PHYSICAL REVIEW D VOLUME 23, NUMBER 11 1 JUNE 1981

Angular zeros of Brown, Mikaelian, Sahdev, and Samuel and the factorization of tree amplitudes in gauge theories

C. J. Goebel and F. Halzen
Physics Department, University of Wisconsin—Madison, Madison, Wisconsin 53706

J. P. Leveille
Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan 48109
(Received 1 December 1980)

The zero in the angular distribution of the process $q\bar{q}\to\gamma W$ (discovered by Brown, Mikaelian, Sahdev, and Samuel) when the magnetic moment of the W has the Yang-Mills value, is shown to be a consequence of a factorizability of the amplitude into one factor which contains the dependence on the charge or other internal-symmetry indices, and another which contains the dependence on the spin or polarization indices. In gauge theories generally, this factorization is found to hold for any four-particle tree-approximation amplitude, when one or more of the four particles is a gauge-field quantum. The factorization hinges on a "spatial generalized Jacobi identity" obeyed by the polarization-dependent factors of the vertices, in analogy to the generalized Jacobi identity obeyed by the charge-index-dependent factors. We emphasize that observation of the process $q\bar{q}\to\gamma W$ in $p\bar{p}$ collisions or the decay $W\to q\bar{q}\gamma$ provides a direct test of the prediction of gauge (Yang-Mills) theories for vector-vector-vector couplings, just as much as would $e^+e^-\to Z\to W^+W^-$.

Brown, Mikaelian, Sahdev, and Samuel[1] (BMSS) discovered that the angular distribution of the process $q\bar{q}\to W\gamma$ in lowest order has a zero (at an angle which depends on the charge of the quark q) if the magnetic moment of the weak boson W has the value given by the standard electroweak model. This value is equivalent to the fact that in the model the W-W-γ vertex, like all vector-vector-vector vertices in gauge (Yang-Mills) theories, has the Yang-Mills form, symmetric in the three vector particles. Therefore, the study of the angular distribution of the process[1,2] $q\bar{q}\to W\gamma$, or the energy distribution[3] in $W\to q\bar{q}\gamma$, directly probes the gauge structure of the theory in the same way that measurements of processes such as $e^+e^-\to Z\to W^+W^-$ would.

The BMSS zero is surprising, because the differential cross section of the process $q\bar{q}\to W_\gamma$ is the sum of many partial cross sections for the different spin (polarization) states; these must all vanish together for the (unpolarized) cross section to vanish. Why does that happen? The answer is that the lowest-order (tree-approximation) amplitude for the process can be factored so that all the charge-index dependence is in one factor and all the polarization dependence is in another factor. The BMSS zero is in the first factor, and is thus common to the amplitudes of all polarization states.

We shall show below[4] that in any gauge theory such a factorization of the internal-symmetry-(charge-) index dependence and the polarization (spin) dependence into separate factors holds for any tree-approximation four-particle amplitude when one or more of the four particles are gauge-field bosons. For clarity, before doing the general case, we first do a simple example.

Consider a vertex of three charged scalar particles. To get the scattering amplitude for these particles plus a photon, we attach the photon to each leg in turn and sum the diagrams, getting

$$A=\sum_i \frac{A_iB_i}{C_i}\,. \tag{1}$$

We have separated each diagram into a charge factor (A_i), a polarization-dependent factor (B_i), and the propagator denominator (C_i). Their explicit expressions are listed in Table I, using the kinematics defined in Fig. 1. [In Eq. (1), as throughout this paper, we omit from amplitudes overall constant factors, such as the coupling constant of the three-vertex of the three scalar particles.] Notice that

$$\sum_i A_i=\sum_i B_i=\sum_i C_i=0 \tag{2}$$

from charge conservation, energy-momentum conservation $\sum_i p_i=0$, and the massless $p^2=0$ and transverse $p\cdot\epsilon=0$ properties of the photon. The relations of Eq. (2) allow us to write A in factored form,

TABLE I. Factors for the processes of Fig. 1 for three charged scalar particles and a photon. The amplitude is given by $A=\sum_i A_iB_i/C_i$.

