Infrared divergences in quark potential scattering

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We reexamine the cancellation of infrared divergences in massive quark potential scattering in a non-Abelian gauge theory. We improve on previous arguments and extend the proof of cancellation to cases of gauge-nonsinglet potentials and quarks of specific color. We also observe that cancellation fails in the case of the scattering of quarks of specific color by a nonsinglet potential.

I. INTRODUCTION

The work of Doria, Frenkel, and Taylor[1] on infrared (IR) noncancellation in Drell-Yan processes dressed by quantum chromodynamics (QCD) has led to renewed interest in the perturbative IR structure of this theory. Some time ago, an argument was presented that IR divergences cancel to all orders in the scattering of massive quarks from a gauge singlet potential.[2] It is the purpose of this paper to reexamine that problem, correct a technical difficulty in the argument of Ref. 2, and extend the proof of cancellation to cases where either (i) the potential is not a gauge singlet or (ii) the color of the incoming quark is not averaged over. We also find by an explicit example from the one-loop level that if both (i) and (ii) are taken to be the case, IR cancellation fails in general.

The paper is organized as follows. In Sec. II we deal with preliminaries which enable us to identify the origin of IR divergences in the momentum-space integrals from which we calculate cross sections for massive quark potential scattering. Most of this is a review of arguments already given in Ref. 2. In Sec. III, we give the modified argument for cancellation in the singlet potential case with non-color-averaged quarks. In Sec. IV we examine the cases of nonsinglet potentials and nonaveraged quark color.

II. PRELIMINARIES

We will deal with a cross section in which a massive fermion p_i scatters once from a potential which supplies momentum q. We sum over all final states which include gluons whose total energy is less than some value $E_{\max}$. The argument is unaffected by the production of massive fermion pairs by the potential.

Our strategy for demonstrating the cancellation of IR divergences is the following.

(1) We begin by identifying those points in momentum space which give rise to divergences in the contribution of individual graphs to cross sections. At these "pinch singular points" loop-momentum contour integrals must be trapped between singularities. These will include "collinear" as well as IR divergences.

(2) We combine graphical contributions of the same topology (i.e., sum over cuts) and observe that this results in the cancellation of all collinear divergences.

(3) Finally, we combine graphical contributions differing only in terms of the attachments of gluons to the incoming quark line to effect the cancellation of IR divergences.

Before going on, we should indicate how we recognize when divergences associated with any given point in momentum space cancel. There are two ways, one illustrated by step (2) above, and one by step (3). In the former case we find, after summing over cuts, a contour integral which is not pinched between singularities at the point in question. The divergences associated with this point in each of the individual contributions may thus be interpreted as having canceled because the resulting integral gets no divergence from that point. In the latter case, the cancellation is more direct, consisting in combining a set of integrals in which the integrands cancel at the point in question. If, as is the case in potential scattering, the individual integrals are logarithmically divergent, their sum will be finite.

Let us begin with step (1). Graphical contributions to quark scattering cross sections may be represented schematically as in Fig. 1. Γ_L is the graphical contribution to the amplitude, and Γ_R^* to its complex conjugate. Each pinch singular point in the integral is associated with a reduced diagram which includes only the lines which are on-shell at that point. Figure 2 shows the most general of such reduced diagrams. According to the theorem of Coleman and Norton,[3] the on-shell propagation of its lines between vertices must represent a physical process in both Γ_L and Γ_R^*. V_L and V_R are contracted vertices describing the nontrivial momentum transfer in Γ_L and Γ_R^*,

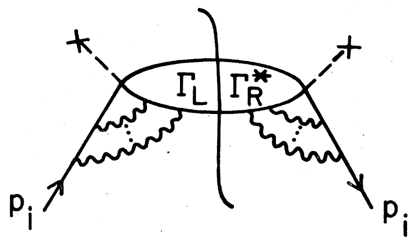

FIG. 1. Graphical contribution to potential scattering.

respectively, from which the final quark p_f and some number of gluon jets (in this case one, represented by J and the heavy lines) emerge. All nonjet uncontracted gluon lines, represented by S_L and S_R^*, carry zero momentum at the pinch singular point. C is the final state connecting the two graphs.

