model EBH spacetime is initially flat, empty Minkowski space for all $v<0$. Then, at $v=0$, an imploding δ-functional shell of null fluid with total (positive) mass m forms a black hole. The results of this paper do not, however, depend on the precise manner in which the black hole is formed; a collapsing ball of matter can replace the imploding null-fluid shell without affecting the results. Negative-energy-density null fluid then falls into the hole at a greater or lesser rate, depending on the choice of $M(v)$, such that the mass of the black hole is reduced to zero at $v=v_0$. The final state is again flat, empty Minkowski space for all $v>v_0$.

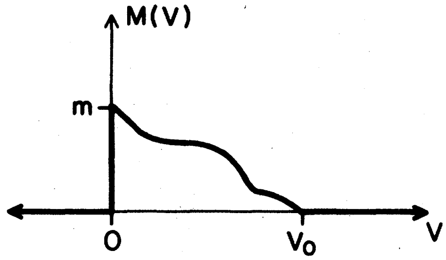

FIG. 1. Possible choice of $M(v)$ for an EBH model.

A Penrose diagram for such a model EBH is illustrated in Fig. 2. Regions I and III are Minkowski space, while region II is the decreasing-mass Vaidya metric section of the model. The null hypersurface labeled $\mathcal{H}$ is the event horizon of the EBH. The radius of the event horizon decreases from some value r_p at $v=0$ to zero at $v=v_0$. The exact value of r_p is a complicated function of $M(v)$, but it is always less than $2m$. The timelike hypersurface labeled $\mathcal{A}$ in Fig. 2 is the apparent horizon of the EBH. Its radius decreases from $2m$ at $v=0$ to zero at $v=v_0$. The apparent horizon lies outside the event horizon, contrary to Proposition 9.2.8 of Hawking and Ellis,[12] because the area of the event horizon monotonically decreases (the weak energy condition is violated). The apparent horizon and the event horizon coalesce at the zero-mass naked singularity (point q in Fig. 2), which gives rise to the Cauchy horizon, labeled $\mathcal{C}$ in Fig. 2.

III. PARTICLE CREATION BY EVAPORATING BLACK HOLES

How does one determine whether a classical model EBH bears any resemblence to a quantum-mechanical real-world EBH? Ideally one could proceed by guessing a classical model EBH metric, then computing the complete four-dimen-

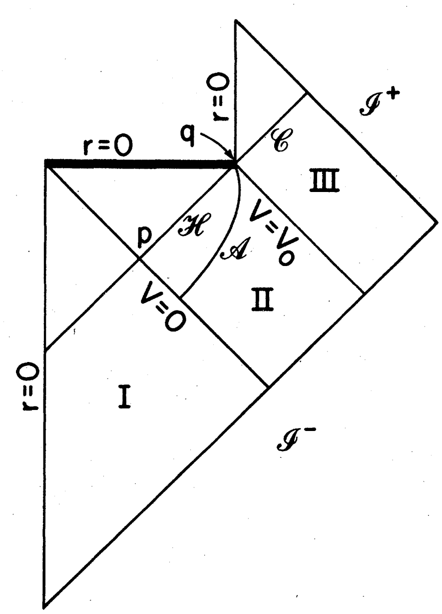

FIG. 2. Vaidya model of an EBH spacetime. $\mathcal{H}$ is the event horizon, $\mathcal{A}$ the apparent horizon, and $\mathcal{C}$ the Cauchy horizon of the spacetime. p is the point at which the event horizon enters the initial Minkowski region, and q is the naked singularity.

sional stress-energy tensor of the quantized fields one is interested in on that background spacetime. If the quantized field's $\langle T_{\mu\nu}\rangle$ happens to be equal to the $T_{\mu\nu}$ of the classical model, then the classical model EBH is in fact a solution of the semiclassical Einstein equations. If the classical and quantum stress-energy tensors differed by a small amount, then one could hope to find a semiclassical EBH solu- by perturbation theory on the original model.

In this section (and those which follow) I will settle for a much more timid goal, namely, trying to determine which classical model EBH's produce an infinite flux of created particles at $\mathscr{I}^+$ and which produce a finite flux. If a model EBH produces an infinite flux of created particles, then clearly the gravitational back reaction to those particles is important. Therefore, that particular model EBH does not resemble a self-consistent semiclassical EBH. On the other hand, a model EBH with finite particle production may be only a perturbation away from a self-consistent semiclassical EBH.

This sort of consideration may be applied to Hawking's original calculation of particle creation by black holes.[2] Since the constant thermal flux emitted by a Schwarzschild black hole persists forever, an infinite total amount of energy is radiated from the Schwarzschild black hole. Obviously the fixed Schwarschild metric is not a good model EBH. The models in this paper, if somewhat *ad hoc* due to their Vaidya nature, at least have event horizons whi which disappear after only a finite time and are thus better models of EBH's than the Schwarzschild metric.