Diagram	A_i	B_i	C_i
1	Q_1	$p_1\cdot\epsilon$	$p_1\cdot p$
2	Q_2	$p_2\cdot\epsilon$	$p_2\cdot p$
3	$-(Q_1+Q_2)$	$p_3\cdot\epsilon$	$p_3\cdot p$

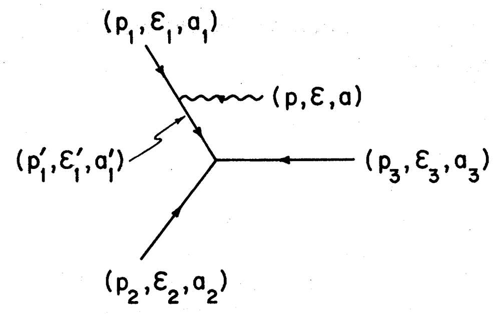

FIG. 1. A tree diagram; the p_i are (incoming) four-momenta, the ϵ_i are polarization vectors and the a_i are internal symmetry charges. In each of the other two tree diagrams the photon, labeled (p, ϵ, a), is attached to another particle, 2 or 3, respectively.

$$\begin{aligned} A &= \frac{A_1 B_1}{C_1} + \frac{A_2 B_2}{C_2} + \frac{A_3 B_3}{C_3} \\ &= \frac{1}{C_3}(A_1 C_2 - A_2 C_1)\left(\frac{B_2}{C_2} - \frac{B_1}{C_1}\right), \end{aligned} \tag{3}$$

or in equivalent forms in which 1, 2, 3 are permuted. Equation (3) exhibits the factorization described above, in which the charge dependence (in the A_i) and the polarization dependence (in the B_i) of the amplitude occur in separate factors. Inserting the expressions from Table I, we have

$$A = \frac{1}{p_3 \cdot p}[(p_2 \cdot p)Q_1 - (p_1 \cdot p)Q_2]\left(\frac{p_2 \cdot \epsilon}{p_2 \cdot p} - \frac{p_1 \cdot \epsilon}{p_1 \cdot p}\right). \tag{4}$$

The amplitude and therefore the cross section vanish at the angle

$$\frac{p_1 \cdot p}{p_2 \cdot p} = \frac{Q_1}{Q_2}. \tag{5}$$

For massless particles this can be simply written in terms of Mandelstam variables as

$$A = (Q_1 u - Q_2 t)\left[\frac{(p_2 \cdot \epsilon)t - (p_1 \cdot \epsilon)u}{stu}\right], \tag{4$'$}$$

which vanishes at the angle $t/u = Q_1/Q_2$.

As we will show further on, the structure of Eqs. (1) and (2) remains unchanged when the particles are given arbitrary spins 0, $\frac{1}{2}$, or 1, and thus the factorization Eq. (3) continues to hold. In the process[1] $q\bar{q} \to \gamma W$, if the $q\bar{q}$ and W are labeled 1, 2, and 3, respectively, the B_i are

$$\begin{aligned} B_1 &= \bar{v}(p_2)\Gamma_\mu(\not{p}_1 + \not{p})\gamma_\nu u(p_1)\epsilon_3^\mu \epsilon^\nu, \\ B_2 &= \bar{v}(p_2)\gamma_\nu(\not{p}_2 + \not{p})\Gamma_\mu u(p_1)\epsilon_3^\mu \epsilon^\nu, \\ B_3 &= \bar{v}(p_2)\Gamma^\lambda u(p_1) C_{\lambda\mu\nu}(-(p_3 + p), p_3, p)\epsilon_3^\mu \epsilon^\nu, \end{aligned} \tag{6}$$

where

$$\begin{aligned} C_{\lambda\mu\nu}(p_1, p_2, p_3) = {} & g_{\lambda\mu}(p_1 - p_2)_\nu + g_{\mu\nu}(p_2 - p_3)_\lambda \\ & + g_{\nu\lambda}(p_3 - p_1)_\mu, \end{aligned}$$

