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# How unique are SU(5) and SO(10) grand unified theories?

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We find that SU(5) and SO(10) grand unified theories are almost uniquely chosen under the major ansatz: only left-handed particles in SU(2) doublets and their antiparticles in SU(2) singlets. We have not used any information on the color group, family structures, or quark-lepton correspondence, etc., at all. The consequence of grand unification is that the color group is SU(3), quarks come in color triplet, and their electric charges are $\frac{2}{3}$ and $-\frac{1}{3}$.

Because of our general belief in the gauge principle, many people have been attempting to construct grand unified models of strong, weak, and electromagnetic interactions[1]; among these, the original SU(5) (Ref. 2) and its extension SO(10) (Ref. 3) are the most popular ones. However, aside from the requirement of renormalizability (hence, the anomaly-free condition[4]), there seems to be a lack of principles for grand unification, which has resulted in a proliferation of models. In this paper, we show that the quantum-number assignments in the Glashow-Weinberg-Salam SU(2) ⊗ U(1) (Ref. 5) alone yield the surprising results: The possible grand unified simple groups are only SU(5) and SO(10). The color group is SU(3) with quarks in color triplets. Quark electric charges are $\frac{2}{3}$ and $-\frac{1}{3}$. Consequently, no simple group exists for unifying the family structure with the grand unified theories. There is a good contrast between our approach and that given by Gell-Mann, Ramond, and Slansky,[6] who have investigated the embedding of the color SU(3) group into a grand unification group.

For particle unification, just fixing the electric charges of quarks and leptons is enough to pick up possible unification groups *without specifying the structures of the electroweak group or the color group.*[7] Here, we are attempting the more grand scheme, i.e., unification of particles and antiparticles as Georgi and Glashow[2] have attempted.

Our motivation is provided by the work of Georgi,[8] which has shown that we cannot unify the electron family if the color group is not SU(3). Our results are the generalization of his and are more general.

Our main ansatz is as follows: *There exist only left-handed particles in SU(2) doublets and their antiparticles in SU(2) singlets.* We have neither right-handed particles in SU(2) doublets nor their antiparticles in SU(2) singlets. We do not put any restrictions on the number of quark or lepton doublets. We denote leptons and quarks as $N$, $E$, $U$, and $D$ and their numbers by $n_l$ and $n_q$, respectively. The quantum-number assignments under $G(\text{color}) \otimes \text{SU}(2) \otimes \text{U}(1)$ are as follows:

$$\begin{aligned} &n_l \quad (N,E)_L\colon\ (1,2,-\tfrac{1}{2}),\\ &n_l \quad E_L^c\colon\ (1,1,1),\\ &n_0 \quad N_L^c\colon\ (1,1,0),\\ &n_q \quad (U,D)_L\colon\ (m,2,\alpha/2m),\\ &n_q \quad U_L^c\colon\ (m,1,-(m+\alpha)/2m),\\ &n_q \quad D_L^c\colon\ (m,1,(m-\alpha)/2m), \end{aligned} \tag{1}$$

where $\alpha$ denotes the ratio of $n_l$ to $n_q$ and the electric charge operator is defined by $Q = I_3 + Y$. The U(1) quantum numbers for quarks are chosen to cancel the triangle anomaly.[4,8] We include the possibilities such as $n_l \neq n_q$ (violation of quark-lepton correspondence), $n_l \neq n_0$ (possibility of having Majorana neutrinos), and $n_l \neq 1$ and $n_q \neq 1$ (more than one family in a single multiplet). If we have the case of $n_q \neq 1$, then we might have a simple group to unify the family structure. Our other ansatz is that no quarks can have zero electric charges.

Using the fact that $m \neq \alpha$, since otherwise the electric charge of the $D$ quark is zero, we have $\text{Tr}\, Y^5 \neq 0$. However, any group except $E_6$, SO(10), or SU($N$) ($N \geqslant 3$) satisfies $\text{Tr} X^5 = 0$ for any (irreducible or reducible) representation $X$ of its Lie-algebra element.[9] Thus, we can use only $E_6$, SO(10), or SU($N$) ($N \geq 3$) to unify quarks and leptons in (1) (and, moreover, the use of non-self-contragredient representations of these groups is necessary, otherwise we have $\mathrm{Tr}X^5=0$). At this point we can say more about the highest weight of the representation: They are, if fully reduced, one of $\Lambda_j$ or $\Lambda_j+\Lambda_k$, where $\Lambda_j$ denotes the fundamental weight for $G$, since the weak $i$-spins are only doublets and singlets. The derivation is similar to the one given in Ref. 10.