In order to estimate particle-creation effects in the model EBH spacetimes, I will calculate the regularized two-dimensional stress-energy tensor of a quantized massless scalar field. The two-dimensional background spacetime is obtained by taking a $\theta=$ constant, $\phi=$ constant slice of the four-dimensional model EBH spacetime. The regularization of two-dimensional stress-energy tensors was first accomplished by Davies, Fulling, and Unruh[13] and by Davies[14]. Ford and Parker[15] have shown that the two-dimensional stress-energy tensor is related to the geometric optics approximation in four-dimensional spherically symmetric spacetimes. Many other techniques, such as utilizing Bogoliubov transformations, will not work in model EBH spacetimes because of the Cauchy horizon which prevents one from Fourier analyzing functions along $\mathscr{I}^+$. The two-dimensional stress-energy tensor calculations do not have this weakness. The stress-energy tensor may be found for the entire two-dimensional spacetime up to the Cauchy horizon after defining the vacuum state on $\mathscr{I}^-$.

The metric of a two-dimensional spacetime can in general be written in the double-null form

$$ds^2=-C(u,v)du\,dv\,, \tag{3}$$

where $C(u,v)$ is a conformal factor. Since all two-dimensional spacetimes are conformally flat, the scalar wave equation is simply

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\Phi=0\,. \tag{4}$$

A special set of null coordinates $\bar{u},\bar{v}$ are defined to be those in which the solutions to Eq. (4) take the form of the usual Minkowski normal-mode solutions $\exp(-i\omega\bar{u})$, $\exp(-i\omega\bar{v})$. The in-vacuum state is then defined as the state annihilated by the field operators with $\omega>0$. Since the model EBH's are asymptotically flat, the in-vacuum state is uniquely defined by requiring the modes to be plane waves near past null infinity.

Davies, Fulling, and Unruh first regularized the two-dimensional in-vacuum stress-energy tensor using geodesic point separation.[13] The now well-known result is

$$T_{\mu\nu}=\theta_{\mu\nu}+\frac{\mathscr{R}}{48\pi}g_{\mu\nu}\,, \tag{5}$$

where

$$\theta_{\bar{u}\bar{u}}=-(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{u}\bar{u}}\,, \tag{6}$$

$$\theta_{\bar{v}\bar{v}}=-(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{v}\bar{v}}\,, \tag{7}$$

$$\theta_{\bar{u}\bar{v}}=\theta_{\bar{v}\bar{u}}=0\,, \tag{8}$$

C is the conformal factor from the metric in Eq. (3), and $\mathscr{R}$ is the two-dimensional scalar curvature.

Since the model EBH spacetime is past asymptotically flat ($\mathscr{I}^-$ exists), scalar field modes will have the form $\exp(-i\omega v)$ near past null infinity. This gives the relation

$$\bar{v}=v\,, \tag{9}$$

valid everywhere in the spacetime. In Region I ($v<0$), spacetime is Minkowskian and the two-dimensional metric may be written as

$$ds^2=-du\,d\bar{v}\,, \tag{10}$$

where $u=v-2r$, to establish the connection to Eq. (1). Similarly, in region III ($v>v_0$) the metric is

$$ds^2=-dU\,d\bar{v}\,, \tag{11}$$

where $U=v-2r$.

Unfortunately it does not appear possible to explicitly construct the double-null form of the two-dimensional metric for region II, except for very special choices of $M(v)$ (treated in the Appendices). The two-dimensional metric for region II will be left in the form

$$ds^2 = -\left(1 - \frac{2M(V)}{r}\right)dv^2 + 2dv\,dr\,. \qquad (12)$$

The usual reflection boundary condition at $r = 0$ in region I, combined with Eq. (9), yields

$$\bar{u} = u \qquad (13)$$

everywhere in region I ($v < 0$). This is enough to determine $T_{\mu\nu}$ in region I, yielding the obvious answer

$$T_{\mu\nu} = 0 \quad (v < 0)\,. \qquad (14)$$

Since an explicit double-null form for the region II metric does not appear to be constructable in terms of known functions, there is no way to precisely determine $T_{\mu\nu}$ in regions II or III. It will be possible, however, to determine whether the stress-energy tensor component T_{UU} in region III diverges as $U \to v_0$, i.e., as the Cauchy horizon is approached.