where $\Gamma_\mu = \gamma_\mu(a + b\gamma_5)$, with a and b arbitrary. For simplicity we have taken the quarks to be massless; in the massive case, B_3 would have an additional term $\sim m_W^{-2}$ [see Eq. (11b)] because of nonconservation of the axial-vector current. Note that in B_1 and B_2 the coupling of γ to a quark has been taken to be the minimal (Dirac) coupling, and in B_3 the coupling of γ to W has been taken to be the Yang-Mills coupling. The relation $\sum_i B_i = 0$ and the consequent zero of the differential cross section would not hold if we used different couplings, e.g., different magnetic moments of the quark or W, rather than the minimal couplings required by the Yang-Mills gauge theory. In this sense a reaction such as $q\bar{q} \to W\gamma$ provides a direct test of the underlying gauge theory.

As another example covered by our general theorem, we mention $q\bar{q} \to gg$, where q is a quark and g is the color gauge boson (gluon). The color-charge factors of the three tree diagrams of Fig. 1 are $(T^a T^b)/t$, $(T^b T^a)/u$, and $[T^a, T^b]/(u+t)$, respectively, and the cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{T^b T^a u + T^a T^b t}{u + t}\right)^2 \left.\frac{d\sigma}{d\Omega}\right|_{\text{Abelian}} \tag{7}$$

which again displays the structure of Eqs. (3)—(5). σ_{Abelian} is the $e\bar{e} \to \gamma\gamma$ cross section.

We now turn to the general result,[4] which we state as follows:

Let 1, 2, 3 be three particles which have arbitrary masses, spins ≤ 1, and (belong to) representations of a local (gauged) semisimple internal-symmetry group G. (Any of the three particles may be the massless gauge boson g of G.) Let the three particles have a coupling (three-vertex) which is invariant under G as well as the Lorentz group, is minimal (= renormalizable), and which factorizes. Then the tree-diagram S-matrix (mass-shell) amplitude for the four particles $123g$, constructed from the minimal gauge-boson–particle couplings and the 123 three-vertex specified above, factorizes.

The demonstration of this result hinges on a property of the three-vertices which has not been previously recognized, to our knowledge. It is a kind of Jacobi identity for the polarization-dependent parts of the three-vertices [Eq. (14) below], analogous to the Jacobi identity of the charge-index-dependent parts [Eq. (12) below]. No physical significance or general demonstration (as opposed to case-by-case verification) of this identity is known to us.

By our assumptions, the three-vertex of particles 1, 2, 3 has the form

$$\Gamma^{123}_{a_1a_2a_3} V^{123}(p_1p_2p_3, \epsilon_1\epsilon_2\epsilon_3)\,, \tag{8}$$

where the a_i are internal-symmetry ("charge") indices (each a_i stands for several numbers if G is of rank higher than one), the p_i are incoming four-momenta ($\sum p_i = 0$), and the ϵ_i are polarization "vectors", viz., four-vectors for spin-1 particles, spinors for spin-$\frac{1}{2}$ particles, and (trivially) scalars for spin-0 particles. The factor $\Gamma^{123}_{a_1a_2a_3}$ is a Clebsch-Gordan (CG) coefficient of G. The factor V^{123} is a Lorentz invariant and is linear in each of the ϵ_1. For the various spin possibilities we give the minimal V^{123} in Table II.