Now, we consider the unification of particles in (1) as a *single irreducible representation.* For SU($N$) ($N \geq 3$), the representation should be completely antisymmetric with the highest weight $\Lambda_j$ ($1 \leq j \leq N-1$); otherwise it would contain more states other than doublets and singlets in SU(2) as can be shown using the argument used by Gell-Mann *et al.*[6] Then, the anomaly coefficient, which is in proportion to the third-order Casimir invariant, does not vanish, except for the self-contragredient case. For the case of $E_6$, we have the quartic trace identity[11]

$$\mathrm{Tr}X^4 = K(\rho)(\mathrm{Tr}X^2)^2 \tag{2}$$

for any irreducible representation $\rho$ and for any generic element $X$ of the Lie algebra of $E_6$. Applying this identity to $X=Y+tI_3$, where $t$ is an arbitrary number, we find that $m=\alpha$ in order to satisfy Eq. (2), which leads to the neutral electric charge of the $D$ quark. Therefore, we have only SO(10) as a possible candidate for grand unification of a single irreducible multiplet.

The nonexactness of the quark-line rule demands that the color group should be one of $E_6$, SO($4m+2$) ($m \geq 2$), or SU($m$) ($m \geq 3$) (see Ref. 12). The examination of subgroups in SO(10) shows that we can have only *SU(3) as the color group.* After some calculation, using the result noted in an earlier paragraph, the possible representations are of the type $\Lambda_5$, $\Lambda_1+\Lambda_5$, or $2\Lambda_5$ (as well as their complex conjugates). It turns out that the last two representations under the decomposition into SU(3) ⊗ SU(2) ⊗ U(1) contain triplets in SU(2). Therefore, we have found that the spinor representation $\Lambda_5$ is the only candidate in SO(10), which is the standard model: $n_l=n_q=n_0=1$, $m=3$, with the electric charges $Q_U=\frac{2}{3}$ and $Q_D=-\frac{1}{3}$. We can prove the same result, using the identity[11]

$$\mathrm{Tr}X^7 = D(\rho)\,\mathrm{Tr}X^2\,\mathrm{Tr}X^5 \tag{3}$$

which is a consequence of the fact that SO(10) (also $E_6$) does not possess fundamental seventh-order Casimir invariants. Actually, we can show that the quark-line rule is used only to eliminate SU(2) as the color group.

Next, we consider the unification of particles as *two irreducible representations* $\rho_1$ and $\rho_2$. We can show that the only two multiplet structures possible are

$$\begin{aligned}&(n_l(N,E)_L, n_0^{(1)}N_L^c, n_q D_L^c),\\ &(n_l E_L^c, n_0^{(2)}N_L^c, n_q(U,D)_L, n_q U_L^c)\end{aligned} \tag{4}$$

for the case where given leptons and quarks with the same $Y$ and $I$ will not split, except possibly $N_L^c$. Other cases will contradict $\mathrm{Tr}^{(i)}Y=0$ or $\mathrm{Tr}^{(i)}Q=0$, since $m \neq \alpha$, where $\mathrm{Tr}^{(i)}$ denotes the trace operation in the representation space $\rho_i$. Note that in order to have the multiplet structure (4), we must have $m=3\alpha$, which gives $Q_U=\frac{2}{3}$ and $Q_D=-\frac{1}{3}$. Then $\mathrm{Tr}^{(1)}Y^3 \neq 0$ tells us that the unification group should be SU($N$) ($N \geq 3$), since $\mathrm{Tr}X^3=0$ for any group except SU($N$) ($N \geq 3$).[9] The representations to be used are of the types $\Lambda_j$ and $\Lambda_k$ ($1 \leq j,\ k \leq N-1$).