To determine $T_{\mu\nu}$ in region III ($v > v_0$), we need to know how to identify an outgoing null geodesic $\bar{u} = u =$ constant in region I with its corresponding null geodesic $U =$ constant in region III. If we could determine this function $U(\bar{u})$, then $T_{\mu\nu}$ in region III could be completely determined, as the metric there could be written

$$ds^2 = -\frac{dU}{d\bar{u}}\,d\bar{u}\,d\bar{v}\,, \qquad (15)$$

and $T_{\mu\nu}$ could be calculated using Eqs. (5)–(8) with $C(\bar{u},\bar{v}) = dU/d\bar{u}$. Since, by the orthogonal nature of the double-null coordinate system in Eq. (11), it is clear that $\partial U/\partial\bar{v} = 0$, we can immediately write down two of the three stress-energy tensor components in region III,

$$T_{U\bar{v}} = T_{\bar{v}\bar{v}} = 0\,. \qquad (16)$$

Thus the only stress-energy present in region III is the outgoing flux of energy created by the EBH, T_{UU}.

The limiting behavior of T_{UU} near the Cauchy horizon clearly depends on the form of $dU/d\bar{u}$ as $U \to v_0$, by Eqs. (15) and (6); thus we must determine how to identify an outgoing $\bar{u} =$ constant null geodesic with its corresponding $U =$ constant null geodesic in region III near the Cauchy horizon.

Outgoing null geodesics in the Vaidya metric of Eq. (12) must satisfy

$$\frac{dr}{dv} = \tfrac{1}{2}\left(1 - \frac{2M(v)}{r}\right). \qquad (17)$$

The function $U(\bar{u})$ may in principle be found as follows. Note that at $v = 0$, $\bar{u} = -2r$. Choose a particular value r_0 at $v = 0$, and then integrate Eq. (17) from $v = 0$ to $v = v_0$ for the chosen $M(v)$, with $r(v = 0) = r_0$ as the initial condition. The final value of r at v_0, which we call R, may be related to U by noting that $U = v - 2r$ at $v = v_0$. Thus, if we consider R as a function of r_0, we can trivially construct $U(\bar{u})$. In particular, with the above relations we notice that

$$\frac{dU}{d\bar{u}} = \frac{dR}{dr_0}\,. \qquad (18)$$

For the moment consider r in Eq. (17) as a function of two variables, v and r_0, i.e., $r = r(v, r_0)$. Then if a prime is used to denote $\partial/\partial v$, Eq. (17) may be written

$$r' = \tfrac{1}{2}\left(1 - \frac{2M(v)}{r}\right), \qquad (19)$$

and taking the derivative with respect to r_0 one finds

$$\left(\frac{\partial r}{\partial r_0}\right)' = \frac{M(v)}{r^2}\left(\frac{\partial r}{\partial r_0}\right), \qquad (20)$$

This may be rearranged into the form

$$\left[\ln\left(\frac{\partial r}{\partial r_0}\right)\right]' = \frac{M(v)}{r^2}\,, \qquad (21)$$

which may be integrated with respect to v to find

$$\ln\left(\frac{\partial r}{\partial r_0}\right) = \int_0^v \frac{M(v)}{r^2}\,dv\,. \qquad (22)$$

As $v \to v_0$, $r \to R$, and Eq. (22) becomes

$$\ln\left(\frac{dR}{dr_0}\right) = \int_0^{v_0} \frac{M(v)}{r^2}\,dv\,, \qquad (23)$$

where the partial derivative has been replaced by a total derivative since R does not depend on v, and it is understood that r in the integrand is equal to $r(v, r_0)$.

Since the integrand in Eq. (23) is positive definite for any arbitrary function $M(v)$ used in a model EBH, the integral must be positive (although possibly divergent). This suggests that, in a neighborhood of $U = v_0$ (the Cauchy horizon), we should be able to approximate the limiting form of $dU/d\bar{u}$ by

$$\frac{dU}{d\bar{u}} = A(v_0 - U)^a + A_1(v_0 - U)^{a+1} + \cdots\,. \qquad (24)$$

Here $a \leq 0$ since, if $a > 0$, then by Eq. (18) the left-hand side of Eq. (23) would approach negative infinity as $U \to v_0$ (equivalently, as $r_0 \to r_p$, see Fig. 2), while we know the right-hand side of Eq. (23) to be positive definite. For an arbitrary function $dU/d\bar{u}$, a can be defined by

$$a = \lim_{U \to v_0} \frac{d[\ln(dU/d\bar{u})]}{d[\ln(v_0 - U)]}\,. \qquad (25)$$

Evaluating T_{UU} from Eq. (6) with the asymptotic form of Eq. (24), one finds that

$$12\pi T_{UU} = a(a-2)(v_0 - U)^{-2} - \frac{A_1}{4A} a(a-2)(v_0-U)^{-1} + O(1) \quad (26)$$

in some neighborhood of the Cauchy horizon. Note that unless either $a=0$ or $a=2$ both the energy density and the integrated energy density [which will be proportional to $(v_0 - U)^{-1}$] diverge as $U \to v_0$. We already know that $a \neq 2$ for any choice of $M(v)$. Let us now see what choices of $M(v)$ yield $a=0$, and hence finite total particle creation, and what choices of $M(v)$ have $a<0$ and divergent particle creation along the Cauchy horizon.