We can now sum over final states C (step 2), grouping together contributions with different Γ_L and Γ_R^* but the same over-all topology. Unitarity requires that the sum of all such contributions integrated over all loop momenta gives the imaginary part of the uncut diagram. We can, in fact, use a much stronger result.[4] Consider a pinch singular point such as the one represented by Fig. 2. *Fix* the three-momenta of all lines in Fig. 2 and integrate only over energies, then sum over cuts. The result is the imaginary part of the uncut Feynman integral integrated over all energies of its loops, but still with all *fixed* three-momenta. We can thus apply unitarity in the neighborhood of any point. We can now reexamine the resulting integral to see whether the point, which is a pinch singular point in each cut graph, is still one in the uncut graph.

Suppose for the moment that the momentum transfer is above threshold in the final state, i.e., $(q+p_i)^2 > m^2$. Then the possible pinch singular points of the resulting integral are quite simple,

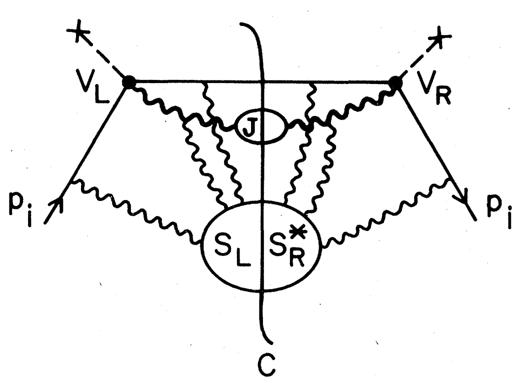

FIG. 2. Reduced diagram representing most general pinch singular point in a graphical contribution.

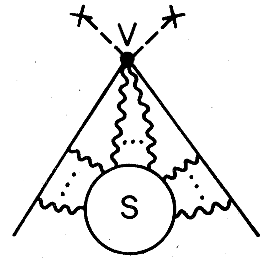

FIG. 3. Most general pinch singular point after the sum over cuts.

much simpler than in the individual terms.[2] The most general reduced diagram is shown in Fig. 3. There are now no jets, only zero-momentum gluons attached to the *initial* fermion line p_i or to the contracted vertex V, and to each other in S. The final fermion has also disappeared into V. Evidently all divergences associated with gluon jets or the attachment of soft lines to the final quark must cancel in the sum because there are no pinch singular points corresponding to such processes left, at least when we are above threshold. We will deal with the special case of threshold scattering, $(q+p_i)^2 \to m^2$ below, but for now we note that step 2 has eliminated much of the complications in the problem, and has left us with only true IR divergences with which to deal.

In fact, above threshold, the situation is even simpler than indicated by Fig. 3. If we count powers in the neighborhood of such points we find that actual divergence is possible only when no gluon lines are attached to V. Thus we are reduced to examining the graphs of Fig. 4. We also note that the soft gluons attach to p_i only at three-point vertices.[3] Above threshold, vertex V is a smooth function of the momenta $\{k\}$ of the gluons in Fig. 4 so that if we can exhibit cancellation between some set of terms which are singular in the k's we will have demonstrated cancellation without having to worry about the details of V above threshold.

When $q+p$ actually reaches threshold there is a bit more of a problem. The subgraph involving the two vertices at which the potential acts can

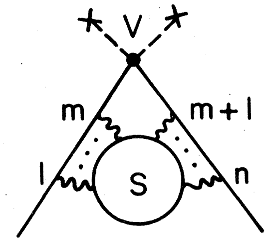

FIG. 4. Pinch singular point associated with divergence above threshold.

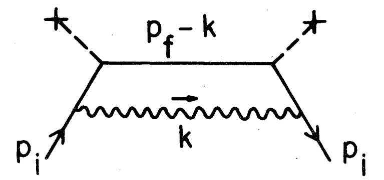

FIG. 5. Graph with extra divergence at threshold.

no longer be considered a smooth function of the k's. This, of course, is the case in the lowest-order example, Fig. 5. But there again we can show—not that V is independent of k—but rather that it is a smooth function of k once q is averaged over a neighborhood of the threshold. Taking the lowest-order example, we choose

$$p=((p_z^2+m^2-\delta^2)^{1/2},0,0,p_z)\,,$$
$$q=q_z\hat{z}+\tilde{q},\quad |\tilde{q}^{\mu}|\ll|q_z|,\quad \mu=0,1,2 \tag{2.1}$$
$$\delta\ll\Delta\ll|p_z|\,.$$

Then, defining V as an average over q,

$$V(p,k)=\int d^4q\,V_L(q)\frac{\not p+\not q-\not k+m}{(p+q-k)^2-m^2+i\epsilon}V_R^*(q)$$
$$\sim\int d^3\tilde{q}\int_{-2p_z-\Delta}^{(q_z)_{\max}}dq_z V_L(q)(\not p+\not q-\not k+m)\,[2p\cdot k+2(p-k)\cdot\tilde{q}-2(p_z-k_z)q_z-q_z^{\,2}-\delta^2+i\epsilon]^{-1}V_R^*(q)\,. \tag{2.2}$$

