Borel transform technique and the n-bubble-diagram contribution to the lepton anomaly

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By using the Borel transform technique we calculate analytically the muon anomaly from the mass-dependent n-bubble diagram in the limit where the mass ratio m_μ/m_e is large. For large n the sign oscillates, in contrast with the correct anomaly. The analysis shows that the validity of expanding in $L \equiv \ln(m_\mu/m_e)$ is strongly dependent on the order $N=2n+2$, in which we calculate.

I. INTRODUCTION

In calculating the mass-dependent contribution to g-2 of the muon, it has been customary for many years to use the large mass ratio $m_\mu/m_e \sim 207$ as a good expansion parameter.[1,2] We restrict ourselves to the class of diagrams with electron vacuum polarization insertions into the lowest-order muon vertex (see Fig. 1).

One considers the asymptotic part of the photon's self-energy $d_R^\infty(q^2/m_e^2)$, that is, terms of order $O(m_e^2/q^2)$ are neglected.[3] From this one can, in principle, calculate the anomaly to $O(1)$. Another possibility is to use the Kinoshita method.[4] For low-order perturbation theory this approximation seems to work very well. The question is whether this will be valid in high order $n \gg 1$, and how strongly the approximation depends on m_μ/m_e.

It is the purpose of this paper to investigate this question for a simple class of diagrams, namely the mass-dependent n-bubble diagram (see Fig. 2).

Our analysis shows that the expansion breaks down for $n \geq n_0$, where n_0 is dependent on the mass ratio m_μ/m_e. In particular we show that the answer starts oscillating like $(-1)^n$ in disagreement with the exact anomaly which is positive for all n.

It is possible to explain why this so-called "false expansion" breaks down. We have neglected terms such as m_e^2/q^2. Now to get the full anomaly one must integrate $d_R(q^2/m_e^2)$ with $q^2=-m_\mu^2x^2/(1-x)$ over the range $0 \leq x \leq 1$. Clearly, the term m_e^2/q^2 contains a singularity at $x=0$, and so the neglected terms may become important. The full anomaly does not have such a problem since d_R goes to zero for $x \to 0$.

It was shown earlier that d_R^∞ satisfies a homogenous Callan-Symanzik equation,[3] and since the asymptotic anomaly is a linear functional of d_R^∞, it itself satisfies a CS equation. This equation is then solved to all orders, but in view of the above, one might question the validity of this. That is, one cannot neglect the right-hand-side function $\Delta(q^2/m_e^2) \to 0$.

In Sec. II we calculate the anomaly exactly for all n, in the limit $m_\mu/m_e \gg 1$, by making use of the Borel transform technique.[5] For large n an approximate expression is obtained. The exact anomaly is evaluated numerically and is compared to the above-mentioned anomaly for different mass ratios. We also compare with Lautrup's asymptotic estimate.[6]

II. MUON ANOMALY FROM THE MASS-DEPENDENT n-BUBBLE DIAGRAM

The exact muon anomaly from the mass-dependent n-bubble diagram is $a_n(\alpha/\pi)^{n+1}$, where

$$a_n = \int_0^1 dx(1-x)\left[-\pi^{(2)}\left(-\frac{x^2}{1-x}\frac{m_\mu^2}{m_e^2}\right)\right]^n, \tag{1}$$

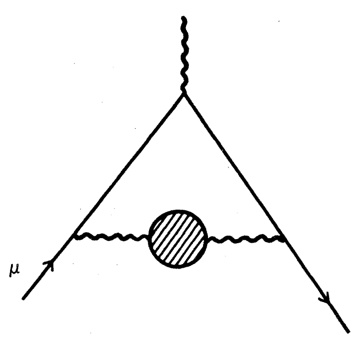

FIG. 1. Electron vacuum polarization insertion into the lowest muon vertex.