A divergent flux of outgoing radiation is present if $a<0$, and hence

$$\lim_{r_0 \to r_p}\left[\ln\left(\frac{dR}{dr_0}\right)\right] = +\infty . \quad (27)$$

Thus infinite particle production requires that the integral in Eq. (23) diverge. Clearly the integral can only diverge if $r \to 0$ such that $M(v)/r^2 \to \infty$. This can only happen near $v=v_0$, since only there (and only along the event horizon $r_0 = r_p$) can r approach zero such that the integral will diverge. Thus only the asymptotic behavior of $r(v, r_0)$ near $v=v_0$ is important.

I will assume that the mass function $M(v)$ may be approximated near $v=v_0$ by

$$M(v) \approx m_1\left(\frac{v_0 - v}{v_0}\right)^b + \text{higher-order terms}, \quad (28)$$

where b may be defined for an arbitrary $M(v)$ in a manner analogous to Eq. (25).[16]

Inspection of Fig. 2 shows that, for any nonzero value of R, there is a finite neighborhood [range of $(v_0 - v)$] for which

$$\frac{1}{2} > \frac{M(v)}{r}, \quad (29)$$

namely, that area outside the apparent horizon. As an approximation to $r(v, r_0)$ near $v=v_0$, let us assume that $\frac{1}{2} \gg M(v)/r$, so that Eq. (17) becomes

$$\frac{dr}{dv} \approx \frac{1}{2}; \quad \frac{v_0 - v}{v_0} \ll 1 \quad (30)$$

yielding $r \approx \frac{1}{2}(v_0 - v) + R$ near $v=v_0$. Substituting this expression back into the right-hand side of Eq. (17) one obtains

$$\frac{dr}{dv} \approx \frac{1}{2} - \frac{2m_1}{v - v_0 + 2R}\left(\frac{v_0 - v}{v_0}\right)^b \quad (31)$$

and we see that Eq. (30) is justified for all $b>0$ as $v - v_0 \to 0$. A further check can be made by assuming $r \approx \frac{1}{2}(v - v_0) + R_0 + f(v)$ and using this on the left-hand side of Eq. (31) to find $f(v)$, the first correction term. This yields $f(v) \sim (v_0 - v)^{b+1}$, which is less than $(v_0 - v)$ as $v \to v_0$, so the approximation is self-consistent. Thus, in a neighborhood of any final value R, $r(v, r_0)$ can be approximated by

$$r = \tfrac{1}{2}(v - v_0) + R + O((v_0 - v)^{b+1}) . \quad (32)$$

Substituting this expression for r into Eq. (23) yields

$$\ln\left(\frac{dR}{dr_0}\right) = \frac{4m_1}{v_0^b}\int_{v_1}^{v_0} \frac{(v_0 - v)^b dv}{(v_0 - v)^2 - 4(v_0 - v)R + 4R^2} + \int_0^{v_1} \frac{M(v)}{r^2} dv , \quad (33)$$

where v_1 is smaller than the radius of convergence of our approximations, i.e., $v_0 - v_1 \ll v_0$. The second term in Eq. (33) can never diverge, since r can only approach zero as $v \to v_0$. To determine whether the first term diverges as $R \to 0$ $(U \to v_0)$, take the limit

$$\lim_{R \to 0}\left[\ln\left(\frac{dR}{dr_0}\right)\right] = \frac{4m_1}{v_0^b}\lim_{R\to 0}\int_{v_1}^{v_0} \frac{(v_0 - v)^b dv}{(v_0 - v)^2 - 4(v_0 - v)R + 4R^2} + K , \quad (34)$$

where K is the necessarily finite limit of the second term in Eq. (33). Interchanging the order of the limit and integration we find

$$\lim_{R\to 0}\ln\left(\frac{dR}{dr_0}\right) = \frac{4m_1}{v_0^b}\int_{v_1}^{v_0}(v_0 - v)^{b-2} dv + K . \quad (35)$$

This integral converges for all $b>1$ and diverges for all $b \leq 1$. Thus any arbitrarily chosen mass function $M(v)$ which approaches zero with nonzero slope (limit of dM/dv as $v \to v_0$ nonzero) will produce an infinite flux of created particles, while only a mass function which has $dM/dv = 0$ at $v = v_0$ can produce a finite total energy flux.

In terms of the parameters a and b defined by Eqs. (24) and (28), any $b>1$ has $a=0$, while $0<b\leq 1$ has $a<0$. This guarantees, via Eq. (26), that the diverging energy flux always diverges in the positive direction (positive-energy-density particles are radiated to $\mathscr{I}^+$).