Here, as in the two-particle irreducible case, the relevant q_z pole is in the lower half-plane, at least for $|k_z|<|p_z|$. [Note $(q_z)_{\max}<0$.] So everywhere in the neighborhood of $k=0$ we can deform the q_z integral as in Fig. 6 so that

$$V(p,k)=\int d^3\tilde{q}\int_{-2p_z-\Delta}^{(q_z)_{\max}}dq_z V_L(q)\frac{\not p+\not q+m}{2p\cdot\tilde{q}-2p_zq_z-q_z^{\,2}-\delta^2+i\epsilon}V_R^*(q)+O\left(\frac{k}{\Delta}\right)\,. \tag{2.3}$$

One way to look at this result is to observe that the discontinuity in V, which is all that contributes to the cross section, comes from the δ-function part of $(\not p+\not q-\not k-m)^{-1}$. Small changes in k near $k=0$ only shift the position of the δ function slightly; they do not change its value in the q_z integral much.

It is easy to generalize this argument to an arbitrary graph. This is done by applying the Coleman-Norton theorem[3] again—now at threshold $(q+p)^2=m^2$. The only pinch singular points that are two-particle irreducible in the vertical channel have reduced diagrams of the form of Fig. 7, where all the $l_i=0$ at the pinch singular point. Now in the q_z integral there are $N+1$ poles, all in the lower half-plane near $q_z=-2p_z$. So the same argument as above shows that $V(p,k)$ is still a smooth function of k near $k=0$.

Before turning to the cancellation of IR divergences in the next section we should add a few words on the nature of the integral near the pinch singular point. The general reduced diagram of Fig. 4 can be decomposed as in Fig. 8, where all two-particle cuts in the vertical channel are exhibited explicitly. The number of (not necessarily connected) gluon Green's functions S_i gives the number of logarithmic divergences associated with each such reduced diagram. We can see this by counting powers: Within each S_i, all components of all momenta must vanish together to get divergence, but the momenta in S_j may vanish independently of S_k for $j<k$. There are only n independent ways of scaling momenta to zero while retaining logarithmic divergence, and thus there are only n such logarithms. This pattern is already present in QED.[5] There is one apparent exception to this rule however, having to do with

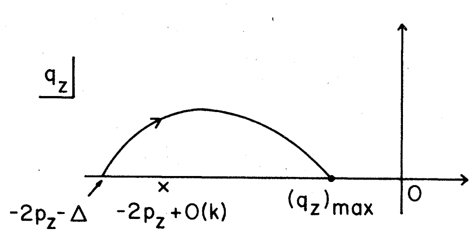

FIG. 6. Deformation of q_z integral.

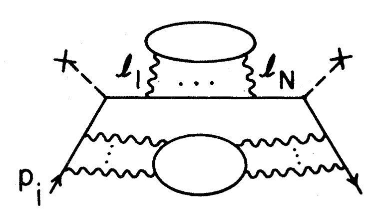

FIG. 7. Higher-order example.

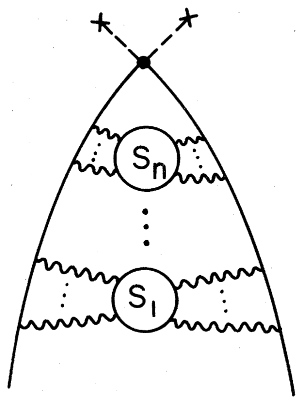

FIG. 8. Decomposition of Fig. 4 by two-fermion states.

the quark lines. Consider Fig. 9, in which a particular quark propagator $p+k$ is isolated. We choose for definiteness the rest frame of p. The relevant denominator is thus

$$2p_0k_0 + k^2 + i\epsilon \ . \tag{2.4}$$

If all components of k are of the same order, only the first term in (2.4) need be kept. This is the eikonal approximation. But suppose k_0 is allowed to decrease independently of the other components of k. The order of magnitude of none of the denominators in S will be affected because they are all quadratic in all components of the k's. Locally, the k_0 integral will look like

$$\int_{O(|\vec{k}|^2/p_0)}^{O(|\vec{k}|)} \frac{dk_0}{2p_0k_0 + i\epsilon} \ , \tag{2.5}$$