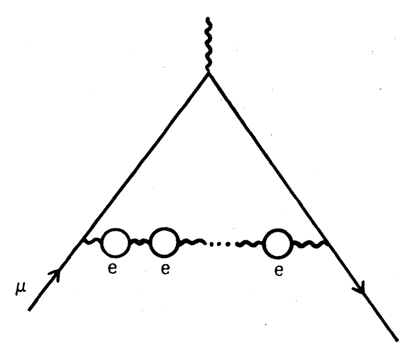

FIG. 2. The mass-dependent n-bubble diagram contributing to g-2 of the muon.

$\pi^{(2)}$ being the standard second-order vacuum polarization function[6] given by

$$\pi^{(2)}\left(\frac{t}{m_e^2}\right) = \frac{8}{9} - \frac{B^2}{3} + \left(\frac{1}{2} - \frac{B^2}{6}\right) B \ln\frac{B-1}{B+1}$$

and

$$B = (1 - 4m_e^2/t)^{1/2}.$$

The anomaly (evaluated in the limit $m_\mu/m_e >> 1$) is denoted b_n and uses the asymptotic vacuum polarization function

$$\pi_\infty^{(2)} = \frac{5}{9} - \frac{2}{3}\ln\frac{m_\mu}{m_e} - \tfrac{1}{3}\ln\frac{x^2}{1-x}. \tag{2}$$

Furthermore, let c_n stand for the anomaly with the x^2 in $\pi_\infty^{(2)}$ replaced by 1. c_n represents the true asymptotic value of a_n for large n.

In the following let L stand for $\ln(m_\mu/m_e)$, $a = \frac{5}{9} - \frac{2}{3}L$, and $b = -\frac{1}{3}$. In order to evaluate b_n we will consider the Borel transform $B(\kappa)$ of the series

$$\sum_{n=0}^{\infty} b_n \kappa^n, \tag{3}$$

which is defined as[5]

$$B(\kappa) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\,\kappa^n. \tag{4}$$

Using Eqs. (1), (2), and (4) one finds

$$B(\kappa) = e^{-\kappa a}\,\frac{(1+\kappa b)}{(2-\kappa b)(1-\kappa b)}\,\frac{\Gamma(1+\kappa b)\Gamma(1-2\kappa b)}{\Gamma(1-\kappa b)}. \tag{5}$$

To obtain b_n one now differentiates $B(\kappa)$ n times with respect to κ:

$$b_n = \left.\frac{d^n B(\kappa)}{d\kappa^n}\right|_{\kappa=0} \equiv B^{(n)}(0). \tag{6}$$

Since it is easier to differentiate $\ln\Gamma(Z)$, we find it convenient to define $G(\kappa) = \ln B(\kappa)$. Using the fact that the Euler function $\psi(Z)$ satisfies

$$\psi(Z) = \frac{d}{dZ}\ln\Gamma(Z),$$
$$\psi^{(n)}(Z)\big|_{Z=1} = (-1)^{n+1} n!\,\zeta(n+1), \tag{7}$$

we find

$$G^{(1)}(0) = -a + \tfrac{5}{2}b = \tfrac{2}{3}L - \tfrac{25}{18},$$
$$G^{(n)}(0) = b^n(n-1)!\left\{(-1)^{n-1} + \frac{1}{2^n} + 1 + \zeta(n)[(-1)^n + 2^n - 1]\right\}, \quad n \geq 2. \tag{8}$$

Asymptotically for large n, $G^{(n)}(0)$ approaches

$$G^{(n)}(0) \simeq (2b)^n (n-1)!\,. \tag{9}$$

To obtain b_n we first notice that the following recursion formula holds (easily proved by differen-

TABLE I. Coefficients $b_{n,m}$ up to $n=5$.