IV. CONCLUSIONS

What can we conclude if we believe that the results derived here are valid in the generic, realistic EBH case? An immediate consequence is that the extrapolation of Hawking's fixed Schwarzschild background result (temperature $\sim 1/M$) cannot be expected to hold all the way down to $M=0$. A naive application of Hawking's temperature formula, using v as the time coordinate, predicts that $M \sim (v_0 - v)^{1/3}$. While this should clearly be true in the early stages of the evaporation, the above result shows that it cannot hold all the way down to $M=0$. Instead, at some time close to $M=0$, perhaps on the order of several times, the Planck time, the mass-decrease function must turn over and approach $M=0$ with $dM/dv=0$ in order to have finite total particle creation (see Fig. 3).[16]

Of course, we must consider just how qualitatively correct the results of this paper are when extrapolated to the generic EBH case. Clearly an actual EBH is not well modeled by the cut-and-past Vaidya model presented here. The Vaidya model has negative-energy-density matter present all the way out to past null infinity, whereas a realistic EBH probably violates the energy condition only near the black hole. On the other hand, the negative energy density tends to zero like r^{-2}, and is only important in determining the finiteness of the particle creation near the event horizon, where negative-energy- density matter would be found in a real EBH spacetime.

There are, however, three more serious weaknesses in these models. First, any null fluid Vaidya metric has a classical stress-energy tensor with zero trace, i.e., $T^\alpha_\alpha=0$, while a correct semiclassical EBH spacetime has a nonzero trace for the stress-energy tensor, quadratic in the curvature tensors of the spacetime.[17] I do not see any easy way to correct this weakness. Null fluids and classical electromagnetic fields invariably yield zero trace stress-energy tensors. Any crude attempt to mock up a nonzero trace, perhaps by adding a non-null perfect fluid, will still not have a trace related to the square of the curvature tensors.

The second weakness is the lack of outgoing radiation in the model spacetimes. The shrinkage of the event horizon is controlled by sending in negative energy from $\mathscr{I}^-$, rather than by having positive energy radiated to $\mathscr{I}^+$. One must at least question whether the self-gravitational field of the outgoing created particles might not damp out the infinite particle creation found when $dM/dv \neq 0$ as $M\to 0$ (of course, it could also make it worse). This sort of effect can be modeled with classical Vaidya metrics. A negative-energy-density ingoing Vaidya spacetime may be matched along a timelike hypersurface (ideally near the apparent horizon) to an outgoing positive-energy-density Vaidya spacetime. The timelike boundary surface may then be thought of as a spherical shell of pair creation events; a negative-energy null particle falls down the hole, and a positive-energy null particle escapes to $\mathscr{I}^+$. Such models of EBH spacetimes, and particle-creation effects within such spacetimes, will be considered in another paper.

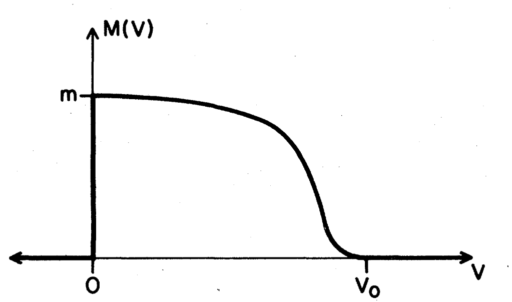

FIG. 3. Mass function which has $M\sim(v_0-v)^{1/3}$ for large M, to agree with Hawking's result, and has $dM/dv\to 0$ as $M\to 0$, to yield finite total particle creation.

The third possible weakness is that the results found here might depend on the use of two-dimensional stress-energy tensor calculations. I feel that it is extremely unlikely that a full four-dimensional calculation would have qualitatively different results. In previous work, where two- and four-dimensional stress-energy tensor calculations can be compared (e.g., the fixed Schwarzschild background[2,6,13] or the instability of the Reissner-Nordström Cauchy horizon[18,19]), the qualitative results (particularly, the question of diverging stress-energy along some null surface) have been independent of the dimensionality of the calculation.

The main result of this paper also has relevance to how one thinks about cosmic censorship. While usually considered as a conjecture within classical general relativity, there are two ways in which semiclassical relativity has impinged on cosmic censorship. First, Hawking showed that black holes evaporate,[2] and Gowdy proved that a future event horizon which lasts only a finite time implies the existence of a naked singularity.[20,21,22] This shows that the effects of quantized matter fields can create naked singularities. Secondly, particle-creation effects have been used to argue that the back reaction from particle creation might prevent the formation of counterexamples to cosmic censorship, such as shell-focusing singularities,[23] Reissner-Nordström singularities with $|Q|>M$,[15,24] and (for strong cosmic censorship) the timelike singularities inside the Reissner-Nordström or Kerr-Newman black hole.[18,19,25] Thus, quantized matter fields may also prevent the formation of naked singularities.