TABLE II. Minimal (=point=renormalizable) couplings.

s_1	s_2	s_3	$V^{123}(p_1p_2p_3, \epsilon_1\epsilon_2\epsilon_3)$
0	0	0	1
0	0	1	$(p_1 - p_2)\cdot \epsilon_3$
0	1	1	none
1	1	1	$(p_1-p_2)\cdot\epsilon_3\,\epsilon_1\cdot\epsilon_2$ $+(p_2-p_3)\cdot\epsilon_1\,\epsilon_2\cdot\epsilon_3$ $+(p_3-p_1)\cdot\epsilon_2\,\epsilon_3\cdot\epsilon_1$
$\frac{1}{2}$	$\overline{\frac{1}{2}}$	0	$\overline{\epsilon_2}\gamma\epsilon_1\,,\quad \gamma = a + b\gamma_5$
$\frac{1}{2}$	$\overline{\frac{1}{2}}$	1	$\overline{\epsilon_2}\not{\epsilon}_3\gamma\epsilon_1$

The three-vertex of the gauge-boson minimal coupling to, say, particle 1, is a special case of (8), namely,

$$\Gamma^{1g1}_{a_1'aa_1} V^{1g1}(p_1'pp_1, \epsilon_1'\epsilon\epsilon_1)\,. \tag{9}$$

Since g belongs to the adjoint representation of G, the CG coefficient $\Gamma^{1g1}_{a_1'aa_1}$ is a matrix of the generator J_a of G.

The tree-diagram amplitude for the $123g$ four-vertex is

$$A_{123g} = \sum_{(3)} \Gamma^{123}_{xa_2a_3}\Gamma^{1g1}_{xaa_1}[(2p_1\cdot p + p^2 + p_1{}^2 - m_1{}^2)^{-1} V^{123}((p_1+p)p_2p_3, \epsilon_1'\epsilon_2\epsilon_3) V^{1g1}(-(p_1+p)pp_1, \overline{\epsilon}_1{}'\epsilon\epsilon_1)$$
$$-\epsilon\cdot\partial_{p_1} V^{123}(p_1p_2p_3, \epsilon_1\epsilon_2\epsilon_3]\,, \tag{10}$$

where $\sum_{(3)}$ means the sum over the three terms resulting from replacing the role of particle 1 (to which g is attached) by particles 2 and 3, respectively. There is also an implied sum over the repeated indices of the intermediate particle, namely, the charge index x and the polarization ϵ_1'; the sum over the latter means a sum over a complete set of polarizations, with the results

$$\epsilon_1'\overline{\epsilon}_1' = m_1 + \not{p}_1 \quad \text{for Dirac particle, spin } \tfrac{1}{2}$$
$$= -m_1 + \not{p}_1 \quad \text{for antiparticle, spin } \overline{\tfrac{1}{2}} \tag{11a}$$

and

$$\epsilon_1'\overline{\epsilon}_1' = 1 - p_1p_1/m_1{}^2 \quad \text{for vector particle, spin 1.} \tag{11b}$$

The bars on $\overline{x}$ and $\overline{\epsilon}_1'$ are relevant only if non-self-conjugate bases are used. The last term in (10) (which has no propagator factor) is a seagull, resulting from the momentum dependence of V^{123}

One can verify gauge invariance, i.e., that if ϵ is replaced by p the amplitude (10) vanishes when the particles 1, 2, 3 are real, i.e., on the mass shell ($p_i{}^2 = m_i{}^2$) and with physical polarization. In all cases, this vanishing of (3) comes about as follows: When $\epsilon = p$ and 1, 2, 3 are real, the quantity in the square bracket is the same for all three terms; consequently (10) is proportional to the sum over the charge-index factor, which vanishes,

$$\Gamma^{123}_{xa_2a_3}\Gamma^{1g1}_{\overline{x}aa_1} + \Gamma^{123}_{a_1xa_3}\Gamma^{2g2}_{\overline{x}aa_2} + \Gamma^{123}_{a_1a_2x}\Gamma^{3g3}_{\overline{x}aa_3}$$
$$\equiv \sum_{(3)} \Gamma^{123}_{xa_2a_3}\Gamma^{1g1}_{xaa_1} = 0\,. \tag{12}$$

This vanishing is the statement that $\Gamma^{123}_{a_1a_2a_3}$ is an invariant coupling of the three representations, in view of the fact that the $\Gamma^{jgj}_{xaa_j}$ are matrix elements of the generators. In the special case that all particles are gauge bosons, the $\Gamma^{123}_{a_1a_2a_3} = \Gamma^{ggg}_{a_1a_2a_3}$ are the structure constants and (12) is the Jacobi identity. Hence we can call (12) the generalized Jacobi identity.