$b_{0,0}$:	$\frac{1}{2}$
$b_{1,0}$:	$-\frac{25}{36}$
$b_{1,1}$:	$\frac{1}{3}$
$b_{2,0}$:	$\frac{2}{9}\zeta(2) + \frac{317}{324}$
$b_{2,1}$:	$-\frac{25}{27}$
$b_{2,2}$:	$\frac{2}{9}$
$b_{3,0}$:	$-\frac{2}{9}\zeta(3) - \frac{25}{27}\zeta(2) - \frac{8609}{5832}$
$b_{3,1}$:	$\frac{4}{9}\zeta(2) + \frac{317}{162}$
$b_{3,2}$:	$-\frac{25}{27}$
$b_{3,3}$:	$\frac{4}{27}$
$b_{4,0}$:	$\frac{16}{27}\zeta(4) + \frac{8}{27}\zeta^2(2) + \frac{100}{81}\zeta(3) + \frac{634}{243}\zeta(2) + \frac{64613}{26244}$
$b_{4,1}$:	$-\frac{16}{27}\zeta(3) - \frac{200}{81}\zeta(2) - \frac{8602}{2187}$
$b_{4,2}$:	$\frac{16}{27}\zeta(2) + \frac{634}{243}$
$b_{4,3}$:	$-\frac{200}{243}$
$b_{4,4}$:	$\frac{8}{81}$
$b_{5,0}$:	$-\frac{40}{27}\zeta(5) - \frac{80}{81}\zeta(3)\zeta(2) - \frac{1000}{243}\zeta(4) - \frac{500}{243}\zeta^2(2) - \frac{3170}{729}\zeta(3) - \frac{43045}{6561}\zeta(2) - \frac{2182775}{472392}$
$b_{5,1}$:	$\frac{160}{81}\zeta(4) + \frac{80}{81}\zeta^2(2) + \frac{1000}{243}\zeta(3) + \frac{6340}{729}\zeta(2) + \frac{323065}{39366}$
$b_{5,2}$:	$-\frac{80}{81}\zeta(3) - \frac{1000}{243}\zeta(2) - \frac{43045}{6561}$
$b_{5,3}$:	$\frac{160}{243}\zeta(2) + \frac{6340}{2187}$
$b_{5,4}$:	$-\frac{500}{729}$
$b_{5,5}$:	$\frac{16}{243}$

TABLE II. Check of asymptotic expression for b_n for the mass ratio $m_\mu/m_e = 10$.

n	b_n	b_n (asymptotic)
0	0.500	0.230
1	0.072	−0.153
2	0.390	0.205
3	−0.180	−0.409
4	1.36	1.09
5	−3.23	−3.64
6	1.51×10^1	1.45×10^1
7	-6.70×10^1	-6.78×10^1
8	3.64×10^2	3.62×10^2
9	-2.17×10^3	-2.17×10^3
10	1.45×10^4	1.45×10^4
11	-1.06×10^5	-1.06×10^5
12	8.52×10^5	8.49×10^5
13	-7.38×10^6	-7.36×10^6
14	6.89×10^7	6.87×10^7
15	-6.89×10^8	-6.87×10^8

TABLE III. The quantities a_n, b_n, and c_n up to $n=15$ for the physical mass ratio $m_\mu/m_e=207$.

n	a_n	b_n	c_n
0	0.500	0.5	3.27×10^7
1	1.09	1.08	5.45×10^6
2	2.72	2.72	1.82×10^6
3	7.23	7.19	9.10×10^5
4	2.02×10^1	2.02×10^1	6.06×10^5
5	5.85×10^1	5.81×10^1	5.05×10^5
6	1.75×10^2	1.75×10^2	5.05×10^5
7	5.40×10^2	5.34×10^2	5.90×10^5
8	1.71×10^3	1.71×10^3	7.86×10^6
9	5.53×10^3	5.40×10^3	1.18×10^6
10	1.83×10^4	1.89×10^4	1.97×10^6
11	6.20×10^4	5.65×10^4	3.60×10^6
12	2.14×10^5	2.54×10^5	7.21×10^6
13	7.55×10^5	3.96×10^5	1.56×10^7
14	2.71×10^6	6.06×10^6	3.64×10^7
15	9.93×10^6	-2.36×10^7	9.11×10^7

tiation of $B(\kappa)=\exp[G(\kappa)]$):

$$B^{(n)}(\kappa)=\sum_{k=0}^{n-1}\binom{n-1}{k}G^{(n-k)}(\kappa)B^{(k)}(\kappa) \tag{10}$$