Until now, the naked singularity produced by an EBH has not really been taken seriously; it is often looked on as a "benign" naked singularity, since it has zero mass and presumably only exists for an instant (of course, it could exist longer, and become a negative mass singularity, but it need not). The calculations in this paper show that this sort of zero-mass naked singularity is not always benign and may produce a diverging energy flux of created particles, just as a positive mass (shell-focusing) or negative mass (Reissner-Nordström near $r = 0$) might.

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APPENDIX A: SELF-SIMILAR MODEL OF AN EVAPORATING BLACK HOLE

In this appendix I shall examine the self-similar model of an EBH, where $M(v) \sim (v_0 - v)$. This model is of particular interest since, owing to its self-similarity (or homothesy), it will be possible to explicitly construct the quantum stress-energy tensor for the entire spacetime.

The metric for the entire spacetime is given by Eq. (1), where now

$$M(v) = \begin{cases} 0, & v < 0 \\ m(1 - v/v_0), & v_0 > v > 0 \\ 0, & v > v_0. \end{cases} \tag{A1}$$

The homothesy of the linearly decreasing mass Vaidya metric allows us to explicitly construct a double-null form for the metric; this in turn allows us to find $\langle T_{\mu\nu} \rangle$ everywhere explicitly. Taking a $\theta =$ constant, $\phi =$ constant slice through the model EBH spacetime to get a two-dimensional metric, we are left with

$$ds^2 = -\left[1 - \frac{2m}{r}\left(\frac{v_0 - v}{v_0}\right)\right] dv^2 + 2dv\,dr. \tag{A2}$$

Considerable simplification results from adopting new coordinates z and ζ defined by

$$z = \frac{v_0 - v}{r}, \tag{A3}$$

$$\zeta = -\ln(v_0 - v). \tag{A4}$$

The metric then can be written in the form

$$ds^2 = -e^{-2\zeta}\left[\left(1 - 2\mu z + \frac{2}{z}\right) d\zeta^2 + \frac{2}{z^2}\, d\zeta\, dz\right], \tag{A5}$$

where $\mu = m/v_0$. The metric can now be reduced to a double-null form by defining a new null coordinate η by

$$\eta = \zeta + 2z^*, \tag{A6}$$

$$z^* = \int (z^2 - 2\mu z^3 + 2z)^{-1} dz. \tag{A7}$$

This yields the metric form

$$ds^2 = -e^{-2\zeta}\left(1 - 2\mu z + \frac{2}{z}\right) d\zeta\, d\eta, \tag{A8}$$

valid throughout region II of Fig. 2, from $\zeta = -\ln(v_0)$ to $\zeta = +\infty$. The coordinate z varies from zero (on $\mathscr{I}^-$ and $v = v_0$) to infinity (along the curvature singularity).

The apparent and event horizons are easily defined in terms of the new coordinates. The apparent horizon $\mathscr{A}$ is now the surface $z = \mu/2$. The event horizon is also a homothetic Killing horizon, where the homothetic Killing vector, given by

$$\xi^\zeta = -1, \quad \xi^z = 0, \tag{A9}$$

which satisfies the homothetic Killing equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2 g_{\mu\nu}, \tag{A10}$$

becomes null. The coordinate equation for the event horizon is then simply

$$z = z_+ = \frac{1}{4\mu}\left[1 + (1 + 16\mu)^{1/2}\right]. \tag{A11}$$

This implies that the radius of the horizon at $v = 0$ is

$$r_p = 4m\left[1 + (1 + 16\mu)^{1/2}\right]^{-1} \tag{A12}$$

and hence, if we adopt u and v as null coordinates in the initial Minkowski region ($v < 0$), then the event horizon in that region is given by $u = -2r_p$.

The metric in region I is

$$ds^2 = -du\, dv \quad (v < 0), \tag{A13}$$

and similarly, in region III, the final Minkowski spacetime segment is

$$ds^2 = -dU\, dv \quad (v > v_0). \tag{A14}$$

The two-dimensional stress-energy tensor for a quantized massless scalar field may now be computed by relating these three sets of null coordinates [Eqs. (A8), (A13), and (A14)] to the canonical set $(\bar{u}, \bar{v})$ in which the vacuum state is defined.