which seems to give another logarithmic divergence, this time associated with the independent vanishing of k_0 for fixed k. When k_0 is literally zero we no longer have

$$2p\cdot k \gg k^2 \tag{2.6}$$

and the eikonal approximation fails. What are we to do with such noneikonal contributions? The thing is to notice that the k_0 integral is not trapped at $k_0 = 0$ for k fixed, *even when we make the eikonal approximation*. Thus the k_0 integration can be performed without ever coming nearer than $O(|\vec{k}|)$ of $k_0 = 0$ by deforming the k integral into the complex plane. The direction of deformation is specified by the $i\epsilon$ on the fermion line. Thus the potential extra logarithms associated with noneikonal quarks never actually develop, and we can, if we want to, do all the integrations in such a way that the eikonal approximation is always valid. However, in comparing different integrals for possible cancellation between their integrands (step 3), we must verify that the k_0 integral has been deformed in the same way in each. This possible source of difficulty, which was not discussed in Ref. 2, is dealt with in the next section.

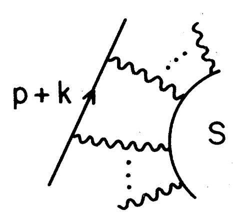

FIG. 9. Graph for discussion of eikonal approximation.

III. CANCELLATION FOR GAUGE-SINGLET POTENTIALS

We are now ready to implement step 3 of the procedure outlined above, and demonstrate cancellation of integrands between singular contributions from various graphs. We have already reduced the possible sources of trouble to points corresponding to reduced diagrams such as Fig. 8. All momenta are nearly zero near the pinch singular point and, in accordance with the comments made at the end of Sec. II, we must pay attention to the possibility that not all quark denominators are eikonal before contours are deformed. We must also be careful that our deformations do not ruin the cancellation which we exhibit.

In this section our potential is taken to be a gauge singlet. We need not average over the color of the incoming quark.

The set of graphs to be combined is exhibited in Fig. 10, with momenta labeled as shown. The Green's function S is the same in each graph, as are the loop momenta $k_1, \ldots, k_n$. S need not be connected, and the whole graph may be two-particle reducible in the vertical channel. We notice that the first and last terms in the sum are self-

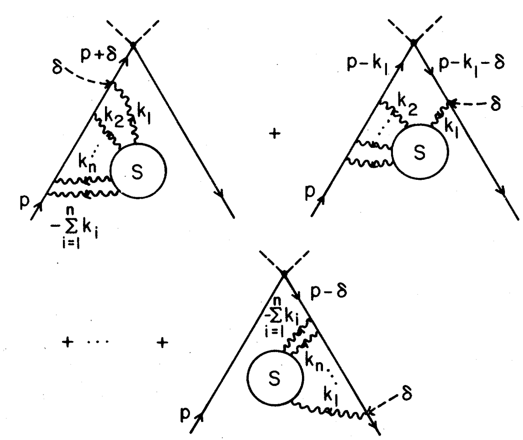

FIG. 10. Contributions added to derive cancellation.

energies, so that the top-most quark propagators are formally on-shell. We know that these poles are canceled by zeros of the self-energy on-shell after mass renormalization. Also, only one of the two self-energies is actually counted in the cross section. As in Ref. 2, we have introduced a "fake" momentum δ (attached to k_1 in Fig. 10) to solve these problems. We will exhibit cancellation at finite δ, which is defined to be real and positive semidefinite. The physical result is recovered in the $\delta \to 0$ limit if we use the fact that

$$\bar{u}(p)\Sigma(p)\frac{1}{\not p+\not\delta-m}=0\,,\quad p^2=m^2\,. \tag{3.1}$$

Equation (3.1) shows that there are *internal* cancellations in the first graph of Fig. 10. Thus, showing cancellation for an arbitrary S in Fig. 10 is enough to demonstrate cancellation for every possible source of divergence in the physical cross section.

With the choice of loop-momentum flow shown in Fig. 10, the problem of noneikonal quarks is trivial. In each graph the orientation of each loop momentum k relative to the quark momentum p is fixed. Poles due to the quark lines are always in the upper half-plane in the k_0 integrals. As a result we can deform all the k_0 integrals into the lower half-plane in the same way in all the terms in Fig. 10. The k_0 integrals will encounter singularities in the lower half-plane only from denominators in S, at a scale $O(|\vec{k}|)$. Thus we may take $(k_i)_0$ to be of the order of $|\vec{k}_i|$ for all i, so that (2.6) is satisfied by all the k_i, and the eikonal approximation is justified. Since k_0 contours are deformed the same way in the different terms, we can cancel them algebraically between corresponding contributions.