The anomaly-free condition is

$$d(\Lambda_j)I_3(\Lambda_j)+d(\Lambda_k)I_3(\Lambda_k)=0\ , \tag{5}$$

where $d(\Lambda_j)$ and $I_3(\Lambda_j)$ denote the dimension and the value of the third-order Casimir invariant for the irreducible representation $\Lambda_j$. We have to satisfy one more condition:

$$\frac{\mathrm{Tr}^{(1)}(I_3)^2}{\mathrm{Tr}^{(2)}(I_3)^2}=\frac{d(\Lambda_j)I_2(\Lambda_j)}{d(\Lambda_k)I_2(\Lambda_k)} \tag{6}$$

which is derived from the well-known identity

$$\mathrm{Tr}(X_\mu X_\nu)=\frac{d(\lambda)}{d(\lambda_0)}I_2(\lambda)g_{\mu\nu}\ , \tag{7}$$

where $I_2(\lambda)$ denotes the second-order Casimir invariant and $\lambda_0$ denotes the adjoint representation. Then, Eqs. (5) and (6) yield

$$2N=j+3k\ . \tag{8}$$

Up to SU(500), we have found only two solutions to Eqs. (5) and (6): $\underline{\bar{5}} \oplus \underline{10}$ in SU(5) and $\underline{\overline{4368}} \oplus \underline{11440}$ in SU(16). By counting the number of $i$-spin doublets in each representation $\Lambda_j$, which is given by the binomial coefficient $\binom{N-2}{j-1}$, we can dispose the case of SU(16). Thus, we have only one SU(5) with $n_l=n_q=1$, $n_0=0$, $m=3$, $\Lambda=\underline{\bar{5}} \oplus \underline{10}$, which is the standard model. We conjecture that this holds without any restriction on $N$, such as $N \leq 500$ used here.

Actually, we can prove the same *if* $n_0=0$ from the beginning. The proof goes as follows: As is shown in Ref. 10, the form of diagonal operators, which has only two eigenvalues $x$ and $y$ and satisfies $x-y=1$, can be given explicitly in terms of fundamental weights, apart from a constant multiplication. For the representation $\Lambda_j$ ($2 \leq j \leq N-2$), the diagonal operator above has the unique form $X \propto \Lambda_1(H)$, where $H$ denotes the Cartan subalgebra element. However, we can define two operators $\tilde{Y}$ and $Z$ as $\tilde{Y}=\frac{5}{6}Y$ and $Z=I_3+\frac{3}{5}Y$, whose eigenvalues for $\rho_1$

are $(\frac{2}{5}, -\frac{3}{5})$ and $(\frac{1}{5}, -\frac{4}{5})$, respectively. Thus, the fact that eigenvalues for the same operator differ from each other implies that we cannot use the representation $\Lambda_j$ $(2 \leq j \leq N-2)$. For $\Lambda_1$ (or $\Lambda_{N-1}$), the diagonal operator does not have a unique form (see details in Ref. 10) and Eqs. (5) and (6) yield the unique solution $N=5$, $j=1$, and $k=3$, which is the standard SU(5) model. In the foregoing, we have assumed the cancellation of triangle anomaly. However, for the uniqueness of SU(5), it turns out that this assumption is not necessary, but emerges as a consequence. Since the argument for this is quite involved, it will be given elsewhere.

A few comments are in order: Why can we not use groups larger than SU(5) or SO(10)? The reason is that we always have particles with opposite chirality in addition to those in Eq. (1) for bigger groups. That is why $\mathrm{Tr}X^5=0$ can be satisfied trivially. Thus, for groups larger than SU(5) or SO(10), we must invent some mechanism to make those particles heavy.

Recently, Zee has found the uniqueness of SU(5) and SO(10) by examining the maximal simple subgroup of $SU(N_f) \otimes U(1)$, which is free from anomaly and free from bare masses.[13] The choice of $N_f=45$ yields only SU(5) and SO(10). Note that our conclusion is far more general than those obtained by Georgi[8] and Zee.[13] We have assumed only the quantum number of $G(\mathrm{color}) \otimes SU(2) \otimes U(1)$.

As a consequence of the uniqueness of the SU(5) and SO(10) grand unified models, we see that no simple group exists for unifying the family structures, using just ordinary quarks and leptons. Our result may provide some justification why the color group is SU(3) and why the electric charges of $U$ and $D$ quarks are $\frac{2}{3}$ and $-\frac{1}{3}$, respectively.

Many details of the results presented here will be discussed elsewhere.[14]

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