We now come to the factorization. The tree amplitude (10) has the form of Eq. (1), where the A_i are the charge-index factors $\Gamma^{123}\Gamma^{igi}$, the B_i are the polarization-dependent factors V^{123} V^{igi}, and the C_i are the propagator denominators $2p_i\cdot p + p^2 + p_i{}^2 - m_i{}^2$. When all particles (including the gauge boson) are on the mass shell, i.e., $p_i{}^2 = m_i{}^2$ and $p^2 = 0$, the sum of the C_i vanishes,

$$\sum C_i = \sum 2p_i\cdot p = -2p^2 = 0\,. \tag{13}$$

According to Eq. (12), the sum of the A_i vanishes. When all particles (including the gauge boson) are physical, i.e., on the mass shell, and the vector particles are transverse, $p_i\cdot\epsilon_i = p\cdot\epsilon = 0$, the sum of the B_i also vanishes, i.e.,

$$\sum_{(3)}[V^{123}((p_1+p)p_2p_3,\epsilon_1'\epsilon_2\epsilon_3)V^{1g1}(-(p_1+p)pp_1,\epsilon_1'\epsilon\epsilon_1)-\epsilon\cdot\partial_{p_1}V^{123}(p_1p_2p_3,\epsilon_1\epsilon_2\epsilon_3)]=0\,. \tag{14}$$

This "spatial generalized Jacobi identity" is easily verified for all cases, but we do not know a more abstract demonstration. From the preceding, namely

$$\sum A_i=\sum B_i=\sum C_i=0\,, \tag{15}$$

factorization follows, as previously described in Eqs. (1), (2), and (3), namely,

$$\sum_{i=1}^{3}A_iB_i/C_i=f(A_i,C_i)g(B_i,C_i)\,, \tag{16}$$

i.e., the dependence on the A_i (containing the charge indices) and the B_i (containing the polarizations) occurs in separate factors.

We make a brief comment on many-particle amplitudes. Consider a five-particle tree amplitude A_{1234g}, which we construct as before by attaching a gauge boson to a four-particle tree amplitude A_{1234}. The simplest case would be that all four particles are spin-0, and are coupled by a point four-vertex (i.e., we assume they have no three-vertices). The above considerations go through with the replacement of Γ^{123} by Γ^{1234} and V^{123} by V^{1234} $(=1)$; the sum $\sum_{(3)}$ becomes $\sum_{(4)}$ in Eqs. (10, (12), (13), (14), and the left-hand side of Eq. (16). The vanishing of the sums, Eq. (15), holds; the sole difference is that now (with i running over four values instead of three) factorization, Eq. (16), does not follow.

One also easily verifies that factorization does not occur for the case that the four spin-0 particles are coupled by three-vertices, so that the tree amplitude A_{1234} is a sum of three terms. In the case that some of the particles have spin greater than 0, the tree amplitude A_{1234} itself does not factor in general (if one or more of the particles are gauge bosons, the tree amplitude A_{1234} factors, but *only* when all four particles are real), so it is not unexpected that the amplitude A_{1234g} does not factor.

ACKNOWLEDGMENTS

This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contract No. DE-AC02 76ER00881-172.

[1]R. W. Brown, D. Sahdev, and K. O. Mikaelian, Phys. Rev. D 20, 1164 (1979); K. O. Mikaelian, M. A. Samuel, and D. Sahdev, Phys. Rev. Lett. 43, 746 (1979).

[2]C. Rubbia, in Proceedings of the XIth International Symposium on Multiparticle Dynamics, Brugge, Belgium, 1980 (unpublished).

[3]K. O. Mikaelian, private communication.

[4]See also Zhu Dongpei, Phys. Rev. D 22, 2266 (1980).