If the initial vacuum state on $\mathscr{I}^-$ is to be the usual Minkowski vacuum, then the scalar field modes will be proportional to $\exp(-i\omega v)$ near $\mathscr{I}^-$. This immediately requires that

$$\bar{v} = v \tag{A15}$$

throughout the spacetime. Reflection of these modes through $r = 0$ in region I yields

$$\bar{u} = u. \tag{A16}$$

Matching $\bar{u}$ and η across the boundary $v = 0$, using Eqs. (A3), (A6), (A7), and (A16), and $u = v - 2r$, one finds

$$\frac{d\bar{u}}{d\eta} = v_0 - \bar{u} + \frac{4mv_0}{\bar{u}}. \tag{A17}$$

Performing a second match between η and U at $v=v_0$ and using Eq. (A18) yields

$$\frac{d\bar{u}}{dU}=\frac{\bar{u}^2-\bar{u}v_0-4mv_0}{\bar{u}(U-v_0)}. \tag{A18}$$

Finally, noting that

$$\frac{d\bar{v}}{d\zeta}=e^{-\zeta}=v_0-\bar{v}, \tag{A19}$$

the three metrics [Eqs. (A8), (A13), and (A14)] may be written in terms of $\bar{u}$ and $\bar{v}$:

$$ds^2=-d\bar{u}\,d\bar{v}\quad (v<0), \tag{A20}$$

$$ds^2=\frac{\bar{u}(v_0-\bar{v})(1-2\mu z+2z^{-1})}{\bar{u}^2-v_0\bar{u}-4mv_0}\,d\bar{u}\,d\bar{v}\quad (v_0>v>0), \tag{A21}$$

$$ds^2=\frac{-(U-v_0)\bar{u}}{\bar{u}^2-\bar{u}v_0-4mv_0}\,d\bar{u}\,d\bar{v}\quad (v>v_0), \tag{A22}$$

where z in Eq. (A21) is considered to be a function of $\bar{u}$ and $\bar{v}$, and U in Eq. (A22) is a function of $\bar{u}$.

The two dimensional stress-energy tensor may now be calculated by the usual procedure. The results are, in region I ($v<0$),

$$T_{\mu\nu}=0, \tag{A23}$$

in region II ($v_0>v>0$),

$$T_{\eta\eta}=(12\pi)^{-1}\left(\tfrac{3}{4}\mu^2z^4-\tfrac{1}{2}\mu z^3-\tfrac{3}{2}\mu z^2+\frac{6mv_0}{\bar{u}^2}-\frac{4mv_0^2}{\bar{u}^3}-\frac{12m^2v_0^2}{\bar{u}^4}\right), \tag{A24}$$

$$T_{\zeta\zeta}=(12\pi)^{-1}(\tfrac{3}{4}\mu^2z^4-\tfrac{1}{2}\mu z^3), \tag{A25}$$

$$T_{\zeta\eta}=\frac{-z^3}{24\pi}\left(1-2\mu z+\frac{2}{z}\right), \tag{A26}$$

and in region III ($v>v_0$),

$$T_{UU}=\frac{mv_0(3\bar{u}^2-2v_0\bar{u}-6mv_0)}{6\pi\bar{u}^4(U-v_0)^2}, \tag{A27}$$

$$T_{vv}=T_{Uv}=0. \tag{A28}$$

The stress-energy tensor in region II is observed to be finite everywhere except $z=+\infty$ and/or $\bar{u}=0$, at the curvature singularity. The (η,ζ) coordinate system behaves poorly as $z\to z_+$, $\eta\to\infty$ (the event horizon), but examination of the stress-energy tensor components in a Kruskal-type coordinate system regular on $\mathcal{H}$ shows that they are finite there.

The stress-energy in region III consists solely of a stream of outgoing radiation whose energy density diverges as $U\to v_0$, i.e., as one approaches the Cauchy horizon. Note also that the integrated energy density diverges as $U\to v_0$. The energy density is always positive for $U<v_0$. Since the stress-energy tensor for region II is finite all along the event horizon, it is natural to associate this diverging energy flux with the naked singularity.

APPENDIX B: "STEP-FUNCTION" MODEL OF AN EVAPORATING BLACK HOLE

This model is of interest primarily because of its extreme simplicity and secondarily because $\langle T_{\mu\nu}\rangle$ may again be calculated explicitly for the entire spacetime.

The spacetime's metric is given by Eq. (1), with

$$M(v)=\begin{cases}0, & v<0\\ M, & v_0>v>0\\ 0, & v>v_0\,.\end{cases} \tag{B1}$$

Thus the spacetime consists of a positive mass null shell collapsing at $v=0$ to form a Schwarzschild black hole, followed by an equal-but-negative mass null shell at $v=v_0$ which "erases" the black hole, leaving Minkowski space.