and, therefore,

$$b_n=\sum_{k=0}^{n-1}\binom{n-1}{k}G^{(n-k)}(0)b_k. \tag{11}$$

If we further write

$$b_n=\sum_{m=0}^{n}b_{n,m}L^m, \tag{12}$$

Eq. (11) gives easily

$$b_{n,m}=\tfrac{2}{3}b_{n-1,m-1}-\tfrac{25}{18}b_{n-1,m}+\sum_{k=m}^{n-2}\binom{n-1}{k}G^{(n-k)}(0)\,b_{k,m} \tag{13}$$

with the requirement $b_{0,m}=\frac{1}{2}\delta_{0,m}$. We now have a recursion relation allowing us to calculate the coefficients $b_{n,m}$ of L^m for arbitrary n. Using REDUCE,[7] we have calculated $b_{n,m}$ up to $n=18$. Table I shows the results up to $n=5$. The $n=0,1,2,3$ values are well known.[1-4]

To get an asymptotic estimate for b_n for $n\gg1$, we go back to Eq. (2). We notice that the singularity at $x=0$ is stronger than the $x=1$ singularity. Setting $x=0$, and using the method of steepest descents we obtain for large n

$$b_n\simeq(-\tfrac{2}{3})^n\,n!\,e^{5/6}\left(\frac{m_e}{m_\mu}\right),\quad n\gg1. \tag{14}$$

Notice that the answer is of $O(m_e/m_\mu)$ and so is comparable with the neglected terms. For a mass ratio $m_\mu/m_e=10$, we checked that this estimate was good to within 2% for $n\gtrsim6$ (see Table II). To see how well b_n approximates a_n, we evaluated a_n by numerical integration. The results for a_n, b_n, and c_n, where[6]

$$c_n\simeq(\tfrac{1}{6})^n n!\,e^{-10/3}\left(\frac{m_\mu}{m_e}\right)^4, \tag{15}$$

are shown in Tables III and IV for the mass ratios $m_\mu/m_e=207$ and $m_\mu/m_e=10$.

We see that for the physical mass ratio $m_\mu/m_e=207$ the approximation $a_n\simeq b_n$ is good up to $n\simeq10$, while for the ratio $m_\mu/m_e=10$, the approximation is totally wrong for all $n\geq1$. That is, in the latter case, the neglected terms of $O(m_e/m_\mu)$ are now bigger than the logarithmic terms and the $O(1)$ terms together. On the other hand, for $n\gtrsim18$, the approximation $a_n\simeq c_n$ is very good.

To summarize, for very large mass ratios, b_n provides a good approximation for low n, while c_n is good for very large n. In the region in between, neither is valid, and one must therefore use the full anomaly. This might have some relevance for the τ-lepton anomaly with muon bubble insertions since $(m_\tau/m_\mu)=16.9$.

TABLE IV. The quantities a_n, b_n, and c_n up to $n=20$ for the mass ratio $m_\mu/m_e=10$.

n	a_n	b_n	c_n
0	0.500	0.500	1.78×10^2
1	0.248	0.072	2.97×10^1
2	0.217	0.390	9.90
3	0.236	-0.180	4.95
4	0.293	1.36	3.30
5	0.405	-3.23	2.75
6	0.610	1.51×10^1	2.75
7	0.990	-6.70×10^1	3.21
8	1.72	3.64×10^2	4.28
9	3.19	-2.17×10^3	6.42
10	6.30	1.45×10^4	1.07×10^1
11	1.31×10^1	-1.06×10^5	1.96×10^1
12	2.92×10^1	8.52×10^5	3.93×10^1
13	6.84×10^1	-7.38×10^6	8.50×10^1
14	1.69×10^2	6.69×10^7	1.98×10^2
15	4.42×10^2	-6.89×10^8	4.96×10^2
16	1.21×10^3	7.35×10^9	1.32×10^3
17	3.54×10^3	-8.33×10^{10}	3.75×10^3
18	1.08×10^4	9.99×10^{11}	1.12×10^4
19	3.45×10^4	-1.27×10^{13}	3.56×10^4
20	1.15×10^5	1.69×10^{14}	1.17×10^5

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