The logarithmically divergent part of the integral comes only from the factor $\not p+m$ in each quark numerator. The complete numerator contributing to the divergent part is then independent of m, and is given by

$$\psi^{\dagger}_{i,a}(p)\times T_1\times T_2\times\cdots\times T_m\times\psi_{i,a}(p)\prod_{i=1}^{n}(2p_{\mu_i})\,, \tag{3.2}$$

where $\psi_{i,a}(p)$ is the quark wave function of spin i and color a. p_{μ_i} contracts into the ith vector index of S, and T_i into the ith color index. Notice that the spin factors $2p_\mu$, which follow from the Dirac equation, are the same as in QED. The group factors are the same in each graph of Fig. 10 because the potential is assumed to be a singlet.

In the cross section, (3.1) is multiplied by the product of eikonal denominators. For $n=1$, their sum over the three terms in Fig. 10 is

$$\begin{aligned}E_1=&\frac{1}{(2p\cdot\delta)(-2p\cdot k_1+i\epsilon)}\\&+\frac{1}{(-2p\cdot k_1+i\epsilon)[-2p\cdot(k_1+\delta)+i\epsilon]}\\&+\frac{1}{(-2p\cdot\delta)[-2p\cdot(k_1+\delta)+i\epsilon]}=0\,.\end{aligned} \tag{3.3}$$

Notice that since δ is positive semidefinite we can drop the $i\epsilon$ in the denominators with only $p\cdot\delta$. We can verify the cancellation for $n>1$ iteratively. We write the sum of eikonal terms for arbitrary n as

$$E_n=L_n+S_n+R_n\,, \tag{3.4}$$

with

$$\begin{aligned}L_n&=\frac{1}{2p\cdot\delta}\prod_{m=1}^{n}\frac{1}{-2\sum_{i=1}^{m}p\cdot k_i+i\epsilon}\,,\\ S_n&=\sum_{s=1}^{n-1}\prod_{m=s}^{n}\frac{1}{-2\sum_{i=1}^{m}p\cdot k_i+i\epsilon}\prod_{h=1}^{s}\frac{1}{-2p\cdot\left(\sum_{i=1}^{h}k_i+\delta\right)+i\epsilon}\,,\\ R_n&=\frac{1}{-2p\cdot\delta}\prod_{m=1}^{n}\frac{1}{-2p\cdot\left(\sum_{i=1}^{m}k_i+\delta\right)+i\epsilon}\,.\end{aligned} \tag{3.5}$$

L_n is the contribution of the first term in Fig. 9, and R_n is of the last term. Now we also have

$$E_n=\frac{1}{-2p\cdot\left(\sum_{i=1}^{n}k_i+\delta\right)+i\epsilon}R_{n-1}+\frac{1}{\sum_{i=1}^{n}p\cdot k_i+i\epsilon}\left[L_{n-1}+S_{n-1}-\frac{2p\cdot\delta}{-2p\cdot\left(\sum_{i=1}^{n}k_i+\delta\right)+i\epsilon}R_{n-1}\right]. \tag{3.6}$$

Substituting $R_{n-1}=-L_{n-1}-S_{n-1}$ into (3.6) shows that $E_n=0$. So, as claimed, the sum of eikonal contributions vanishes identically and the cancellation of IR divergences is demonstrated. Note that the identity here properly incorporates the $i\epsilon$ terms in the denominators, unlike the corresponding identity of Ref. 2.

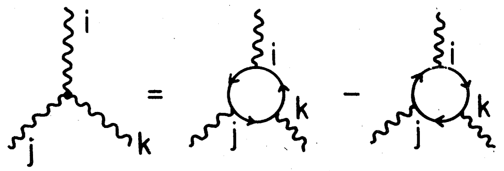

FIG. 11. Group identity relating gluon coupling to quark color flow.

IV. CANCELLATION FOR GAUGE-NONSINGLET POTENTIALS

In the previous section the cancellation of infrared divergences for gauge-singlet potentials was shown. For the gauge-nonsinglet case the only difference for the graphs is the color factors contained in Eq. (3.1). For the $(m+1)$st term of Fig. 10, after averaging over quark color it is now given by

$$\mathrm{Tr}(T_1 \times T_2 \times \cdots \times T_m V T_{m+1} \times \cdots \times T_n)\,, \tag{4.1}$$

where V refers to the color matrix for the vertex V in Fig. 4. Our aim now is to find the cases where this factor is independent of m, so the various graphs can still be added, as in Sec. III, to give cancellation.