It is trivial to set up double-null coordinates in the three regions:

$$ds^2=-du\,dv,\quad v<0, \tag{B2}$$

$$ds^2=-\left(1-\frac{2m}{r}\right)du^*\,dv,\quad v_0>v>0, \tag{B3}$$

$$ds^2=-dU\,dv,\quad v>v_0. \tag{B4}$$

As usual, to calculate the stress-energy tensor of a quantized massless scalar field we must relate these null coordinates to the canonical set $(\bar{u},\bar{v})$ which defines the usual vacuum state on $\mathscr{I}^-$. The canonical coordinate $\bar{v}$ is simply equal to v by virtue of asymptotic flatness along $\mathscr{I}^-$. Reflection through $r=0$ in region I ($v<0$) then gives

$$\bar{u}=u. \tag{B5}$$

Matching the $\bar{u}$ coordinate to u^* and U is simple because of the simplicity of the Schwarzschild metric. Noting that $u=v-2r$, $u^*=v-2r^*$, where $r^*=r+2m\ln|r/2m-1|$, and $U=v-2r$, the coordinate matches across $v=0$ and $v=v_0$ yield the following differential relations:

$$\frac{d\bar{u}}{du^*}=\frac{\bar{u}+4m}{\bar{u}}, \tag{B6}$$

$$\frac{d\bar{u}}{dU}=\frac{(\bar{u}+4m)(v_0-U)}{\bar{u}(v_0-U-4m)}. \tag{B7}$$

The spacetime metrics of the three regions may now be written in $(\bar{u},\bar{v})$ coordinates as

$$ds^2=-d\bar{u}\,d\bar{v},\quad (v<0), \tag{B8}$$

$$ds^2=-\left(1-\frac{2m}{r}\right)\frac{\bar{u}}{\bar{u}+4m}\,d\bar{u}\,d\bar{v}\quad (v_0>v>0), \tag{B9}$$

$$ds^2 = -\frac{\bar{u}(v_0 - U - 4m)}{(\bar{u}+4m)(v_0 - U)}\, d\bar{u}\, d\bar{v} \quad (v > v_0). \tag{B10}$$

The stress-energy tensor of the quantized field may now be calculated by the usual procedure. The result in region I is the obvious one,

$$T_{\mu\nu} = 0 \quad (v < 0). \tag{B11}$$

In region II ($v_0 > v > 0$) the stress-energy is

$$T_{u^*u^*} = (24\pi)^{-1}\left(-\frac{m}{r^3} + \frac{3}{2}\frac{m^2}{r^4} - \frac{8m}{\bar{u}^3} - \frac{24m^2}{\bar{u}^4}\right), \tag{B12}$$

$$T_{vv} = (24\pi)^{-1}\left(-\frac{m}{r^3} + \frac{3}{2}\frac{m^2}{r^4}\right), \tag{B13}$$

$$T_{u^*v} = (24\pi)^{-1}\left(-\frac{m}{r^3}\right)\left(1 - \frac{2m}{r}\right). \tag{B14}$$

Note that T_{vv} and T_{u^*v} are exactly the expressions given by Davies, Fulling, and Unruh[13] for the two-dimensional Schwarzschild black hole. This is to be expected, since the black hole in this model is exactly Schwarzschild until the moment of evaporation. Equation (B12) differs from Ref. 13 since their result was the limiting form of $T_{u^*u^*}$ along the horizon $r = 2m$. I have not taken that limit here (although, when taken, the results agree, as they must) since in the present model the $r = 2m$ hypersurface is only an apparent horizon lying outside the true event horizon. As Ref. 13 concluded, $\langle T_{\mu\nu}\rangle$ is finite and regular everywhere except at the curvature singularity.

Finally, in region III ($v > v_0$), the stress-energy tensor is

$$T_{vv} = T_{Uv} = 0, \tag{B15}$$

$$T_{UU} = \frac{m}{3\pi(v_0 - U - 4m)^2}\left(\frac{3m}{(v_0 - U)^2} + \frac{1}{(v_0 - U)} - \frac{(v_0 - U)^2}{\bar{u}^3} - \frac{3m(v_0 - U)^2}{\bar{u}^4}\right). \tag{B16}$$

The apparent divergence of T_{UU} at $U = v_0 - 4m$ is illusory. The null surface $U = v_0 - 4m$ is the continuation of the apparent horizon $r = 2m$ into region III. As $U \to v_0 - 4m$, the numerator of Eq. (B16) vanishes quickly enough to keep T_{UU} finite. More precisely, the limiting value of T_{UU} on the apparent horizon (extended into region III) is

$$\lim_{U \to v_0 - 4m} T_{UU} = (128\pi m^2)^{-1}[1 - \exp(-v_0/2m)]. \tag{B17}$$

The divergence at $U = v_0$, the Cauchy horizon, is real. The energy density diverges as $(v_0 - U)^{-2}$, and hence the integrated energy outflow diverges as $(v_0 - U)^{-1}$. Note that this is a positive energy divergence; specifically

$$T_{UU} = (16\pi)^{-1}(v_0 - U)^{-2} + O((v_0 - U)^{-1}) \tag{B18}$$

near $U = v_0$.

Since $T_{u^*u^*}$ is regular along the event horizon, one is forced to conclude that the infinite flux of outgoing radiation is produced by the naked singularity.

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