To be specific we consider the quarks to belong to the fundamental representation of the gauge group. For the argument we consider SU(3). Let the color factor of the blob S be $S^{1,\ldots,m,\ldots,n}$ in which case the overall color factor is

$$\mathrm{Tr}(\lambda_1\lambda_2 \ldots \lambda_m V \lambda_{m+1} \ldots \lambda_n) S^{1,2,\ldots,m,m+1,\ldots,n}\,, \tag{4.2}$$

the trace being taken for color-averaged quarks.

To proceed with the argument we need to decompose the gluon color factors into the fundamental representation. This is done for instance in Ref. 6, and two useful results are

$$iC_{ijk} = \tfrac{1}{4}\,\mathrm{Tr}(\lambda_i\lambda_j\lambda_k - \lambda_j\lambda_i\lambda_k)\,, \tag{4.3}$$

which is shown in Fig. 11, and

$$\lambda^i_{ab}\lambda^i_{cd} = 2\delta_{ad}\delta_{bc} - \tfrac{2}{3}\delta_{ab}\delta_{cd}\,, \tag{4.4}$$

shown in Fig. 12. The solid lines give us the flow of color only. Thus as far as color is concerned we can replace gluon lines by quark lines.

If we make the above replacements, the color

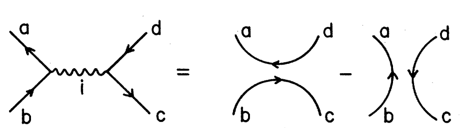

FIG. 12. Group identity relating gluon to quark color flow.

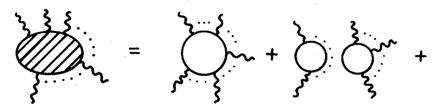

FIG. 13. Decomposition of group factors in multigluon Green's function.

factors of the gluon n-point function S can be decomposed into the n gluons attaching to fermion loops, with no internal gluons remaining (Fig. 13). This is because the internal gluons can all be replaced by quark-antiquark lines. The gluons which attach to a particular fermion line are determined then by the topology of the graph. In general there will be many such terms for each S.

The gluon n-point function couples to the external quark line. If we move the gluon line m across V then the color factor is changed to

$$\mathrm{Tr}(\lambda_1\lambda_2 \ldots \lambda_{m-1} V \lambda_m \lambda_{m+1} \ldots \lambda_n) S^{1,2,\ldots,m-1,m,m+1,\ldots,n} \tag{4.5}$$

and we must show the equality of Eqs. (4.2) and (4.5).

The color factor for one particular term is shown in Fig. 14. In general there will be more terms but we can treat each separately. Now we once again replace all the other gluon lines by quark-antiquark lines and we arrive at the case where the gluon m now attaches itself to only one remaining quark line or to a quark line and a loop (Fig. 15). The latter is zero as $\mathrm{Tr}\lambda_i = 0$. Thus the trace reduces to the form

$$\mathrm{Tr}(\lambda_m\lambda_m V) \tag{4.6}$$

and we get the same results in either case as it does not matter whether the λ_m is located before or after the V. The argument of Sec. II now applies.

This leaves us only with the case when the color of the incoming quark is not averaged over and the potential is not a singlet. We choose an octet potential. There we see that there is *no* cancel-

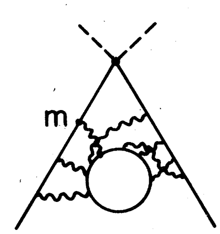

FIG. 14. Result of applying Fig. 13 and Fig. 4.

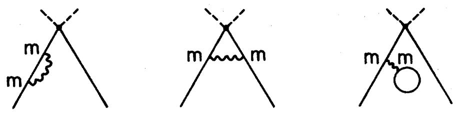

FIG. 15. Reduction of Fig. 14 using Fig. 12.

FIG. 16. One-loop contributions to scattering.

lation even at the one-loop level. For instance, the color factors associated with the diagrams of Fig. 16 are $\lambda_i\lambda_m\lambda_n\lambda_i$ and $\lambda_m\lambda_n\lambda_i\lambda_i$ whose diagonal terms are simply not the same. With equal weights these two contributions cancel identically in the soft region; with unequal weights they cannot cancel. Thus the noncancellation found in the color-averaged Drell-Yan process in Ref. 1 at the two-loop level occurs in single quark scattering if color is not averaged over and the potential is not a singlet.

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